A Solution Manual For

Collection of Kovacic problems

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May 15, 2024

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1 Introduction to Kovacic algorithm

The following is a small introduction to Kovacic algorithm used in solving the ode's listed in this chapter. The algorithm was implemented based on the original paper by Kovacic.[1]

The algorithm is implemented as a Maple module with 4 submodules. The code took about two months to implement and is about 3,000 lines which includes the generation of the latex for each step. Without the need to generate the latex showing all the steps, the code would have been much shorter. Case one of the algorithm was the hardest to implement.

Given an ode of the form $y'' + ay' + by = 0$, it is first converted to $z'' = rz$ by transformation to remove the first derivative which is $y = ze^{\frac{1}{2}\int a\,dx}$. This results in the ode

$$
\begin{aligned}
z'' &= rz \\
&= \left(\frac{1}{4}a^2 + \frac{1}{2}a' - b\right) z
\end{aligned}
\tag{1}
$$

It is eq. (1) which is solved by Kovacic algorithm. Then the first solution y_1 to the original ode is found by inverse transformation using the solution z_1 to (1).

The second solution y_2 is found by reduction of order.

Kovacic algorithms finds a Liouvillian solution to (1) if one exists. There are 4 cases. From now on eq. (1) will be called the DE.

1. DE has solution $z = e^{\int \omega\, dx}$ where $\omega \in \mathbb{C}(x)$
2. DE has solution $z = e^{\int \omega\, dx}$ where ω is polynomial over $\mathbb{C}(x)$ of degree and case (1) does not hold.
3. Solutions of DE are algebraic over $\mathbb{C}(x)$ and case 1,2 do not hold.
4. DE has no Liouvillian solution.

Before showing the algorithm itself and describing how it works, there are necessary (but not sufficient) conditions that should be checked to determine which of the above cases the DE satisfies.

The following are the necessary conditions for each case. To check each case, let $r = \frac{s}{t}$ where $\gcd(s,t) = 1$. This means there is no common factor between s, t. The order of

[1] An Algorithm for Solving Second Order Linear Homogeneous Differential Equations (1985 version). By JERALD J. KOVACIC

r at ∞ is defined as deg (t) − deg (s).

For example, if $r = \frac{1}{x^2}$ then $O(\infty) = 2 - 0 = 2$. And if $r = \frac{1+x}{3x^2}$ then $O(\infty) = 2 - 1 = 1$. The poles of r and the order of each pole needs to be determined.

The poles of r are the zeros of t. For example if $t = x(1 - x)^2$ then there is one pole is at $x = 1$ of order 2 and one pole at $x = 0$ of order 1.

Knowing these two pieces of information all what is needed to determine the necessary conditions for each case and these are the following

1. Every pole of r must have even order or its order is 1. And $O(\infty)$ is even or greater than 2. No poles are allowed in this case. For example, $r = (x^2 + 3)$ has a pole of order zero. Since zero is even number, then this r qualifies as case 1 because it has $O(\infty) = 0 - 2 = -2$ which is even.

2. r have at least one pole of order 2 or the order is odd and greater than 2. There are no conditions related to $O(\infty)$ for this case.

3. r have only poles of order 1 or 2. And $O(\infty)$ must be at least 2.

If the conditions are not satisfied then there is no need to try that specific case as there will be no solution. However if the conditions are satisfied, this does not necessarily mean a solution exists for that case. This is what necessary but not sufficient conditions means.

The following table summarizes the above conditions and the possible L list for each case.

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Some observations: In case one, no odd order pole is allowed except for pole of order 1. For case 3, only poles of order 1,2 are allowed. If $O(\infty) = 0$, which means s and t have same degree, then only possibility is case 1 or case 2. Case 3 is not possible.

For case 1, if $O(\infty)$ is negative, then it has to be even. For example if $r = \frac{x^6}{(x-1)^2}$ then now $O(\infty) = 2 - 6 = -4$. But if $r = \frac{x^5}{(x-1)^2}$ then $O(\infty) = 2 - 5 = -3$ and hence this can not be case 1.

If a pole is of order 2 and $O(\infty)$ is say 2, then all three cases are met. In this case $L = [1, 2, 4, 6, 12]$. If case 1 and 2 are met only then $L = [1, 2]$.

The algorithm is described in details in another document here.

The following diagrams give summary of the algorithm for each case. In the implementation, it was found that majority of ode's used in this chapter fell into case one.

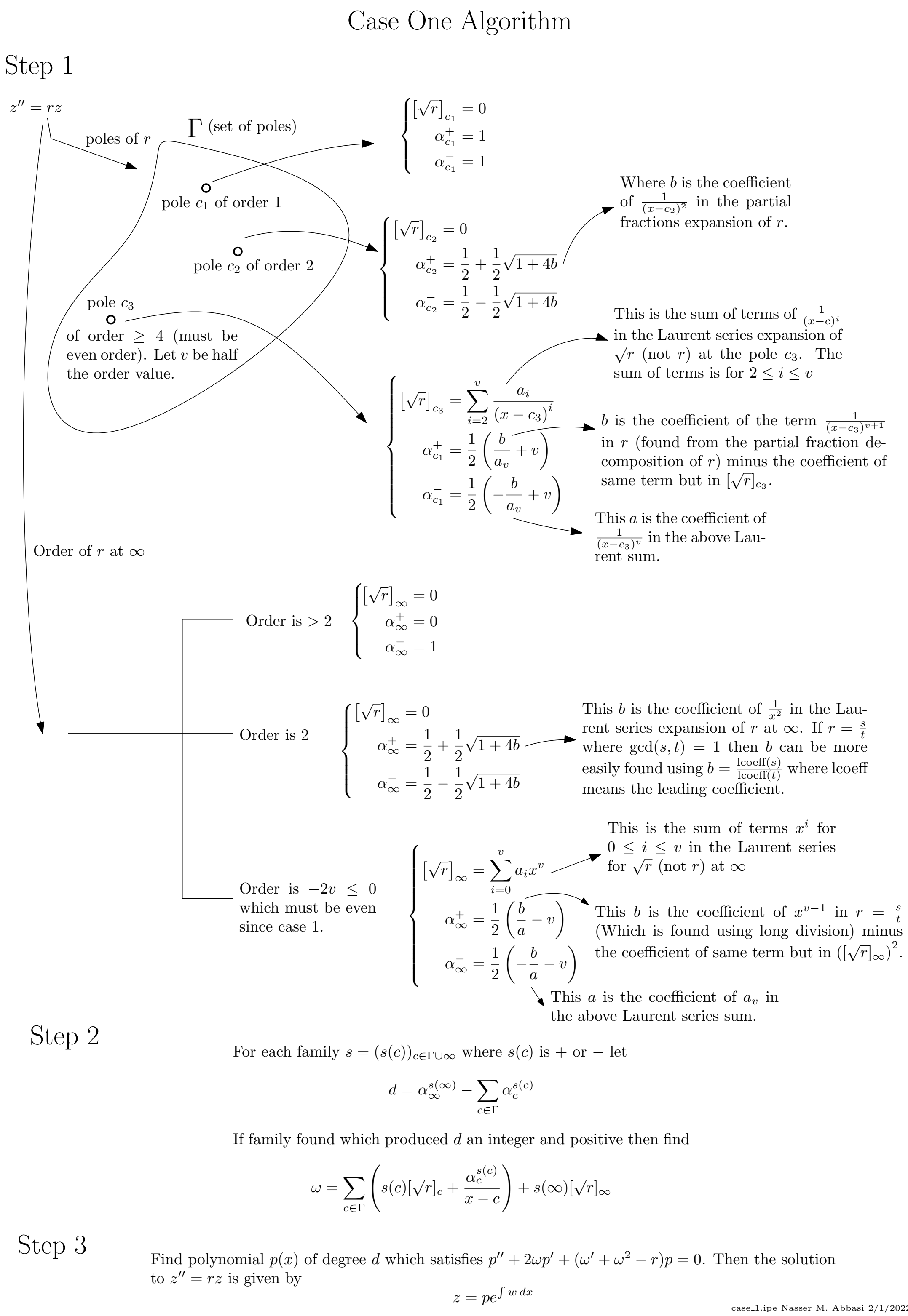


Figure 1: Case 1 algorithm

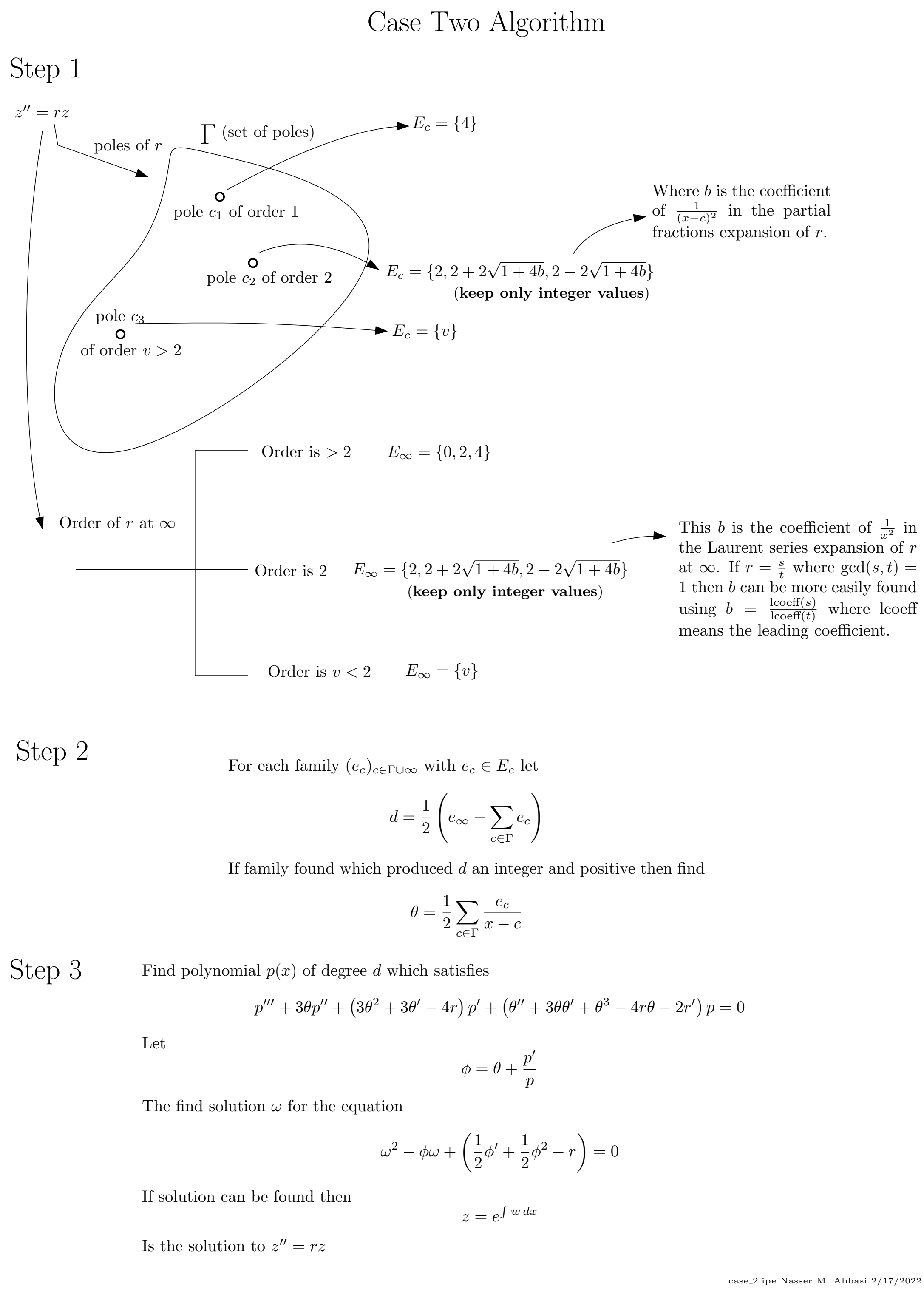
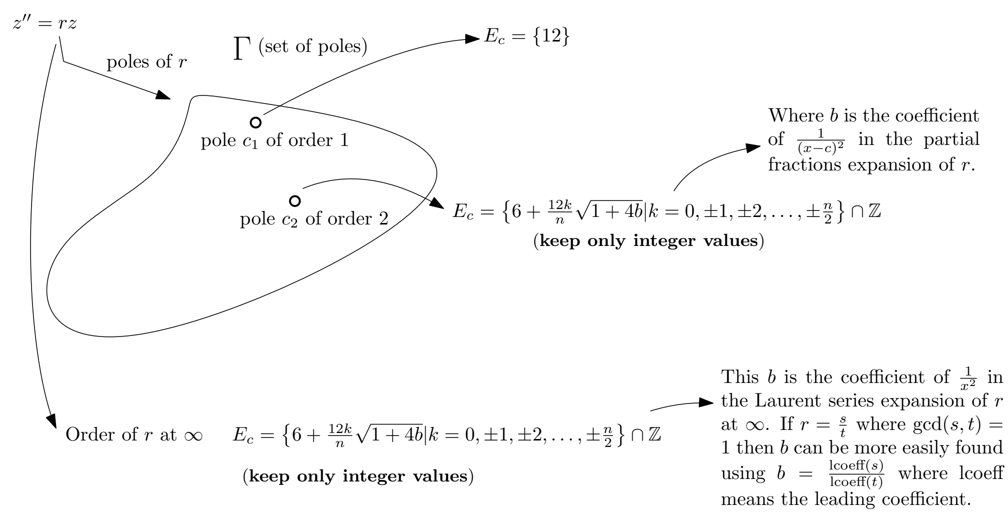


Figure 2: Case 2 algorithm

Case 3 Algorithm

Step 1



Step 2

For each family $(e_c)_{c \in \Gamma \cup \infty}$ with $e_c \in E_c$ let

$$d = \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

If family found which produced d an integer and positive then find

$$\theta = \frac{n}{12}\sum_{c \in \Gamma} \frac{e_c}{x - c}$$

Step 3

Let polynomial $p(x)$ of degree d with coefficients a_i. . The following set of equations are set up in order to determine the coefficients a_i of the above polynomial

$$\begin{aligned} p_n &= -p \\ p_{i-1} &= -p_i' - \theta p_i - (n-1)(i+1) r p_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned}$$

Where n above is either 4,6 or 12.
The coefficients a_i are solved for from

$$p_{-1} = 0$$

By using method of undetermined coefficients. Now generate equation for ω using the equation

$$\sum_{i=0}^{n} \frac{p_i}{(n-i)!}\omega^i = 0$$

If solution ω can be found then

$$z = e^{\int w \, dx}$$

Is the solution to $z'' = rz$

case_3.ipe Nasser M. Abbasi 2/17/2022

Figure 3: Case 3 algorithm

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2.1 problem 1

2.1.1 Maple step by step solution 35

Internal problem ID [7491]
Internal file name [`OUTPUT/6424_Sunday_June_05_2022_04_51_56_PM_50821051/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2-1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-1\\ B&=-2x\\ C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{(x^2 - 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= (x^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{(x^2 - 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2} + \frac{3}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{3}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \frac{3}{2(1+x)} + (-)(0) \\ &= -\frac{1}{2(x-1)} + \frac{3}{2(1+x)} \\ &= \frac{x-2}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(1+x)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{3}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\ &= z_1 \left(\sqrt{x-1}\,\sqrt{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(1+x)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^2\right) + c_2\left((1+x)^2\left(-\frac{x}{(1+x)^2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1(1+x)^2 - c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^2 - c_2 x
$$

Verified OK.

2.1.1 Maple step by step solution

Let's solve

$\left(x^2-1\right) y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' - 2xy' + 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-1)+a_k(k+r-1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r-2)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{a_0}{4}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$

- Revert the change of variables $u=1+x$

  $\left[y=\frac{a_0(x-1)^2}{4}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k k}{2(k+3)}$

- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2},a_{k+1}=\frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u=1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 (x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2-1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 + 1)$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 39

```
DSolve[(x^2-1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x^2-1}\left(c_1 (x-1)^2 + c_2 x\right)}{\sqrt{1-x^2}}$$

2.2 problem 2

2.2.1 Maple step by step solution 44

Internal problem ID [7492]
Internal file name [OUTPUT/6425_Sunday_June_05_2022_04_52_00_PM_73451663/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(x^2-1\right)y''-6xy'+12y=0}$$

Writing the ode as

$$\left(x^2-1\right)y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-1\\B&=-6x\\C&=12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= \left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 3: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(1+x)^2} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)} + \frac{15}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} + (-)(0) \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)}\right)(0) + \left(\left(\frac{3}{2\,(x-1)^2} - \frac{5}{2\,(1+x)^2}\right) + \left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{5}{2}}}{(x-1)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(1+x)}{2}} \\ &= z_1 \left((x-1)^{\frac{3}{2}} (1+x)^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(x-1) + 3\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{x(x^2+1)}{(1+x)^4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^4\right) + c_2\left((1+x)^4\left(-\frac{x(x^2+1)}{(1+x)^4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)^4 - c_2 x\left(x^2+1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^4 - c_2 x\left(x^2+1\right)
$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$(x^2-1)\,y'' - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -3$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''-6xy'+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-3)+a_k(k+r-3)\,(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-3)\left(\left(-2k-2r-2\right)a_{k+1}+a_k(k+r-4)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-2a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{3a_0}{2}$

- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$

- Express in terms of a_0

  $a_3=-\frac{a_0}{2}$

- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{8}$

- Express in terms of a_0

  $a_4=\frac{a_0}{16}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$

- Revert the change of variables $u=1+x$

  $\left[y=\frac{a_0(x-1)^4}{16}\right]$

- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k k}{2(k+5)}$

- Solution for $r=4$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\frac{a_0 (x-1)^4}{16}+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+4}\right), b_{k+1}=\frac{b_k k}{2(k+5)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\left(x^3+x\right)+c_2\left(x^4+6x^2+1\right)$$

✓ Solution by Mathematica
Time used: 0.132 (sec). Leaf size: 45

```
DSolve[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{x^2-1}\left(c_2 x\left(x^2+1\right)+c_1(x-1)^4\right)}{\sqrt{1-x^2}}$$

2.3 problem 3

Internal problem ID [7493]
Internal file name [`OUTPUT/6426_Sunday_June_05_2022_04_52_02_PM_39037889/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+3)\,y''-7xy'+16y=0}$$

Writing the ode as

$$(x^2+3)\,y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3 \\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 234}{4\left(x^2+3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 234 \\ t &= 4\left(x^2+3\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 234}{4\left(x^2+3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 5: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x-i\sqrt{3}\right)^2} + \frac{77}{16\left(x+i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x-i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x+i\sqrt{3}\right)}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-234}{4\left(x^2+3\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} + (-)\left(0\right) \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2+6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{7}{4\left(x-i\sqrt{3}\right)^2}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) e^{\int \left(-\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)}\right) dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left(x^2+3\right)^{\frac{7}{4}}} \\
&= \frac{8x^4 - 72x^2 + 27}{8\left(x^2+3\right)^{\frac{7}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-7x}{x^2+3}\,dx} \\
&= z_1 e^{\frac{7\ln\left(x^2+3\right)}{4}} \\
&= z_1\left(\left(x^2+3\right)^{\frac{7}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 9x^2 + \frac{27}{8}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{7\ln\left(x^2+3\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-256\left(x^4-9x^2+\frac{27}{8}\right)\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\ln\left(\sqrt{x^2+3}-x\right)+\left(-3840x^5-720\right.}{256\left(x^4-9x^2+\frac{27}{8}\right)\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+\right.}\right.
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + c_2\left(x^4 - 9x^2\right. \\
&\quad + \frac{27}{8}\left(\frac{-256\left(x^4-9x^2+\frac{27}{8}\right)\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\ln\left(\sqrt{x^2+3}-x\right)+\left(-3840x^5-7\right.}{256\left(x^4-9x^2+\frac{27}{8}\right)\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x\right.}\right.
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + \frac{c_2\left(-256\left(x^4 - 9x^2 + \frac{27}{8}\right)\left(\left(-x^3 - \frac{3}{2}x\right)\sqrt{x^2+3} + x^4 + 3x^2 + \frac{9}{8}\right)\ln\left(\sqrt{x^2+3} - x\right) + \left(-3840x^5 - 720x\right.\right.}{\left(-256x^3 - 384x\right)\sqrt{x^2+3} + 256x^4 + 768x^2 +} \tag{1}$$

Verification of solutions

$$y = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + \frac{c_2\left(-256\left(x^4 - 9x^2 + \frac{27}{8}\right)\left(\left(-x^3 - \frac{3}{2}x\right)\sqrt{x^2+3} + x^4 + 3x^2 + \frac{9}{8}\right)\ln\left(\sqrt{x^2+3} - x\right) + \left(-3840x^5 - 720x\right.\right.}{\left(-256x^3 - 384x\right)\sqrt{x^2+3} + 256x^4 + 768x^2 +}$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 101

```
dsolve((x^2+3)*diff(y(x),x$2)-7*x*diff(y(x),x)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + c_2\left(\frac{\ln\left(\sqrt{x^2+3} - x\right)x^4}{64} + \frac{25\sqrt{x^2+3}\,x^3}{768} + \frac{25x^4}{768} - \frac{9\ln\left(\sqrt{x^2+3} - x\right)x^2}{64} - \frac{55\sqrt{x^2+3}\,x}{512} - \frac{75x^2}{256} + \frac{27\ln\left(\sqrt{x^2+3} - x\right)}{512} + \frac{225}{2048}\right)$$

✓ Solution by Mathematica

Time used: 0.523 (sec). Leaf size: 492

```
DSolve[(x^2+3)*y''[x]-7*x*y'[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24}c_2\Big(12960x^2\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$+5248800x^2\text{RootSum}\Big[210880720572480000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-2723852861741002572000\#1^2-410617$$
$$-4860\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$-1968300\text{RootSum}\Big[210880720572480000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-2723852861741002572000\#1^2-410617$$
$$-1440x^4\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$-583200x^4\text{RootSum}\Big[210880720572480000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(273530830607325028080000000\#1^3-2723852861741002572000\#1^2-410617$$
$$+165\sqrt{x^2+3}x+216x^2\log\left(\sqrt{x^2+3}-x\right)-81\log\left(\sqrt{x^2+3}-x\right)$$
$$-24x^4\log\left(\sqrt{x^2+3}-x\right)-50\sqrt{x^2+3}x^3\Big)+c_1\left(x^4-9x^2+\frac{27}{8}\right)$$

2.4 problem 4

2.4.1 Maple step by step solution 61

Internal problem ID [7494]
Internal file name [OUTPUT/6427_Sunday_June_05_2022_04_52_07_PM_33490433/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''+8xy'+12y=0}$$

Writing the ode as

$$(x^2-1)\,y''+8xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= 8x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8$$
$$t = (x^2-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 6: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-1} + \frac{2}{1+x} + (-)\,(0) \\ &= -\frac{1}{x-1} + \frac{2}{1+x} \\ &= \frac{-3+x}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{1+x}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(1+x)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{1+x}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1} + \frac{2}{1+x}\right) dx} \\ &= \frac{(1+x)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x}{x^2-1} \, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(1+x)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-3x^2-1}{3\,(1+x)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3} \left(\frac{-3x^2-1}{3\,(1+x)^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3\,(x-1)^3\,(1+x)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3\,(x-1)^3\,(1+x)^3}
$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$(x^2-1)\,y'' + 8xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left.\left((1+x) \cdot P_2(x)\right)\right|_{x=-1} = 4$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left.\left((1+x)^2 \cdot P_3(x)\right)\right|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 8xy' + 12y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (8u - 8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3))\, u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+4)((-2k-2r-2)\, a_{k+1} + a_k(k+r+3)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((x^2-1)*diff(y(x),x$2)+8*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x^2+1)}{(x-1)^3(x+1)^3} + \frac{c_2(x^3+3x)}{(x-1)^3(x+1)^3}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 37

```
DSolve[(x^2-1)*y''[x]+8*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3(x^2-1)^3}$$

2.5 problem 5

Internal problem ID [7495]
Internal file name [`OUTPUT/6428_Sunday_June_05_2022_04_52_10_PM_91733108/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3y'' + xy' - 4y = 0}$$

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 54$$
$$t = 36$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+54}{36} \\ &= Q + \frac{R}{36} \\ &= \left(\frac{x^2}{36} + \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{36} + \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= 4 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{36} + \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18-a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\int \frac{x}{6} dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{3}\, dx} \\
&= z_1 e^{-\frac{x^2}{12}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 18x^2 + 27
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 18x^2 + 27\right) + c_2 \left(x^4 + 18x^2 + 27 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{\left(x^4 + 18x^2 + 27\right)^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(3*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 43

```
DSolve[3*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{6}} \,\mathrm{HermiteH}\left(-5, \frac{x}{\sqrt{6}}\right) + \frac{1}{27}c_2(x^4 + 18x^2 + 27)$$

2.6 problem 6

2.6.1 Maple step by step solution 80

Internal problem ID [7496]
Internal file name [OUTPUT/6429_Sunday_June_05_2022_04_52_14_PM_19398223/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{5y'' - 2xy' + 10y = 0}$$

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 55 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 55}{25} \\ &= Q + \frac{R}{25} \\ &= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\ &= \frac{x^2}{25} - \frac{11}{5} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{5}\right) - (0) \\ &= -\frac{11}{5} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{5} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &= -6 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{x}{5}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^2}{25}\right.\right.$$

$$\frac{2a_4x^4}{5} + \frac{4(25+a_3)\,x^3}{5} + \frac{6(a_2+10a_4)\,x^2}{5} + \frac{2(4a_1+15a_3)\,x}{5} + 2a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{\int -\frac{x}{5} dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{-\frac{x^2}{10}} \\
&= \frac{(4x^5 - 100x^3 + 375x) e^{-\frac{x^2}{10}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x}{5} \, dx} \\
&= z_1 e^{\frac{x^2}{10}} \\
&= z_1 \left(e^{\frac{x^2}{10}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{16\, e^{\frac{x^2}{5}}}{x^2 \left(4x^4 - 100x^2 + 375\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^5 - 25x^3 + \frac{375}{4} x \right) + c_2 \left(x^5 - 25x^3 + \frac{375}{4} x \left(\int \frac{16\, e^{\frac{x^2}{5}}}{x^2 \left(4x^4 - 100x^2 + 375\right)^2} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \left(x^5 - 25x^3 + \frac{375}{4} x \right) \\
+ 16 c_2 x \left(x^4 - 25x^2 + \frac{375}{4} \right) \left(\int \frac{e^{\frac{x^2}{5}}}{x^2 \left(4x^4 - 100x^2 + 375\right)^2} dx \right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(x^5 - 25x^3 + \frac{375}{4} x \right) + 16 c_2 x \left(x^4 - 25x^2 + \frac{375}{4} \right) \left(\int \frac{e^{\frac{x^2}{5}}}{x^2 \left(4x^4 - 100x^2 + 375\right)^2} dx \right)
$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$5y'' - 2xy' + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{5} - 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2xy'}{5} + 2y = 0$

- Multiply by denominators

  $5y'' - 2xy' + 10y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(5a_{k+2}(k+2)\,(k+1) - 2a_k(k-5)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $5(k^2 + 3k + 2)\, a_{k+2} - 2a_k(k-5) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(5*diff(y(x),x$2)-2*x*diff(y(x),x)+10*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\left(x^5-25x^3+\frac{375}{4}x\right)+c_2\left(x^5-25x^3+\frac{375}{4}x\right)\left(\int\frac{\mathrm{e}^{\frac{x^2}{5}}}{\left(4x^4-100x^2+375\right)^2 x^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.174 (sec). Leaf size: 138

```
DSolve[5*y''[x]-2*x*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}\left(4x^4-100x^2+375\right)\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} -\frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

2.7 problem 7

2.7.1 Maple step by step solution 89

Internal problem ID [7497]
Internal file name [OUTPUT/6430_Sunday_June_05_2022_04_52_17_PM_18389270/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2 y' - 3yx = 0}$$

Writing the ode as

$$y'' - x^2 y' - 3yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3+8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3+8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3+8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2} \right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r \right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{x^2}{2} \right)(1) + \left((x) + \left(\frac{x^2}{2} \right)^2 - \left(\frac{x(x^3 + 8)}{4} \right) \right) &= 0 \\ -x a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int \frac{x^2}{2} dx} \\ &= (x) \, \mathrm{e}^{\frac{x^3}{6}} \\ &= x \, \mathrm{e}^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x \, \mathrm{e}^{\frac{x^3}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{\frac{x^3}{3}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)
$$

Verified OK.

2.7.1 Maple step by step solution

Let's solve

$y'' - x^2 y' - 3yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\,(k a_{k+2} - a_{k-1} + a_{k+2}) = 0$$

- Shift index using $k- >k+1$

$$(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^3}{3}} x - \frac{9c_2 e^{\frac{x^3}{3}} 3^{\frac{2}{3}} e^{-\frac{x^3}{6}} \left(x^6 \,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) + 5 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right) x^3 + 10 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right)\right)}{10x^3 \left(x^3\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.139 (sec). Leaf size: 51

```
DSolve[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.8 problem 8

Internal problem ID [7498]
Internal file name [`OUTPUT/6431_Sunday_June_05_2022_04_52_20_PM_21077337/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''+2xy'-2y=0}$$

Writing the ode as

$$(x^2+1)\, y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= 2-(1)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2(x - i)^2} - \frac{1}{2(x + i)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right.\right.
$$
$$
-\frac{2(x^2 + 1)\, a_0}{(-x + i)^2\, (x + i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x)\, \sqrt{x^2 + 1} \\
&= x\sqrt{x^2 + 1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{1}{x} - \arctan\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{1}{x} - \arctan\left(x\right)\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x + c_2(-\arctan\left(x\right) x - 1) \tag{1}
$$

Verification of solutions

$$y = c_1x + c_2(-\arctan(x)\,x - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(\arctan(x)\,x + 1)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 48

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1 - ix) + c_2x\log(1 + ix) + 2ic_2)$$

2.9 problem 9

2.9.1 Maple step by step solution 105

Internal problem ID [7499]
Internal file name [`OUTPUT/6432_Sunday_June_05_2022_04_52_22_PM_20813184/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' - 2y = 0}$$

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 14: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) = 0$$
$$-a_1 x - 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2+1\right) \mathrm{e}^{\int \frac{x}{2} dx} \\ &= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}} \\ &= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2+1\right) + c_2 \left(x^2 + 1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2+1)^2} dx \right) \right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)$$

Verified OK.

2.9.1 Maple step by step solution

Let's solve

$y'' + xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$\left(k^2+3k+2\right) a_{k+2}+a_k(k-2)=0$

- Recursion relation; series terminates at $k=2$

  $a_{k+2}=-\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k=0$

  $a_2=a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y=A_2x^2+A_1x+a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 35

```
DSolve[y''[x]+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2(x^2+1)$$

2.10 problem 10

Internal problem ID [7500]
Internal file name [`OUTPUT/6433_Sunday_June_05_2022_04_52_25_PM_39350206/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-6x+10)\,y''-4(-3+x)\,y'+6y=0}$$

Writing the ode as

$$(x^2-6x+10)\,y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2-6x+10\\ B&=-4x+12\\ C&=6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{\left(x^2 - 6x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= \left(x^2 - 6x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{\left(x^2 - 6x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 16: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x - 3 - i)^2} + \frac{2}{(x - 3 + i)^2} + \frac{2i}{x - 3 - i} - \frac{2i}{x - 3 + i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2 - 6x + 10)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3+i$	2	0	2	-1
$3-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} + (-)(0) \\
&= -\frac{1}{x-3-i} + \frac{2}{x-3+i} \\
&= \frac{-3-3i+x}{x^2-6x+10}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right) dx} \\
&= \frac{(x^2 - 6x + 10)^2}{(ix - 3i + 1)^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x+12}{x^2-6x+10} \, dx} \\
&= z_1 e^{\ln(x^2-6x+10)} \\
&= z_1 \left(x^2 - 6x + 10\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 6x + 10)^3}{(ix - 3i + 1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln(x^2-6x+10)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2 - 6x + \frac{26}{3}}{(x-3+i)^3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \right) + c_2 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2-6x+10)^3}{(ix-3i+1)^3} + \frac{c_2 (x^2-6x+10)^3 \left(x^2-6x+\frac{26}{3}\right)}{(ix-3i+1)^3 (x-3+i)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2-6x+10)^3}{(ix-3i+1)^3} + \frac{c_2 (x^2-6x+10)^3 \left(x^2-6x+\frac{26}{3}\right)}{(ix-3i+1)^3 (x-3+i)^3}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve((x^2-6*x+10)*diff(y(x),x$2)-4*(x-3)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1 \left(\frac{26}{3} + x^2 - 6x \right) + c_2 (x^3 - 30x + 60)
$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 36

```
DSolve[(x^2-6*x+10)*y''[x]-4*(x-3)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}i\left(c_2\left(3x^2 - 18x + 26\right) + 3c_1(x - (3 + i))^3\right)$$

2.11 problem 11

2.11.1 Maple step by step solution 121

Internal problem ID [7501]
Internal file name [`OUTPUT/6434_Sunday_June_05_2022_04_52_27_PM_29056166/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+6x)\,y''+(3x+9)\,y'-3y=0}$$

Writing the ode as

$$\begin{aligned}(x^2+6x)\,y''+(3x+9)\,y'-3y&=0 &(1)\\ Ay''+By'+Cy&=0 &(2)\end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+6x\\ B&=3x+9\\ C&=-3\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x)\tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 90x - 27$$
$$t = 4\left(x^2 + 6x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{11}{16x} - \frac{3}{16\,(x+6)^2} - \frac{3}{16x^2} - \frac{11}{16\,(x+6)}
$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4\,(x^2+6x)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+-\left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\ &= \frac{5}{2}-\left(\frac{3}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2} + \frac{9}{2}}{x\,(x+6)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\,(x+6)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4\,(x^2+6x)^2}\right)\right) = \frac{9 - 3a_0}{x\,(x+6)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+3)\, \mathrm{e}^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\
&= (x+3)\, \mathrm{e}^{\frac{3\ln(x)}{4} + \frac{3\ln(x+6)}{4}} \\
&= (x+3)\, x^{\frac{3}{4}} (x+6)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x+9}{x^2+6x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\
&= z_1\left(\frac{1}{(x\,(x+6))^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-2x^2-12x-9}{81\,(x+3)\,\sqrt{x}\,\sqrt{x+6}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}\right) + c_2\left(\frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}\left(\frac{-2x^2-12x-9}{81\,(x+3)\,\sqrt{x}\,\sqrt{x+6}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}} - \frac{2c_2\left(x^2+6x+\frac{9}{2}\right)(x+6)^{\frac{1}{4}}\,x^{\frac{1}{4}}}{81\,(x\,(x+6))^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}} - \frac{2c_2\left(x^2+6x+\frac{9}{2}\right)(x+6)^{\frac{1}{4}}\,x^{\frac{1}{4}}}{81\,(x\,(x+6))^{\frac{3}{4}}}$$

Verified OK.

2.11.1 Maple step by step solution

Let's solve

$(x^2+6x)\,y''+(3x+9)\,y'-3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{3y}{x(x+6)}-\frac{3(x+3)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(x+3)y'}{x(x+6)}-\frac{3y}{x(x+6)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3(x+3)}{x(x+6)}, P_3(x)=-\frac{3}{x(x+6)}\right]$

  ◦ $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

    $\left((x+6)\cdot P_2(x)\right)\Big|_{x=-6}=\frac{3}{2}$

  ◦ $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

    $\left((x+6)^2\cdot P_3(x)\right)\Big|_{x=-6}=0$

  ◦ $x=-6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-6$

- Multiply by denominators

  $y''x(x+6)+(3x+9)\,y'-3y=0$

- Change variables using $x=u-6$ so that the regular singular point is at $u=0$

$$(u^2-6u)\left(\frac{d^2}{du^2}y(u)\right)+(3u-9)\left(\frac{d}{du}y(u)\right)-3y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$-3a_0 r(1+2r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-3a_{k+1}(k+1+r)\,(2k+3+2r)+a_k(k+r+3)\,(k+r-1)\right)u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-6\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1}+a_k(k+r+3)\,(k+r-1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+3)(k+r-1)}{3(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+1}=\frac{a_k(k+3)(k-1)}{3(2k+3)(k+1)}$$

- Apply recursion relation for $k=0$

$a_1 = -\frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$

- Revert the change of variables $u = x + 6$

$\left[y = a_0\left(-1 - \frac{x}{3}\right)\right]$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Revert the change of variables $u = x + 6$

$\left[y = \sum_{k=0}^{\infty} a_k (x+6)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0\left(-1 - \frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k (x+6)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve((x^2+6*x)*diff(y(x),x$2)+(3*x+9)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+3) + \frac{c_2(2x^2+12x+9)}{\sqrt{x^2+6x}}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 82

```
DSolve[(x^2+6*x)*y''[x]+(3*x+9)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right)+\sqrt{6}c_1(2x^2+12x+9)}{9\sqrt{\pi}\sqrt[4]{-x^2}\sqrt{x+6}}$$

2.12 problem 12

2.12.1 Maple step by step solution 132

Internal problem ID [7502]
Internal file name [OUTPUT/6435_Sunday_June_05_2022_04_52_30_PM_61950386/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + (t^2 - 1) y' + t^2 y = 0}$$

Writing the ode as

$$ty'' + (t^2 - 1) y' + t^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 4t^3 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 4t^3 + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - t + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - 1 - \frac{1}{t} - \frac{2}{t^2} - \frac{17}{4t^3} - \frac{25}{2t^4} - \frac{75}{2t^5} - \frac{117}{t^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= -1 + \frac{t}{2} \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1 - t + \frac{1}{4}t^2$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^4 - 4t^3 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}t^2 - t\right) + \left(\frac{3}{4t^2}\right) \\ &= \frac{t^2}{4} - t + \frac{3}{4t^2}\end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned}b &= (0) - (1) \\ &= -1\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= -1 + \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &= -\frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 4t^3 + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-1 + \frac{t}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + (-)\left(-1 + \frac{t}{2}\right) \\
&= -\frac{1}{2t} + 1 - \frac{t}{2} \\
&= -\frac{(t-1)^2}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)(1) + \left(\left(\frac{1}{2t^2} - \frac{1}{2}\right) + \left(-\frac{1}{2t} + 1 - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 4t^3 + 3}{4t^2}\right)\right) = 0$$

$$\frac{(a_0 + 1)(t - 1)}{t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t - 1) \, \mathrm{e}^{\int \left(-\frac{1}{2t} + 1 - \frac{t}{2}\right) dt} \\
&= (t - 1) \, \mathrm{e}^{t - \frac{t^2}{4} - \frac{\ln(t)}{2}} \\
&= \frac{(t - 1) \, \mathrm{e}^{-\frac{t(t-4)}{4}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2 - 1}{t} \, dt} \\
&= z_1 e^{-\frac{t^2}{4} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t} \, \mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\,dt \\ &= y_1 \left(\int \frac{t\,\mathrm{e}^{\frac{t(t-4)}{2}}}{(t-1)^2}dt\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\right) + c_2\left((t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\left(\int \frac{t\,\mathrm{e}^{\frac{t(t-4)}{2}}}{(t-1)^2}dt\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}} + c_2(t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\left(\int \frac{t\,\mathrm{e}^{\frac{t(t-4)}{2}}}{(t-1)^2}dt\right) \tag{1}$$

Verification of solutions

$$y = c_1(t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}} + c_2(t-1)\,\mathrm{e}^{-\frac{t(t-2)}{2}}\left(\int \frac{t\,\mathrm{e}^{\frac{t(t-4)}{2}}}{(t-1)^2}dt\right)$$

Verified OK.

2.12.1 Maple step by step solution

Let's solve

$ty'' + (t^2 - 1)\, y' + t^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t^2-1)y'}{t} - yt$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t^2-1)y'}{t} + yt = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{t^2-1}{t}, P_3(t) = t\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\bigg|_{t=0} = -1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (t^2 - 1)\, y' + t^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $t^2 \cdot y$ to series expansion

  $t^2 \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+2}$

- Shift index using $k->k-2$

  $t^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} t^{k+r}$

- Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

  $t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)\,(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of t must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}(k+r-1) + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+3}(k+3+r)\,(k+1+r) + a_{k+1}(k+1+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{k a_{k+1} + r a_{k+1} + a_k + a_{k+1}}{(k+3+r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{k a_{k+1}+a_k+a_{k+1}}{(k+3)(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{k a_{k+1}+a_k+a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}\right]$

- Recursion relation for $r = 2$

  $a_{k+3} = -\frac{k a_{k+1}+a_k+3a_{k+1}}{(k+5)(k+3)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+3} = -\frac{k a_{k+1}+a_k+3a_{k+1}}{(k+5)(k+3)}, a_1 = 0, a_2 = -\frac{a_0}{4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+3} = -\frac{k a_{k+1}+a_k+a_{k+1}}{(k+3)(k+1)}, a_1 = 0, a_2 = -\frac{a_0}{2}, b_{k+3} = -\frac{k b_{k+1}+b_k+3b_k}{(k+5)(k+3)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
               <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\` i
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 50

```
dsolve(t*diff(y(t),t$2)+ (t^2-1)*diff(y(t),t)+t^2*y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 e^{t-\frac{1}{2}t^2}(t-1) + c_2 e^{t-\frac{1}{2}t^2}(t-1)\left(\int \frac{t\,e^{\frac{1}{2}t^2-2t}}{(t-1)^2}dt\right)$$

✓ Solution by Mathematica

Time used: 0.434 (sec). Leaf size: 70

```
DSolve[t*y''[t]+(t^2-1)*y'[t]+t^2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{1}{2}e^{-\frac{t^2}{2}+t-2}\left(\sqrt{2\pi}c_2(t-1)\operatorname{erfi}\left(\frac{t-2}{\sqrt{2}}\right)+2e^2c_1(t-1)-2c_2e^{\frac{1}{2}(t-2)^2}\right)$$

2.13 problem 13

2.13.1 Maple step by step solution 140

Internal problem ID [7503]
Internal file name [`OUTPUT/6436_Sunday_June_05_2022_04_52_33_PM_73902378/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - t(t+2)\,y' + (t+2)\,y = 0}$$

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-t^2 - 2t}{t^2} \, dt} \\ &= z_1 e^{\frac{t}{2} + \ln(t)} \\ &= z_1 \left(t \, \mathrm{e}^{\frac{t}{2}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2 - 2t}{t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{t + 2\ln(t)}}{(y_1)^2} \, dt \\ &= y_1 \left(\mathrm{e}^t \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\big(t(\mathrm{e}^t)\big)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 t + c_2 \mathrm{e}^t t \tag{1}$$

Verification of solutions

$$y = c_1 t + c_2 \mathrm{e}^t t$$

Verified OK.

2.13.1 Maple step by step solution

Let's solve

$y'' t^2 + (-t^2 - 2t)\, y' + (t+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left. (t \cdot P_2(t)) \right|_{t=0} = -2$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left. (t^2 \cdot P_3(t)) \right|_{t=0} = 2$

  - $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$y'' t^2 - t(t+2)\, y' + (t+2)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, t^{k+r}$

  ◦ Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r-1)\left(a_{k+1}(k+r)-a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+r}$

- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k+1}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1}=\frac{a_k}{k+1}\right]$

- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+2}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+2}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(t^2*diff(y(t),t$2)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t\, \mathrm{e}^{t}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-t*(t+2)*y'[t]+(t+2)*y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t\left(c_2 e^t + c_1\right)$$

2.14 problem 14

2.14.1 Maple step by step solution 150

Internal problem ID [7504]
Internal file name [`OUTPUT/6437_Sunday_June_05_2022_04_52_35_PM_15423371/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{ty'' - (t+1)\,y' + y = 0}$$

Writing the ode as

$$ty'' + (-t-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 23: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}
$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t-1}{t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{\ln(t)+t}}{(y_1)^2}\,dt \\ &= y_1\left(-(t+1)\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-(t+1)\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t + c_2(-t-1)$$

Verified OK.

2.14.1 Maple step by step solution

Let's solve

$t y'' + (-t-1)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+1)y'}{t} + \frac{y}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- ○ Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$

- ○ Shift index using $k- >k+1$

$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1(t+1) + c_2\mathrm{e}^t$$

✓ Solution by Mathematica
Time used: 0.041 (sec). Leaf size: 19

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1e^t - c_2(t+1)$$

2.15 problem 15

2.15.1 Maple step by step solution 160

Internal problem ID [7505]
Internal file name [`OUTPUT/6438_Sunday_June_05_2022_04_52_37_PM_42707998/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-t+1)\,y''+ty'-y=0}$$

Writing the ode as

$$(-t+1)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t+1 \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4\left(t - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 4t + 6 \\ t &= 4(t - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4\left(t - 1\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 25: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(t-1)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(t-1)} + \frac{3}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4\,(t-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(t-1)} + \left(\frac{1}{2} \right) \\
&= -\frac{1}{2(t-1)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2(t-1)} + \frac{1}{2} \right)(0) + \left(\left(\frac{1}{2(t-1)^2} \right) + \left(-\frac{1}{2(t-1)} + \frac{1}{2} \right)^2 - \left(\frac{t^2 - 4t + 6}{4(t-1)^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-1)} + \frac{1}{2} \right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{-t+1} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t-1)}{2}} \\
&= z_1 \left(\sqrt{t-1}\, \mathrm{e}^{\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{-t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t-1)}}{(y_1)^2}\, dt \\ &= y_1\left(-t\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-t\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t - c_2 t \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t - c_2 t$$

Verified OK.

2.15.1 Maple step by step solution

Let's solve

$(-t+1)\, y'' + t y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t-1} + \frac{ty'}{t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{ty'}{t-1} + \frac{y}{t-1} = 0$

☐ Check to see if $t_0 = 1$ is a regular singular point

  ○ Define functions

  $\left[P_2(t) = -\frac{t}{t-1}, P_3(t) = \frac{1}{t-1}\right]$

  ○ $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\bigg|_{t=1} = -1$

  ○ $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$

  ○ $t = 1$is a regular singular point

  Check to see if $t_0 = 1$ is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(t-1) - ty' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k (t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k (t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 e^t$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 17

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2 t$$

2.16 problem 16

2.16.1 Maple step by step solution 167

Internal problem ID [7506]
Internal file name [`OUTPUT/6439_Sunday_June_05_2022_04_52_40_PM_40817785/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\,x^r + a_1(3 + 2r)\,(1 + 2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/100)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/100)*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.17 problem 17

2.17.1 Maple step by step solution 177

Internal problem ID [7507]
Internal file name [`OUTPUT/6440_Sunday_June_05_2022_04_52_41_PM_90649222/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{ty'' - (t+1)\,y' + y = 0}$$

Writing the ode as

$$ty'' + (-t-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 29: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t - 1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{e^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t-1}{t} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, e^{\frac{t}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\ln(t)+t}}{(y_1)^2}\, dt \\
&= y_1 \left(-(t+1)\,\mathrm{e}^{-t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(-(t+1)\,\mathrm{e}^{-t}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^t + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t + c_2(-t-1)$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$t y'' + (-t-1)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+1)y'}{t} + \frac{y}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\bigg|_{t=0} = -1$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- ○ Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- ○ Shift index using $k- >k+1$

$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1(t+1) + c_2 e^t$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 19

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] ==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

2.18 problem 18

2.18.1 Maple step by step solution 187

Internal problem ID [7508]
Internal file name [`OUTPUT/6441_Sunday_June_05_2022_04_52_44_PM_63468054/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-t+1)\,y''+ty'-y=0}$$

Writing the ode as

$$(-t+1)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t+1 \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{t^2 - 4t + 6}{4\left(t - 1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= t^2 - 4t + 6 \\
t &= 4(t - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{t^2 - 4t + 6}{4\left(t - 1\right)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(t-1)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} - \frac{1}{2(t-1)} + \frac{3}{4(t-1)^2}
$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4(t-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(t-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\,(t-1)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(t-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\,(t-1)^2}\right) + \left(-\frac{1}{2\,(t-1)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4\,(t-1)^2}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-1)} + \frac{1}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{-t+1} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t-1)}{2}} \\
&= z_1 \left(\sqrt{t-1}\,\mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{-t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t-1)}}{(y_1)^2}\, dt \\ &= y_1 \left(-t\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-t\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^t - c_2 t \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^t - c_2 t$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$(-t+1)\, y'' + t y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t-1} + \frac{ty'}{t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{ty'}{t-1} + \frac{y}{t-1} = 0$

□ Check to see if $t_0 = 1$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t}{t-1}, P_3(t) = \frac{1}{t-1}\right]$

- $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Big|_{t=1} = -1$

- $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Big|_{t=1} = 0$

- $t = 1$is a regular singular point

  Check to see if $t_0 = 1$ is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(t-1) - ty' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = t - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = t - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 e^t$$

✓ Solution by Mathematica
Time used: 0.034 (sec). Leaf size: 17

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] ==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2 t$$

2.19 problem 19

2.19.1 Maple step by step solution 197

Internal problem ID [7509]
Internal file name [`OUTPUT/6442_Sunday_June_05_2022_04_52_46_PM_19551459/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x) \, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x \, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi} \, \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\, x}{2} \right) x - 2 \, \mathrm{e}^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.19.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\left(k a_{k+2} + a_k + a_{k+2}\right) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,e^{-\frac{x^2}{2}} + \frac{c_2\sqrt{2}\,e^{-\frac{x^2}{2}}\left(i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{x^2}{2}} - \pi\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\right)}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.20 problem 20

Internal problem ID [7510]
Internal file name [`OUTPUT/6443_Sunday_June_05_2022_04_52_49_PM_69934126/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-4xy'+6y=0}$$

Writing the ode as

$$(x^2+1)\,y''-4xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+1\\ B&=-4x\\ C&=6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-i)^2} + \frac{2}{(x+i)^2} + \frac{2i}{x-i} - \frac{2i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-i} + \frac{2}{x+i} + (-)\,(0) \\ &= -\frac{1}{x-i} + \frac{2}{x+i} \\ &= \frac{x-3i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{x-i}+\frac{2}{x+i}\right)(0)+\left(\left(\frac{1}{(x-i)^2}-\frac{2}{(x+i)^2}\right)+\left(-\frac{1}{x-i}+\frac{2}{x+i}\right)^2-\left(-\frac{8}{(x^2+1)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x-i}+\frac{2}{x+i}\right)dx} \\
&= \frac{(x^2+1)^2}{(ix+1)^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1}\, dx} \\
&= z_1 e^{\ln(x^2+1)} \\
&= z_1 \left(x^2+1\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^3}{(ix+1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2-\frac{1}{3}}{(x+i)^3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^3}{(ix+1)^3} \right) + c_2 \left(\frac{(x^2+1)^3}{(ix+1)^3} \left(\frac{x^2 - \frac{1}{3}}{(x+i)^3} \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 (x^2+1)^3}{(ix+1)^3} + \frac{c_2 (x^2+1)^3 \left(x^2 - \frac{1}{3}\right)}{(ix+1)^3 (x+i)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2+1)^3}{(ix+1)^3} + \frac{c_2 (x^2+1)^3 \left(x^2 - \frac{1}{3}\right)}{(ix+1)^3 (x+i)^3}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve((1+x^2)*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1(-3x^2+1) + c_2(x^3 - 3x)
$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 33

```
DSolve[(1+x^2)*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}i\left(c_2\left(3x^2-1\right)+3c_1(x-i)^3\right)$$

2.21 problem 21

2.21.1 Maple step by step solution 213

Internal problem ID [7511]
Internal file name [`OUTPUT/6444_Sunday_June_05_2022_04_52_51_PM_99832753/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 36: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0)\\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right)\\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(-x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^x - c_2 x$$

Verified OK.

2.21.1 Maple step by step solution

Let's solve

$(1-x)\,y'' + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  - $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica
Time used: 0.033 (sec). Leaf size: 17

```
DSolve[(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.22 problem 22

2.22.1 Maple step by step solution 223

Internal problem ID [7512]
Internal file name [`OUTPUT/6445_Sunday_June_05_2022_04_52_55_PM_88590770/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2y'' + xy' + 3y = 0}$$

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 20$$
$$t = 16$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 38: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{4}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 20}{16} \\
&= Q + \frac{R}{16} \\
&= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\
&= \frac{x^2}{16} - \frac{5}{4}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{4}\right) - (0) \\
&= -\frac{5}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{16} - \frac{5}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{4}\right) \\ &= -\frac{x}{4} \\ &= -\frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) &= 0 \\ 2 + \frac{a_1 x}{2} + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2-2\right) \mathrm{e}^{\int -\frac{x}{4} dx} \\ &= \left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{8}} \\ &= \left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{2} \, dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2-2\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{4}}\right) + c_2 \left(\left(x^2-2\right) \mathrm{e}^{-\frac{x^2}{4}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2-2\right)^2} dx\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}} + c_2(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}} + c_2(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right)$$

Verified OK.

2.22.1 Maple step by step solution

Let's solve

$2y'' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{2} - \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{2} + \frac{3y}{2} = 0$

- Multiply by denominators

  $2y'' + xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ∘ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ∘ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ∘ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)(k+1) + a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(2k^2 + 6k + 4)\, a_{k+2} + a_k(k+3) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(2*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^2}{4}} (x^2 - 2) + c_2 \mathrm{e}^{-\frac{x^2}{4}} (x^2 - 2) \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2 - 2)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.333 (sec). Leaf size: 61

```
DSolve[2*y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8} e^{-\frac{x^2}{4}} \left(\sqrt{\pi} c_2 (x^2 - 2) \operatorname{erfi}\left(\frac{x}{2}\right) + 8c_1 (x^2 - 2) - 2c_2 e^{\frac{x^2}{4}} x \right)$$

2.23 problem 23

2.23.1 Maple step by step solution 232

Internal problem ID [7513]
Internal file name [`OUTPUT/6446_Sunday_June_05_2022_04_52_58_PM_3283845/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2}\right) \\
&= -\frac{x}{2} \\
&= -\frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\
a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x) \, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x \, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.23.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2))x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \, e^{-\frac{x^2}{2}} + \frac{c_2\sqrt{2}\, e^{-\frac{x^2}{2}}\left(i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{x^2}{2}} - \pi \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\right)}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.24 problem 24

2.24.1 Maple step by step solution 241

Internal problem ID [7514]
Internal file name [`OUTPUT/6447_Sunday_June_05_2022_04_53_00_PM_24211602/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x\\ B&=x\\ C&=-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\,(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\,(x-1)^2}\right) + \left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^x - c_2 x$$

Verified OK.

2.24.1 Maple step by step solution

Let's solve

$(1-x)\, y'' + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)(k+r-1)-a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r)-a_k\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k}{k+1+r}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{a_k}{k+1}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

- Revert the change of variables $u=x-1$

$$\left[y=\sum_{k=0}^{\infty}a_k(x-1)^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1}=\frac{a_k}{k+3}$$

- Solution for $r=2$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$

- Revert the change of variables $u=x-1$

$$\left[y=\sum_{k=0}^{\infty}a_k(x-1)^{k+2}, a_{k+1}=\frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+\mathrm{e}^x c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

2.25 problem 25

2.25.1 Maple step by step solution 251

Internal problem ID [7515]
Internal file name [OUTPUT/6448_Sunday_June_05_2022_04_53_03_PM_93245541/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right) x - 2\, \mathrm{e}^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.25.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2))x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \,\mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2\sqrt{2}\,\mathrm{e}^{-\frac{x^2}{2}}\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{x^2}{2}} - \pi \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\right)}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.26 problem 26

2.26.1 Maple step by step solution 260

Internal problem ID [7516]
Internal file name [OUTPUT/6449_Sunday_June_05_2022_04_53_05_PM_78362697/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(-x^2+4)\,y''+xy'+2y=0}$$

Writing the ode as

$$(-x^2+4)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{11x^2 - 24}{4\left(x^2 - 4\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 11x^2 - 24$$
$$t = 4\left(x^2 - 4\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{11x^2 - 24}{4\left(x^2 - 4\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{32\,(x+2)} + \frac{5}{16\,(x+2)^2} + \frac{5}{16\,(x-2)^2} + \frac{17}{32\,(x-2)}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4\,(x^2-4)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{11}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1, 2, 5\}$
-2	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (-1))) \\ &= 2 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(2))} + \frac{-1}{(x-(-2))}\right) \\ &= -\frac{1}{2(x-2)} - \frac{1}{2(x+2)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 2$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 2$, then letting

$$p = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(a_0 + 6)\,x + 36a_1}{(x^2-4)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2-6} - \frac{1}{2(x-2)} - \frac{1}{2(x+2)}\right)w + \frac{-11x^4+74x^2-128}{4x^6-56x^4+256x^2-384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\,x^2\sqrt{x^2-4} + x^3 - 8\sqrt{3}\,\sqrt{x^2-4} - 2x}{2(x^2-6)(x-2)(x+2)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= \mathrm{e}^{\int \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2\left(x^2-6\right)(x-2)(x+2)}dx}\\ &= \frac{\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\left(\sqrt{2}\,\sqrt{3}\,x-4\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{\left(4+\sqrt{2}\,\sqrt{3}\,x\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{(x+2)^{\frac{1}{4}}(x-2)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\,dx}\\ &= z_1 e^{\frac{\ln\left(x^2-4\right)}{4}}\\ &= z_1\left(\left(x^2-4\right)^{\frac{1}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{-x^2+4}\, dx}}{(y_1)^2}\, dx$$

$$= y_1 \int \frac{e^{\frac{\ln(x^2-4)}{2}}}{(y_1)^2}\, dx$$

$$= y_1 \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}}{x^2-6} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \right) + c_2 \left(\sqrt{x^2-6}\,\Big(x + \sqrt{x^2-4}\Big)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}}}{x^2-6}\right.\right.$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} + c_2\sqrt{x^2-6}\,\Big(x + \sqrt{x^2-4}\Big)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\right.}}{x^2-6}\right. \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} + c_2\sqrt{x^2-6}\left(x\right.$$

$$\left.+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} e^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\right.}}{x^2-6}\right.$$

Verified OK.

2.26.1 Maple step by step solution

Let's solve

$(-x^2+4)\,y''+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{xy'}{x^2-4}+\frac{2y}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{xy'}{x^2-4}-\frac{2y}{x^2-4}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-2$

- Multiply by denominators

  $y''(x^2-4)-xy'-2y=0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-u+2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)(2k-1+2r) + a_k(k^2 + 2kr + r^2 - 2k - 2r -$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1} + \left(k^2 + (2r-2)\,k + r^2 - 2r - 2\right)a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k^2+2kr+r^2-2k-2r-2)a_k}{2(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(k^2-2k-2)a_k}{2(k+1)(2k-1)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(k+1)(2k-1)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(k+1)(2k-1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(k+\frac{5}{2})(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(k+\frac{5}{2})(2k+2)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(k+\frac{5}{2})(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-2k-2)a_k}{2(k+1)(2k-1)}, b_{k+1} = \frac{(k^2+k-\frac{11}{4})b_k}{2(k+\frac{5}{2})(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve((4-x^2)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x^2-6}\,\sin\left(\int \frac{\sqrt{-x^2+4}\,\sqrt{3}}{x^2-6}dx\right) + c_2\sqrt{x^2-6}\,\cos\left(\int \frac{\sqrt{-x^2+4}\,\sqrt{3}}{x^2-6}dx\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 58

```
DSolve[(4-x^2)*y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x^2-4)^{3/4}\left(c_1 P^{\frac{3}{2}}_{-\frac{1}{2}+\sqrt{3}}\left(\frac{x}{2}\right) + c_2 Q^{\frac{3}{2}}_{-\frac{1}{2}+\sqrt{3}}\left(\frac{x}{2}\right)\right)$$

2.27 problem 27

2.27.1 Maple step by step solution 267

Internal problem ID [7517]
Internal file name [`OUTPUT/6450_Sunday_June_05_2022_04_53_08_PM_11611305/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 48: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4}$$

Verified OK.

2.27.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (-16x^2 + 3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-16x^2 + 3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k - 16a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh(2x) + c_2\sqrt{x}\,\cosh(2x)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.28 problem 28

2.28.1 Maple step by step solution 277

Internal problem ID [7518]
Internal file name [`OUTPUT/6451_Sunday_June_05_2022_04_53_10_PM_28841233/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 50: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(x-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x - c_2 x$$

Verified OK.

2.28.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$

  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k(x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.29 problem 29

2.29.1 Maple step by step solution 284

Internal problem ID [7519]
Internal file name [`OUTPUT/6452_Sunday_June_05_2022_04_53_13_PM_12655681/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2 + 2) y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2 + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\cos(x) + c_2 x\sin(x)
$$

Verified OK.

2.29.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + (x^2+2)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r)\,(-2 + r)\, x^r + a_1 r(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r - 1)\,(k + r - 2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.30 problem 31

2.30.1 Maple step by step solution 295

Internal problem ID [7520]
Internal file name [`OUTPUT/6453_Sunday_June_05_2022_04_53_15_PM_96475843/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0}$$

Writing the ode as

$$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x \\ B &= -x^2+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4\left(x^2 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{4x} + \frac{3}{4(x-2)^2} + \frac{3}{4x^2} - \frac{1}{4(x-2)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^4-8x^3+24x^2-24x+12}{4x^4-16x^3+16x^2}\\
&= Q+\frac{R}{4x^4-16x^3+16x^2}\\
&= \left(\frac{1}{4}\right)+\left(\frac{-4x^3+20x^2-24x+12}{4x^4-16x^3+16x^2}\right)\\
&= \frac{1}{4}+\frac{-4x^3+20x^2-24x+12}{4x^4-16x^3+16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1)-(0)\\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}}-0\right) &&= -1\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}}-0\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{x^4-8x^3+24x^2-24x+12}{4\left(x^2-2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} - \frac{1}{2(x-2)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln(x-2)}{2}} \\
&= z_1\left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x-2)}}{(y_1)^2} \, dx \\
&= y_1\left(-\mathrm{e}^{-x} x^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}\right) + c_2\left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}\left(-\mathrm{e}^{-x} x^2\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

Verified OK.

2.30.1 Maple step by step solution

Let's solve

$(x^2-2x)\, y'' + (-x^2+2)\, y' + (2x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y'}{x(x-2)} - \frac{2(x-1)y}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-2)y'}{x(x-2)} + \frac{2(x-1)y}{x(x-2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2-2}{x(x-2)}, P_3(x) = \frac{2(x-1)}{x(x-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x(x-2)+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_{k}x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^{m}\cdot y$ to series expansion for $m=0..1$

    $x^{m}\cdot y=\sum_{k=0}^{\infty}a_{k}x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

    $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

    $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}r(-2+r)\,x^{-1+r}+\left(-2a_{1}(1+r)\,(-1+r)+a_{0}(1+r)\,(-2+r)\right)x^{r}+\left(\sum_{k=1}^{\infty}(-2a_{k+1}(k+r+$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term must be 0

  $-2a_{1}(1+r)\,(-1+r)+a_{0}(1+r)\,(-2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1} =$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)(k+r-1)-2(k+1)^2a_{k+2}+(-4ra_{k+2}-a_k)(k+1)-2r^2a_{k+2}-ra_k+3a_k+$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Recursion relation for $r=2$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}, -6a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(2*x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^2 + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 18

```
DSolve[(x^2-2*x)*y''[x]+(2-x^2)*y'[x]+(2*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_2x^2 + c_1e^x$$

2.31 problem 32

2.31.1 Maple step by step solution 302

Internal problem ID [7521]
Internal file name [`OUTPUT/6454_Sunday_June_05_2022_04_53_17_PM_68570584/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 56: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) + c_2 \left(\frac{\mathrm{e}^x}{\sqrt{x}} (x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^x}{\sqrt{x}} + c_2 \sqrt{x}\, \mathrm{e}^x \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^x}{\sqrt{x}} + c_2 \sqrt{x}\, \mathrm{e}^x
$$

Verified OK.

2.31.1 Maple step by step solution

Let's solve

$4x^2 y'' + (-8x^2 + 4x) y' + (4x^2 - 4x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2 - 4x - 1) y}{4x^2} + \frac{(2x-1) y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1) y'}{x} + \frac{(4x^2 - 4x - 1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2 - 4x - 1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' - 4x(2x-1)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r)\, x^r + \left(a_1(3+2r)(1+2r) - 4a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) - 4a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+(4*x-8*x^2)*diff(y(x),x)+(4*x^2-4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{x}}{\sqrt{x}} + c_2 \sqrt{x}\, \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{x}(c_2 x + c_1)}{\sqrt{x}}$$

2.32 problem 33

2.32.1 Maple step by step solution 309

Internal problem ID [7522]
Internal file name [`OUTPUT/6455_Sunday_June_05_2022_04_53_19_PM_88638966/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2}
$$

Verified OK.

2.32.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2} (c_2 x + c_1)$$

2.33 problem 34

2.33.1 Maple step by step solution 318

Internal problem ID [7523]
Internal file name [OUTPUT/6456_Sunday_June_05_2022_04_53_21_PM_96434551/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x+1)\,y''-2y'-(3+2x)\,y=0}$$

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x+1\\B&=-2\\C&=-2x-3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 8x + 6 \\ t &= (2x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= 1 \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = 1
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\
&= Q + \frac{R}{4x^2 + 4x + 1} \\
&= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\
&= 1 + \frac{4x + 5}{4x^2 + 4x + 1}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$
\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x + \frac{1}{2}\right)} + (-)(1) \\ &= -\frac{1}{2\left(x + \frac{1}{2}\right)} - 1 \\ &= -\frac{2(1 + x)}{2x + 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1}\, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1\left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x+1)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(x\,\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(x\,\mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x
$$

Verified OK.

2.33.1 Maple step by step solution

Let's solve

$(2x+1)\,y'' - 2y' + (-2x-3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3+2x)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{2x+1} - \frac{(3+2x)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  - Define functions

$\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{3+2x}{2x+1}\right]$

- ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}} = -1$

- ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}} = 0$

- ◦ $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $(2x+1)\,y''-2y'+(-2x-3)\,y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

- ◦ Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

- ◦ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}r(-2+r)\,u^{-1+r}+\left(2a_{1}(1+r)\,(-1+r)-2a_{0}\right)u^{r}+\left(\sum_{k=1}^{\infty}(2a_{k+1}(k+1+r)\,(k+r-1)-2a_{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,2\}$$

- Each term must be 0

$$2a_{1}(1+r)\,(-1+r)-2a_{0}=0$$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k+1}(k+1+r)\,(k+r-1)-2a_{k}-2a_{k-1}=0$$

- Shift index using $k->k+1$

$$2a_{k+2}(k+2+r)\,(k+r)-2a_{k+1}-2a_{k}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2+r)(k+r)}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2)k}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2)k}$$

- Recursion relation for $r=2$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+4)(k+2)}$$

- Solution for $r=2$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+2},a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+4)(k+2)},6a_{1}-2a_{0}=0\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2*x+1)*diff(y(x),x$2)-2*diff(y(x),x)-(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x} c_1 + c_2 x \, e^{x}$$

✓ Solution by Mathematica
Time used: 0.085 (sec). Leaf size: 29

```
DSolve[(2*x+1)*y''[x]-2*y'[x]-(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x-\frac{1}{2}} \left(c_2 e^{2x+1} x + c_1\right)$$

2.34 problem 35

2.34.1 Maple step by step solution 327

Internal problem ID [7524]
Internal file name [`OUTPUT/6457_Sunday_June_05_2022_04_53_24_PM_8032852/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (2x+2)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3}
$$

Verified OK.

2.34.1 Maple step by step solution

Let's solve

$xy'' + (-2x-2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x-2)\,y' + (x+2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\,x^{-1+r} + (a_1(1+r)\,(-2+r) - 2a_0(-1+r))\,x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0,3\}$

- Each term must be 0

  $a_1(1+r)(-2+r)-2a_0(-1+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)-2a_k k-2a_k r+2a_k+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=3$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-(2*x+2)*diff(y(x),x)+(x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^3$$

✓ Solution by Mathematica
Time used: 0.025 (sec). Leaf size: 23

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{3} e^x \left(c_2 x^3 + 3c_1\right)$$

2.35 problem 36

2.35.1 Maple step by step solution 334

Internal problem ID [7525]
Internal file name [`OUTPUT/6458_Sunday_June_05_2022_04_53_26_PM_71259149/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2+2)\,y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\cos(x) + c_2 x\sin(x)
$$

Verified OK.

2.35.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-2xy'+(x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r+a_1r(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r-1)\,(k+r-2)+a_{k-2})\,x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$

- Each term must be 0

  $a_1r(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.36 problem 38

2.36.1 Maple step by step solution 341

Internal problem ID [7526]
Internal file name [`OUTPUT/6459_Sunday_June_05_2022_04_53_28_PM_53356577/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4}
$$

Verified OK.

2.36.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (-16x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k + r - \frac{1}{2}\right)\left(k + r - \frac{3}{2}\right)a_k - 16a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $4\left(k + \frac{3}{2} + r\right)\left(k + \frac{1}{2} + r\right)a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh(2x) + c_2\sqrt{x}\,\cosh(2x)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.37 problem 39

2.37.1 Maple step by step solution 348

Internal problem ID [7527]
Internal file name [OUTPUT/6460_Sunday_June_05_2022_04_53_30_PM_29330767/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + (4x^2 + 3)\, y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + (4x^2 + 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\sqrt{x}\right) + c_2\left(\cos(x)\sqrt{x}(\tan(x))\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x)\sqrt{x} + c_2 \sin(x)\sqrt{x} \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x)\sqrt{x} + c_2 \sin(x)\sqrt{x}$$

Verified OK.

2.37.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (4x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(4x^2+3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2y''-4xy'+(4x^2+3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\,x^r+a_1(1+2r)\,(-1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r-1)\,(2k+2r-3)+4$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right)a_k+4a_{k-2}=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left(k+\frac{1}{2}+r\right)a_{k+2}+4a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{4a_k}{(2k+3+2r)(2k+1+2r)}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{(2k+4)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}},a_{k+2}=-\frac{4a_k}{(2k+4)(2k+2)},a_1=0\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+2}=-\frac{4a_k}{(2k+6)(2k+4)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}},a_{k+2}=-\frac{4a_k}{(2k+6)(2k+4)},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right),a_{k+2}=-\frac{4a_k}{(2k+4)(2k+2)},a_1=0,b_{k+2}=-\frac{4b_k}{(2k+6)(2k+4)},b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sin(x) + c_2\sqrt{x}\,\cos(x)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(4*x^2+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-ix}\sqrt{x}\left(2c_1 - ic_2e^{2ix}\right)$$

2.38 problem 40

2.38.1 Maple step by step solution 355

Internal problem ID [7528]
Internal file name [`OUTPUT/6461_Sunday_June_05_2022_04_53_32_PM_46515768/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' - (x^2 - 2)\,y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (-x^2 + 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 70: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x} c_2 x}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x} c_2 x}{2}
$$

Verified OK.

2.38.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (-x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} - \frac{(x^2-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (-x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)\,(k+r-2) - a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)-(x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sinh(x) + c_2 x \cosh(x)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-2*x*y'[x]-(x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} x + \frac{1}{2} c_2 e^x x$$

2.39 problem 41

2.39.1 Maple step by step solution 362

Internal problem ID [7529]
Internal file name [`OUTPUT/6462_Sunday_June_05_2022_04_53_34_PM_99875169/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2x(1+x)\,y' + (x^2+2x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + (-2x^2-2x)\,y' + (x^2+2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2 - 2x \\ C &= x^2 + 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\,dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\,\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x\,\mathrm{e}^{x}) + c_2 (x\,\mathrm{e}^{x}(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{x} c_1 x + x^2 \mathrm{e}^{x} c_2 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{x} c_1 x + x^2 \mathrm{e}^{x} c_2
$$

Verified OK.

2.39.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 - 2x)\, y' + (x^2 + 2x + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2x+2)y}{x^2} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x^2+2x+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2x(1+x)\, y' + (x^2+2x+2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + (a_1 r(-1+r) - 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) - 2a_0(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 4a_{k-1} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+1+r)(k+r) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 4a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} + 2a_{k+1}r - a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1} - a_k + 4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{2ka_{k+1} - a_k + 2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2kb_{k+1} - b_k + 4b_{k+1}}{(k+3)(k+2)}, b_1 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \, e^x + c_2 e^x x^2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[x^2*y''[x]-2*x*(x+1)*y'[x]+(x^2+2*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^x x(c_2 x + c_1)$$

2.40 problem 42

2.40.1 Maple step by step solution 369

Internal problem ID [7530]
Internal file name [`OUTPUT/6463_Sunday_June_05_2022_04_53_35_PM_43754617/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2x(x+2)\,y' + \left(x^2+4x+6\right)y = 0}$$

Writing the ode as

$$x^2y'' + \left(-2x^2-4x\right)y' + \left(x^2+4x+6\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 74: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\,dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1\left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 \mathrm{e}^x\right) + c_2\left(x^2 \mathrm{e}^x (x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = x^2 \mathrm{e}^x c_1 + c_2 x^3 \mathrm{e}^x \tag{1}$$

Verification of solutions

$$y = x^2 \mathrm{e}^x c_1 + c_2 x^3 \mathrm{e}^x$$

Verified OK.

2.40.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 - 4x) y' + (x^2 + 4x + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+4x+6)y}{x^2} + \frac{2(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x+2)y'}{x} + \frac{(x^2+4x+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 6$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-2x(x+2)\,y'+(x^2+4x+6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r+(a_1(-1+r)\,(-2+r)-2a_0(-2+r))\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(k+r-2)\,(k+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{2,3\}$

- Each term must be 0

$a_1(-1+r)(-2+r)-2a_0(-2+r)=0$

- Solve for the dependent coefficient(s)

$a_1=\frac{2a_0}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-2)(k+r-3)-2a_{k-1}k-2a_{k-1}r+a_{k-2}+6a_{k-1}=0$

- Shift index using $k->k+2$

$a_{k+2}(k+r)(k+r-1)-2a_{k+1}(k+2)-2a_{k+1}r+a_k+6a_{k+1}=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{2ka_{k+1}+2a_{k+1}r-a_k-2a_{k+1}}{(k+r)(k+r-1)}$

- Recursion relation for $r=2$

$a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r=2$

$\left[y=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)},a_1=2a_0\right]$

- Recursion relation for $r=3$

$a_{k+2}=\frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r=3$

$\left[y=\sum_{k=0}^{\infty}a_kx^{k+3},a_{k+2}=\frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)},a_1=a_0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k+2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+3}\right),a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)},a_1=2a_0,b_{k+2}=\frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+3)(k+2)},b_1\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 + c_2 e^x x^3$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]-2*x*(x+2)*y'[x]+(x^2+4*x+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^x x^2 (c_2 x + c_1)$$

2.41 problem 43

2.41.1 Maple step by step solution 376

Internal problem ID [7531]
Internal file name [`OUTPUT/6464_Sunday_June_05_2022_04_53_38_PM_45926251/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4xy' + (x^2+6)\,y = 0}$$

Writing the ode as

$$x^2y'' - 4xy' + (x^2+6)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 76: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^2 \cos(x)\right) + c_2\left(x^2 \cos(x)\left(\tan(x)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \cos(x) + c_2 \sin(x)\, x^2 \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \cos(x) + c_2 \sin(x)\, x^2$$

Verified OK.

2.41.1 Maple step by step solution

Let's solve

$x^2 y'' - 4xy' + (x^2+6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+6)y}{x^2} + \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{(x^2+6)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 6$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - 4xy' + (x^2 + 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2 + r)\, (-3 + r)\, x^r + a_1(-1 + r)\, (-2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k + r - 2)\, (k + r - 3) + a_{k-2})\, x^{k\cdot}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2 + r)\, (-3 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

  $a_1(-1 + r)\, (-2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)(k+r-3)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+r)(k+r-1)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^2 \sin(x) + c_2 \cos(x)\, x^2$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-4*x*y'[x]+(x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-ix}x^2\left(2c_1 - ic_2e^{2ix}\right)$$

2.42 problem 44

2.42.1 Maple step by step solution 386

Internal problem ID [7532]
Internal file name [`OUTPUT/6465_Sunday_June_05_2022_04_53_40_PM_97722135/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1\\ B &= -x\\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 78: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-x\, e^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x) + c_2 \left(e^x \left(-x\, e^{-x}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 x$$

Verified OK.

2.42.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+\mathrm{e}^x c_2$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

2.43 problem 45

2.43.1 Maple step by step solution 393

Internal problem ID [7533]
Internal file name [`OUTPUT/6466_Sunday_June_05_2022_04_53_42_PM_69220947/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4x(1+x)\,y' + (3+2x)\,y = 0}$$

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (3+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 3 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 80: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\,(\mathrm{e}^{x})\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x}$$

Verified OK.

2.43.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 - 4x)\,y' + (3 + 2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y'}{x} - \frac{(3+2x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{(3+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{3+2x}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2 y'' - 4x(1+x)\, y' + (3+2x)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right) = 0$

- Shift index using $k->k+1$

  $4\left(k+r-\frac{1}{2}\right)\left(\left(k+\frac{1}{2}+r\right)a_{k+1}-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k}{2k+1+2r}$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2a_k}{2k+2}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2a_k}{2k+4}$
- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k}{2k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{2a_k}{2k+2}, b_{k+1}=\frac{2b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\,e^x$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 20

```
DSolve[4*x^2*y''[x]-4*x*(x+1)*y'[x]+(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2e^x + c_1)$$

2.44 problem 46

2.44.1 Maple step by step solution 403

Internal problem ID [7534]
Internal file name [`OUTPUT/6467_Sunday_June_05_2022_04_53_44_PM_53864733/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0}$$

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4 (3x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 81x^2 - 108x + 54$$
$$t = 4(3x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4 (3x - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 82: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x - \frac{1}{3}\right)^2} - \frac{3}{2\left(x - \frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{9x - 6}{6x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x - 1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x - 2}{3x - 1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(3x - 1)}{2}} \\
&= z_1 \left(\sqrt{3x - 1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-3x} x\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\mathrm{e}^{-3x} x\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} - c_2 x\, \mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} - c_2 x\, \mathrm{e}^{-x}$$

Verified OK.

2.44.1 Maple step by step solution

Let's solve

$(3x-1)\, y'' + (-3x-2)\, y' + (-6x+8)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2(3x-4)y}{3x-1} + \frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(3x+2)y'}{3x-1} - \frac{2(3x-4)y}{3x-1} = 0$

□ Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{3x+2}{3x-1}, P_3(x) = -\frac{2(3x-4)}{3x-1}\right]$

- ◦ $\left(x-\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}} = -1$

- ◦ $\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{3}$

$\left(\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}} = 0$

- ◦ $x = \frac{1}{3}$is a regular singular point

Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

$x_0 = \frac{1}{3}$

- Multiply by denominators

$(3x-1)\,y'' + (-3x-2)\,y' + (-6x+8)\,y = 0$

- Change variables using $x = u + \frac{1}{3}$ so that the regular singular point is at $u = 0$

$3u\left(\frac{d^2}{du^2}y(u)\right) + (-3u-3)\left(\frac{d}{du}y(u)\right) + (-6u+6)\,y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k- >k-m$

$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-2+r)\, u^{-1+r} + (3a_1(1+r)(-1+r) - 3a_0(-2+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$3a_1(1+r)(-1+r) - 3a_0(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3a_{k+1}(k+1+r)(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$$

- Shift index using $k->k+1$

$$3a_{k+2}(k+2+r)(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{ka_{k+1}+ra_{k+1}+2a_k-a_{k+1}}{(k+2+r)(k+r)}$$

- Recursion relation for $r=0$

$$a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

$$a_{k+2} = \frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$$

- Recursion relation for $r=2$

$a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 18

```
dsolve((3*x-1)*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-x} c_2 x$$

✓ Solution by Mathematica
Time used: 0.157 (sec). Leaf size: 35

```
DSolve[(3*x-1)*y''[x]-(3*x+2)*y'[x]-(6*x-8)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}(c_1 e^{3x} + 2e c_2 x)}{\sqrt{2}}$$

2.45 problem 47

2.45.1 Maple step by step solution 414

Internal problem ID [7535]
Internal file name [`OUTPUT/6468_Sunday_June_05_2022_04_53_48_PM_22031661/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+2)\,y''+xy'+3y=0}$$

Writing the ode as

$$(x+2)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 12x - 20}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 12x - 20 \\ t &= 4(x+2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 12x - 20}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 84: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{4}{x+2} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{0} a_i x^i \\ = \frac{1}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 12x - 20}{4\left(x+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 4 - (2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{x+2} - \frac{1}{2} \\
&= -\frac{x-2}{2\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{2}{x+2} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{\left(x+2\right)^2}\right) + \left(\frac{2}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 12x - 20}{4\left(x+2\right)^2}\right)\right) = 0
$$

$$
\frac{\left(a_1 + 6\right) x + 2a_0 + 2a_1 + 4}{x+2} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4, a_1 = -6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 6x + 4
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6x + 4\right) \mathrm{e}^{\int \left(\frac{2}{x+2} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 6x + 4\right) \mathrm{e}^{-\frac{x}{2} + 2\ln(x+2)} \\
&= \left(x^2 - 6x + 4\right) \left(x+2\right)^2 \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-6x+4\right)(x+2)^3\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-(x^2-6x+4)\,(x+2)^3\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(x+2)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1+c_2y_2 \\
&= c_1\left(\left(x^2-6x+4\right)(x+2)^3\,\mathrm{e}^{-x}\right)+c_2\Bigg(\left(x^2-6x+4\right)(x \\
&\quad +2)^3\,\mathrm{e}^{-x}\left(\frac{-(x^2-6x+4)\,(x+2)^3\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(x+2)^3}\right)\Bigg)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 (x^2 - 6x + 4) (x+2)^3 e^{-x} + c_2 \left(-\frac{(x^2-6x+4)(x+2)^3 e^{-x-2} \operatorname{expIntegral}_1(-x-2)}{240} - \frac{x^4}{240} + \frac{x^3}{240} + \frac{3x^2}{40} + \frac{11x}{120} - \frac{1}{30} \right) \tag{1}$$

Verification of solutions

$$y = c_1 (x^2 - 6x + 4) (x+2)^3 e^{-x} + c_2 \left(-\frac{(x^2-6x+4)(x+2)^3 e^{-x-2} \operatorname{expIntegral}_1(-x-2)}{240} - \frac{x^4}{240} + \frac{x^3}{240} + \frac{3x^2}{40} + \frac{11x}{120} - \frac{1}{30} \right)$$

Verified OK.

2.45.1 Maple step by step solution

Let's solve

$(x+2) y'' + x y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x+2} - \frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{x+2} + \frac{3y}{x+2} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x}{x+2}, P_3(x) = \frac{3}{x+2}\right]$

  ◦ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = -2$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

- $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\,y''+xy'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - Shift index using $k->k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+3)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)+a_k(k+r+3)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+1)(k-2)}$
- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}$
- Solution for $r=3$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u=x+2$

  $\left[y=\sum_{k=0}^{\infty} a_k(x+2)^{k+3}, a_{k+1}=-\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 115

```
dsolve((2+x)*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}\left(x^5 - 20x^3 - 40x^2 + 32\right) - \frac{c_2(\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\left(-2-x\right)x^5 + \mathrm{e}^{x}x^4 - 20\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\left(-2-x\right)x^3 - \mathrm{e}^{x}x^3 - 40\,\mathrm{e}^{-2}\,\mathrm{expIntegral}}{240}$$

✓ Solution by Mathematica

Time used: 0.68 (sec). Leaf size: 81

```
DSolve[(2+x)*y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{960}e^{-x-1}\left(c_2\left(x^2-6x+4\right)(x+2)^3\,\mathrm{ExpIntegralEi}(x+2) + 3840c_1\left(x^2-6x+4\right)(x+2)^3 - c_2e^{x+2}\left(x^4-x^3-18x^2-22x+8\right)\right)$$

2.46 problem 48

Internal problem ID [7536]
Internal file name [`OUTPUT/6469_Sunday_June_05_2022_04_53_51_PM_8802439/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)\,y''+x(4+x)\,y'+(-x+2)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(-x+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+4x\\C&=-x+2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 36}{4x\left(x - 1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x + 36 \\ t &= 4x(x - 1)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 36}{4x\left(x - 1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 86: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{x} + \frac{35}{4(x-1)^2} - \frac{9}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x(x-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+36}{4x\left(x-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{5}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{5}{2\left(x-1\right)} \\
&= \frac{1}{x} - \frac{5}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x} - \frac{5}{2(x-1)}\right)(2x + a_1) + \left(\left(-\frac{1}{x^2} + \frac{5}{2(x-1)^2}\right) + \left(\frac{1}{x} - \frac{5}{2(x-1)}\right)^2 - \left(\frac{-x+36}{4x(x-1)^2}\right)\right) = 0$$

$$\frac{(a_1 - 6)\, x + 4a_0 - 2a_1}{x(x-1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 6x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x^2 + 6x + 3)\, e^{\int \left(\frac{1}{x} - \frac{5}{2(x-1)}\right) dx} \\
&= (x^2 + 6x + 3)\, e^{\ln(x) - \frac{5\ln(x-1)}{2}} \\
&= \frac{(x^2 + 6x + 3)\, x}{(x-1)^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2}\, dx} \\
&= z_1 e^{-2\ln(x) + \frac{5\ln(x-1)}{2}} \\
&= z_1 \left(\frac{(x-1)^{\frac{5}{2}}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+6x+3}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)+5\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{152x+138}{9x^2+54x+27} + \ln(x) + \frac{1}{9x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+6x+3}{x}\right) + c_2 \left(\frac{x^2+6x+3}{x}\left(\frac{152x+138}{9x^2+54x+27} + \ln(x) + \frac{1}{9x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(1+3(x^3+6x^2+3x)\ln(x)+51x^2+48x)}{3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(1+3(x^3+6x^2+3x)\ln(x)+51x^2+48x)}{3x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(3\ln(x)\,x^3+18x^2\ln(x)+9x\ln(x)+51x^2+48x+1)}{3x^2}$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 53

```
DSolve[x^2*(1-x)*y''[x]+x*(4+x)*y'[x]+(2-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_1x(x^2+6x+3) - c_2(51x^2+3(x^2+6x+3)\,x\log(x)+48x+1)}{3x^2}$$

2.47 problem 49

2.47.1 Maple step by step solution 431

Internal problem ID [7537]
Internal file name [`OUTPUT/6470_Sunday_June_05_2022_04_53_53_PM_42215004/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''+x(2x+1)\,y'-(4+6x)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=2x^2+x\\C&=-6x-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 + 40x + 15$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{2x} - \frac{1}{4(1+x)^2} + \frac{15}{4x^2} - \frac{5}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4(x^2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 3\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^2+40x+15}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right)\\
&= 3-(3)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x+2} + \frac{5}{2x} + (0) \\
&= \frac{1}{2x+2} + \frac{5}{2x} \\
&= \frac{6x+5}{2x\,(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x+2} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2\,(1+x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2x+2} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2+40x+15}{4\,(x^2+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x+2} + \frac{5}{2x}\right) dx} \\
&= \sqrt{1+x}\, x^{\frac{5}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2+x}{x^2(1+x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(1+x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\,x^{\frac{5}{2}}}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{1+x}\,x^{\frac{5}{2}}}{\sqrt{x\,(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\,x^{\frac{5}{2}}}{\sqrt{x\,(1+x)}} \left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{1+x}\,x^{\frac{5}{2}}}{\sqrt{x\,(1+x)}} + \frac{c_2\sqrt{1+x}\,(12\ln(x)\,x^4 - 12\ln(1+x)\,x^4 + 12x^3 - 6x^2 + 4x - 3)}{12x^{\frac{3}{2}}\sqrt{x\,(1+x)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{1+x}\,x^{\frac{5}{2}}}{\sqrt{x\,(1+x)}} + \frac{c_2\sqrt{1+x}\,(12\ln(x)\,x^4 - 12\ln(1+x)\,x^4 + 12x^3 - 6x^2 + 4x - 3)}{12x^{\frac{3}{2}}\sqrt{x\,(1+x)}}$$

Verified OK.

2.47.1 Maple step by step solution

Let's solve

$x^2(1+x)\, y'' + (2x^2+x)\, y' + (-6x-4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(3x+2)y}{x^2(1+x)} - \frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x(1+x)} - \frac{2(3x+2)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x+1}{x(1+x)}, P_3(x) = -\frac{2(3x+2)}{x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\, y'' + x(2x+1)\, y' + (-6x-4)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right) + (2u^2-3u+1)\left(\frac{d}{du}y(u)\right) + (-6u+2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 u^{-1+r} + \left(a_1 (1+r)^2 - a_0 (2r^2 + r - 2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k (2k^2 + 4kr + 2r^2 +\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1 (1+r)^2 - a_0 (2r^2 + r - 2) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 - k - 6)\, a_{k-1} + (-2k^2 - k + 2)\, a_k + a_{k+1}(k+1)^2 = 0$$

- Shift index using $k- > k + 1$

  $$\left((k+1)^2 - k - 7\right) a_k + \left(-2(k+1)^2 - k + 1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1+2*x)*diff(y(x),x)-(4+6*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + \frac{c_2(12x^4 \ln(x) - 12\ln(x+1)x^4 + 12x^3 - 6x^2 + 4x - 3)}{12x^2}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 52

```
DSolve[x^2*(1+x)*y''[x]+x*(1+2*x)*y'[x]-(4+6*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_1 x^2 + \frac{c_2 (12x^4 \log(x) - 12x^4 \log(x+1) + 12x^3 - 6x^2 + 4x - 3)}{12x^2}$$

2.48 problem 50

2.48.1 Maple step by step solution 441

Internal problem ID [7538]
Internal file name [`OUTPUT/6471_Sunday_June_05_2022_04_53_56_PM_23003199/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2x^2+1)\,y''+x(2x^2+4)\,y'+2(1-x^2)\,y=0}$$

Writing the ode as

$$(2x^4+x^2)\,y''+(2x^3+4x)\,y'+(-2x^2+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+x^2 \\ B &= 2x^3+4x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3x^2 - 9}{(2x^2 + 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3x^2 - 9 \\
t &= (2x^2 + 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3x^2 - 9}{(2x^2 + 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 9}{(2x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 9}{(2x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(1) + \left(\left(\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} -\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)}-\frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)dx} \\
&= (x) \frac{1}{\left(4x^2+2\right)^{\frac{3}{4}}} \\
&= \frac{x}{\left(4x^2+2\right)^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2}\, dx} \\
&= z_1 e^{-2\ln(x)+\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(2x^2+1\right)^{\frac{3}{4}}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{\frac{1}{4}}}{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)+\frac{3\ln\left(2x^2+1\right)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{2}\left(2x^2-2\right)\sqrt{2x^2+1}+6\,\operatorname{arcsinh}\left(\sqrt{2}\,x\right)x}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2^{\frac{1}{4}}}{2x}\right) + c_2\left(\frac{2^{\frac{1}{4}}}{2x}\left(\frac{\sqrt{2}\,(2x^2-2)\sqrt{2x^2+1} + 6\ \mathrm{arcsinh}\left(\sqrt{2}\,x\right)x}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 2^{\frac{1}{4}}}{2x} + \frac{c_2\left(\sqrt{2}\,(x^2-1)\sqrt{2x^2+1} + 3\ \mathrm{arcsinh}\left(\sqrt{2}\,x\right)x\right)2^{\frac{1}{4}}}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 2^{\frac{1}{4}}}{2x} + \frac{c_2\left(\sqrt{2}\,(x^2-1)\sqrt{2x^2+1} + 3\ \mathrm{arcsinh}\left(\sqrt{2}\,x\right)x\right)2^{\frac{1}{4}}}{x^2}
$$

Verified OK.

2.48.1 Maple step by step solution

Let's solve

$(2x^4+x^2)\,y'' + (2x^3+4x)\,y' + (-2x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(x^2-1)y}{x^2(2x^2+1)} - \frac{2(x^2+2)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x^2+2)y'}{x(2x^2+1)} - \frac{2(x^2-1)y}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+2)}{x(2x^2+1)}, P_3(x) = -\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(2x^2+1)\,y'' + 2(x^2+2)\,xy' + (-2x^2+2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(1+r)\,x^r + a_1(3+r)\,(2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + 2a_{k-2}(k+r-1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$
- Each term must be 0

  $a_1(3+r)(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + 2a_{k-2}(k+r-1)(k-3+r) = 0$
- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+3+r) + 2a_k(k+r+1)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(x^2*(1+2*x^2)*diff(y(x),x$2)+x*(4+2*x^2)*diff(y(x),x)+2*(1-x^2)*y(x)=0,y(x), singsol=
```

$$y(x)=\frac{c_1}{x}+\frac{c_2\sqrt{2}\left(\sqrt{2}\sqrt{2x^2+1}\,x^2+3\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)x-\sqrt{2}\sqrt{2x^2+1}\right)}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 77

```
DSolve[x^2*(1+2*x^2)*y''[x]+x*(4+2*x^2)*y'[x]+2*(1-x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x)\to -\frac{c_2\sqrt{2x^2+1}}{x^2}+c_2\sqrt{2x^2+1}-\frac{3c_2\log\left(\sqrt{2x^2+1}-\sqrt{2}x\right)}{\sqrt{2}x}+\frac{c_1}{x}$$

2.49 problem 51

2.49.1 Maple step by step solution 451

Internal problem ID [7539]
Internal file name [OUTPUT/6472_Sunday_June_05_2022_04_53_58_PM_4250488/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+2)\,y''+2x(x^2+5)\,y'+2(-x^2+3)\,y=0}$$

Writing the ode as

$$\left(x^4+2x^2\right)y''+\left(2x^3+10x\right)y'+\left(-2x^2+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+2x^2 \\ B &= 2x^3+10x \\ C &= -2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^4 - 5x^2 + 3 \\ t &= (x^3 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + 2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 2 - (0) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} + (0) \\
&= \frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)} \\
&= \frac{3}{x^3+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x-i\sqrt{2}\right)} - \frac{3}{4\left(x+i\sqrt{2}\right)}\right)(2x+a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x-i\sqrt{2}\right)^2} + \frac{3}{4\left(x+i\sqrt{2}\right)^2}\right) + \left(\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2+8)\, \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)}\right) dx} \\
&= (x^2+8)\, \mathrm{e}^{\frac{3\ln(x)}{2} - \frac{3\ln\left(x^2+2\right)}{4}} \\
&= \frac{(x^2+8)\, x^{\frac{3}{2}}}{(x^2+2)^{\frac{3}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+10x}{x^4+2x^2}\, dx} \\
&= z_1 e^{\frac{3\ln\left(x^2+2\right)}{4} - \frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{(x^2+2)^{\frac{3}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+8}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3\ln\left(x^2+2\right)}{2} - 5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-x^2\sqrt{2}\,(x^2+8)\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) + (2x^2-8)\sqrt{x^2+2}}{64x^2\,(x^2+8)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+8}{x}\right) + c_2\left(\frac{x^2+8}{x}\left(\frac{-x^2\sqrt{2}\,(x^2+8)\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)+(2x^2-8)\sqrt{x^2+2}}{64x^2\,(x^2+8)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2+8)}{x} + \frac{c_2\left(-x^2\sqrt{2}\,(x^2+8)\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)+(2x^2-8)\sqrt{x^2+2}\right)}{64x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2+8)}{x} + \frac{c_2\left(-x^2\sqrt{2}\,(x^2+8)\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)+(2x^2-8)\sqrt{x^2+2}\right)}{64x^3}
$$

Verified OK.

2.49.1 Maple step by step solution

Let's solve

$(x^4+2x^2)\,y''+(2x^3+10x)\,y'+(-2x^2+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2(x^2-3)y}{x^2(x^2+2)}-\frac{2(x^2+5)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x^2+5)y'}{x(x^2+2)}-\frac{2(x^2-3)y}{x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2(x^2+5)}{(x^2+2)x}, P_3(x)=-\frac{2(x^2-3)}{x^2(x^2+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

$$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 5$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 3$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x^2(x^2+2)\, y'' + 2x(x^2+5)\, y' + (-2x^2+6)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$2a_0(3+r)(1+r)x^r + 2a_1(4+r)(2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, -1\}$
- Each term must be 0

  $2a_1(4+r)(2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$
- Shift index using $k->k+2$

  $2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$
- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$
- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(k+1)(k-2)}{2(k+4)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(x^2*(2+x^2)*diff(y(x),x$2)+2*x*(x^2+5)*diff(y(x),x)+2*(3-x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1(x^2+8)}{x} - \frac{c_2\sqrt{2}\left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)x^4 - \sqrt{2}\,\sqrt{x^2+2}\,x^2 + 8\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)x^2 + 4\sqrt{2}\,\sqrt{x^2+2}\right)}{64x^3}$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 88

```
DSolve[x^2*(2+x^2)*y''[x]+2*x*(x^2+5)*y'[x]+2*(3-x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{-\sqrt{2}c_2(x^2+8)\,x^2\operatorname{arctanh}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right) + 64c_1x^4 + 2x^2\left(c_2\sqrt{x^2+2} + 256c_1\right) - 8c_2\sqrt{x^2+2}}{64x^3}$$

2.50 problem 52

Internal problem ID [7540]
Internal file name [`OUTPUT/6473_Sunday_June_05_2022_04_54_01_PM_12560383/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''+6xy'+6y=0}$$

Writing the ode as

$$(x^2+1)\,y''+6xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\frac{}{(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2+1}\, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{(x^2+1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(ix+1)^2}\right) + c_2 \left(\frac{1}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{(ix+1)^2} + \frac{c_2 x}{(x-i)^2 (x+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{(ix+1)^2} + \frac{c_2 x}{(x-i)^2 (x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve((1+x^2)*diff(y(x),x$2)+6*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(x^2+1)^2} + \frac{c_2(x^2-1)}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 29

```
DSolve[(1+x^2)*y''[x]+6*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x - c_1(x-i)^2}{(x^2+1)^2}$$

2.51 problem 53

Internal problem ID [7541]
Internal file name [`OUTPUT/6474_Sunday_June_05_2022_04_54_03_PM_23863514/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2xy'-2y=0}$$

Writing the ode as

$$(x^2+1)\,y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2x^2 + 3 \\
t &= (x^2 + 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 94: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 + 3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2(x - i)^2} - \frac{1}{2(x + i)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right.\right.
$$
$$
-\frac{2(x^2 + 1)\, a_0}{(-x + i)^2 (x + i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x)\, \sqrt{x^2 + 1} \\
&= x\sqrt{x^2 + 1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{1}{x} - \arctan(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{1}{x} - \arctan(x)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x + c_2(-\arctan(x)\,x - 1) \tag{1}
$$

Verification of solutions

$$y = c_1x + c_2(-\arctan(x)\,x - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(\arctan(x)\,x + 1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 48

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1-ix) + c_2x\log(1+ix) + 2ic_2)$$

2.52 problem 54

Internal problem ID [7542]
Internal file name [`OUTPUT/6475_Sunday_June_05_2022_04_54_06_PM_83423705/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 54.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-8xy'+20y=0}$$

Writing the ode as

$$(x^2+1)\,y''-8xy'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -8x \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-24}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -24 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{24}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{24}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-i} + \frac{3}{x+i} + (-)(0) \\
&= -\frac{2}{x-i} + \frac{3}{x+i} \\
&= \frac{x-5i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2}\right) + \left(-\frac{2}{x-i} + \frac{3}{x+i}\right)^2 - \left(-\frac{24}{(x^2+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x-i} + \frac{3}{x+i}\right) dx} \\ &= \frac{(x^2+1)^3}{(ix+1)^5} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-8x}{x^2+1} \, dx} \\ &= z_1 e^{2 \ln(x^2+1)} \\ &= z_1 \left((x^2+1)^2 \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^5}{(ix+1)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{4\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^5}{(ix+1)^5}\right) + c_2 \left(\frac{(x^2+1)^5}{(ix+1)^5}\left(\frac{x^4 - 2x^2 + \frac{1}{5}}{(x+i)^5}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2+1)^5}{(ix+1)^5} + \frac{c_2 (x^2+1)^5 \left(x^4 - 2x^2 + \frac{1}{5}\right)}{(ix+1)^5 (x+i)^5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2+1)^5}{(ix+1)^5} + \frac{c_2 (x^2+1)^5 \left(x^4 - 2x^2 + \frac{1}{5}\right)}{(ix+1)^5 (x+i)^5}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((1+x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)+20*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(5x^4 - 10x^2 + 1) + c_2(x^5 - 10x^3 + 5x)$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 38

```
DSolve[(1+x^2)*y''[x]-8*x*y'[x]+20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}ic_2(5x^4 - 10x^2 + 1) + c_1(1 + ix)^5$$

2.53 problem 55

2.53.1 Maple step by step solution 481

Internal problem ID [7543]
Internal file name [`OUTPUT/6476_Sunday_June_05_2022_04_54_08_PM_29003247/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 55.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-8xy'-12y=0}$$

Writing the ode as

$$(1-x^2)\,y''-8xy'-12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -8x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= (x^2-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{(x^2-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 96: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-1} + \frac{2}{1+x} + (-)(0) \\ &= -\frac{1}{x-1} + \frac{2}{1+x} \\ &= \frac{-3+x}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{1+x}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(1+x)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{1+x}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1} + \frac{2}{1+x}\right) dx} \\ &= \frac{(1+x)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x}{1-x^2} \, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(1+x)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-3x^2-1}{3\,(1+x)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{(x-1)^3}\right) + c_2\left(\frac{1}{(x-1)^3}\left(\frac{-3x^2-1}{3\,(1+x)^3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3\,(x-1)^3\,(1+x)^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3\,(x-1)^3\,(1+x)^3}$$

Verified OK.

2.53.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 8xy' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 4$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 8xy' + 12y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (8u - 8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 a_{0} r \left(3+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(-2 a_{k +1} \left(k +1+r \right) \left(k +r +4\right)+a_{k} \left(k +r +4\right) \left(k +r +3\right)\right) u^{k +r}\right) =$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2 r \left(3+r \right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-3, 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(\left(-2 k -2 r -2\right) a_{k +1}+a_{k} \left(k +r +3\right)\right) \left(k +r +4\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k +1} = \frac{a_{k} \left(k +r +3\right)}{2 \left(k +1+r \right)}$

- Recursion relation for $r =-3$

  $a_{k +1} = \frac{a_{k} k}{2 \left(k -2\right)}$

- Series not valid for $r =-3$, division by 0 in the recursion relation at $k =2$

  $a_{k +1} = \frac{a_{k} k}{2 \left(k -2\right)}$

- Recursion relation for $r =0$

  $a_{k +1} = \frac{a_{k} \left(k +3\right)}{2 \left(k +1\right)}$

- Solution for $r =0$

  $\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k}, a_{k +1} = \frac{a_{k} \left(k +3\right)}{2 \left(k +1\right)}\right]$

- Revert the change of variables $u = 1+x$

  $\left[y = \sum_{k =0}^{\infty} a_{k} \left(1+x \right)^{k}, a_{k +1} = \frac{a_{k} \left(k +3\right)}{2 \left(k +1\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 43

```
dsolve((1-x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x^2+1)}{(x-1)^3(x+1)^3} + \frac{c_2(x^3+3x)}{(x-1)^3(x+1)^3}$$

✓ Solution by Mathematica
Time used: 0.045 (sec). Leaf size: 37

```
DSolve[(1-x^2)*y''[x]-8*x*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3(x^2-1)^3}$$

2.54 problem 56

Internal problem ID [7544]
Internal file name [`OUTPUT/6477_Sunday_June_05_2022_04_54_11_PM_69263401/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 56.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+1)\,y''+7xy'+2y=0}$$

Writing the ode as

$$(2x^2+1)\,y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2 + 6 \\ t &= 4\left(2x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 98: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x+\frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x-\frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x+\frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4(2x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x)\, (4x^2 + 2)^{\frac{1}{8}} \\
&= x (4x^2 + 2)^{\frac{1}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve((1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{(2x^2+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 66

```
DSolve[(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{(2x^2+1)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{(2x^2+1)^{3/4}\,\text{Gamma}\left(\frac{1}{4}\right)}$$

2.55 problem 57

2.55.1 Maple step by step solution 499

Internal problem ID [7545]
Internal file name [`OUTPUT/6478_Sunday_June_05_2022_04_54_14_PM_4745276/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 57.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-5xy'-4y=0}$$

Writing the ode as

$$(1-x^2)\,y''-5xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -5x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(1+x)^2} + \frac{5}{16(x-1)^2} - \frac{7}{16(x-1)} + \frac{7}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2+6}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)} \\
&= -\frac{x}{2x^2-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-1\right)^2} + \frac{1}{4\left(1+x\right)^2}\right) + \left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)}\right)^2 - \left(\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(1+x)}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\
&= \frac{x}{\left(x-1\right)^{\frac{1}{4}} \left(1+x\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x-1)}{4}-\frac{5\ln(1+x)}{4}} \\
&= z_1 \left(\frac{1}{(x-1)^{\frac{5}{4}}\,(1+x)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{(x^2-1)^{\frac{1}{4}}\,(x-1)^{\frac{5}{4}}\,(1+x)^{\frac{5}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2}-\frac{5\ln(1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x-\sqrt{x^2-1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x^2-1)^{\frac{1}{4}}\,(x-1)^{\frac{5}{4}}\,(1+x)^{\frac{5}{4}}}\right) \\
&\quad + c_2 \left(\frac{x}{(x^2-1)^{\frac{1}{4}}\,(x-1)^{\frac{5}{4}}\,(1+x)^{\frac{5}{4}}}\left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x-\sqrt{x^2-1}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} + \frac{c_2 \left(\ln \left(x+\sqrt{x^2-1}\right) x - \sqrt{x^2-1}\right)}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} + \frac{c_2 \left(\ln \left(x+\sqrt{x^2-1}\right) x - \sqrt{x^2-1}\right)}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}}$$

Verified OK.

2.55.1 Maple step by step solution

Let's solve

$(1-x^2) y'' - 5xy' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5xy'}{x^2-1} - \frac{4y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5xy'}{x^2-1} + \frac{4y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{5x}{x^2-1}, P_3(x) = \frac{4}{x^2-1}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{2}$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $\left(x^2-1\right) y''+5 x y'+4 y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $\left(u^2-2 u\right)\left(\frac{d^2}{d u^2} y(u)\right)+(5 u-5)\left(\frac{d}{d u} y(u)\right)+4 y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot\left(\frac{d}{d u} y(u)\right)$ to series expansion for $m=0..1$

    $u^m \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r) u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$

  - Convert $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)$ to series expansion for $m=1..2$

    $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1) u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(3+2 r) u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(2 k+5+2 r)+a_k(k+r+2)^2\right) u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(3+2 r)=0$

- Values of r that satisfy the indicial equation

  $r \in\left\{0,-\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)^2-2\left(k+\frac{5}{2}+r\right)(k+1+r) a_{k+1}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r+2)^2}{(2 k+5+2 r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}\right]$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k(k+2)^2}{(2k+5)(k+1)}, b_{k+1} = \frac{b_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve((1-x^2)*diff(y(x),x$2)-5*x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(\ln\left(x+\sqrt{x^2-1}\right)x - \sqrt{x^2-1}\right)}{(x^2-1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 52

```
DSolve[(1-x^2)*y''[x]-5*x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2\sqrt{x^2-1} - c_2 x \log\left(\sqrt{x^2-1}-x\right) + c_1 x}{(x^2-1)^{3/2}}$$

2.56 problem 58

Internal problem ID [7546]
Internal file name [`OUTPUT/6479_Sunday_June_05_2022_04_54_17_PM_67064686/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 58.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-10xy'+28y=0}$$

Writing the ode as

$$(x^2+1)\,y''-10xy'+28y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -10x \\ C &= 28 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 33}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 33$$
$$t = \left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 33}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 33}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{5}{2(x-i)} + \frac{7}{2(x+i)} + (0) \\ &= -\frac{5}{2(x-i)} + \frac{7}{2(x+i)} \\ &= \frac{x-6i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)}\right)(1) + \left(\left(\frac{5}{2\,(x-i)^2} - \frac{7}{2\,(x+i)^2}\right) + \left(-\frac{5}{2\,(x-i)} + \frac{7}{2\,(x+i)}\right)^2 - \Bigg(\frac{2x}{(x^2} \quad - \frac{2(x^2+1)\,(6i}{(-x+i)^2\,(x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 6i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 6i)\, \mathrm{e}^{\int \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right) dx} \\
&= (x - 6i)\, \mathrm{e}^{\frac{\ln\left(x^2+1\right)}{2} - 6i \arctan(x)} \\
&= \frac{(-x + 6i)\,(x^2+1)^{\frac{7}{2}}}{(-x+i)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{5\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\left(x^2+1\right)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-x+6i)(x^2+1)^6}{(-x+i)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{5\ln(x^2+1)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{35x^4-42x^2+3}{105(-x+6i)(x+i)^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(-x+6i)(x^2+1)^6}{(-x+i)^6}\right) + c_2\left(\frac{(-x+6i)(x^2+1)^6}{(-x+i)^6}\left(\frac{35x^4-42x^2+3}{105(-x+6i)(x+i)^6}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(-x+6i)(x^2+1)^6}{(-x+i)^6} + \frac{c_2(x^2+1)^6(35x^4-42x^2+3)}{105(-x+i)^6(x+i)^6} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(-x+6i)(x^2+1)^6}{(-x+i)^6} + \frac{c_2(x^2+1)^6(35x^4-42x^2+3)}{105(-x+i)^6(x+i)^6}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve((1+x^2)*diff(y(x),x$2)-10*x*diff(y(x),x)+28*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(1 + \frac{35}{3}x^4 - 14x^2\right) + c_2\left(x^7 + 21x^5 - 105x^3 + 35x\right)$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 40

```
DSolve[(1+x^2)*y''[x]-10*x*y'[x]+28*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{105}c_2\left(35x^4 - 42x^2 + 3\right) - c_1(x-i)^6(x+6i)$$

2.57 problem 59

2.57.1 Maple step by step solution 516

Internal problem ID [7547]
Internal file name [`OUTPUT/6480_Sunday_June_05_2022_04_54_19_PM_62622680/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 59.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 102: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int -\frac{x}{2} dx} \\
&= (x)\, e^{-\frac{x^2}{4}} \\
&= x\, e^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, e^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right) x - 2\, e^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.57.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,e^{-\frac{x^2}{2}} + \frac{c_2\sqrt{2}\,e^{-\frac{x^2}{2}}\left(i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{x^2}{2}} - \pi\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\right)}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.58 problem 60

Internal problem ID [7548]
Internal file name [`OUTPUT/6481_Sunday_June_05_2022_04_54_22_PM_17222070/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 60.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+1)\,y''-9xy'-6y=0}$$

Writing the ode as

$$(2x^2+1)\,y''-9xy'-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= -9x \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{165x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 165x^2 + 6 \\ t &= 4\left(2x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{165x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 104: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{153}{64\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{153}{64\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{177i\sqrt{2}}{64\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{177i\sqrt{2}}{64\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{165x^2 + 6}{4\left(2x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{165}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{15}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{11}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{165x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{15}{4}$	$-\frac{11}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{15}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{15}{4} - \left(-\frac{9}{4}\right)\\
&= 6
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\ &= -\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)} \\ &= -\frac{9x}{4x^2+2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)}\right)\left(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{3}, a_1 = 0, a_2 = 1, a_3 = 0, a_4 = \frac{5}{3}, a_5 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{9}{8\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{9}{8\left(x+\frac{i\sqrt{2}}{2}\right)}\right) dx} \\
&= \left(x^6 + \frac{5}{3}x^4 + x^2 + \frac{1}{3}\right) \frac{1}{(4x^2+2)^{\frac{9}{8}}} \\
&= \frac{3x^6 + 5x^4 + 3x^2 + 1}{(4x^2+2)^{\frac{1}{8}} (12x^2+6)}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x}{2x^2+1} \, dx} \\
&= z_1 e^{\frac{9\ln(2x^2+1)}{8}} \\
&= z_1 \left((2x^2+1)^{\frac{9}{8}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3x^6 + 5x^4 + 3x^2 + 1)\, 2^{\frac{7}{8}}}{12}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x}{2x^2+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{9\ln(2x^2+1)}{4}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{36(2x^2+1)^{\frac{9}{4}}\, 2^{\frac{1}{4}}}{(3x^6+5x^4+3x^2+1)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(3x^6 + 5x^4 + 3x^2 + 1)\, 2^{\frac{7}{8}}}{12} \right) \\
&\quad + c_2 \left(\frac{(3x^6 + 5x^4 + 3x^2 + 1)\, 2^{\frac{7}{8}}}{12} \left(\int \frac{36(2x^2 + 1)^{\frac{9}{4}}\, 2^{\frac{1}{4}}}{(3x^6 + 5x^4 + 3x^2 + 1)^2} dx \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1(3x^6 + 5x^4 + 3x^2 + 1)\, 2^{\frac{7}{8}}}{12} \\
&\quad + c_2 \left(18x^6 + 30x^4 + 18x^2 + 6\right) 2^{\frac{1}{8}} \left(\int \frac{(2x^2 + 1)^{\frac{9}{4}}}{(3x^6 + 5x^4 + 3x^2 + 1)^2} dx \right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1(3x^6 + 5x^4 + 3x^2 + 1)\, 2^{\frac{7}{8}}}{12} \\
&\quad + c_2 \left(18x^6 + 30x^4 + 18x^2 + 6\right) 2^{\frac{1}{8}} \left(\int \frac{(2x^2 + 1)^{\frac{9}{4}}}{(3x^6 + 5x^4 + 3x^2 + 1)^2} dx \right)
\end{aligned}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve((1+2*x^2)*diff(y(x),x$2)-9*x*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(3x^6+5x^4+3x^2+1) + c_2(3x^6+5x^4+3x^2+1)\left(\int \frac{(2x^2+1)^{\frac{9}{4}}}{(3x^4+2x^2+1)^2(x^2+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 71

```
DSolve[(1+2*x^2)*y''[x]-9*x*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 (2x^2+1)^{13/8} Q_{\frac{11}{4}}^{\frac{13}{4}}\left(i\sqrt{2}x\right) + \frac{64\sqrt[4]{2}c_1(3x^6+5x^4+3x^2+1)}{3\,\text{Gamma}\left(-\frac{9}{4}\right)}$$

2.59 problem 61

2.59.1 Maple step by step solution 534

Internal problem ID [7549]
Internal file name [`OUTPUT/6482_Sunday_June_05_2022_04_54_25_PM_49757253/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 61.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2 - 8x + 11)\, y'' - 16(x-2)\, y' + 36y = 0}$$

Writing the ode as

$$(2x^2 - 8x + 11)\, y'' + (-16x + 32)\, y' + 36y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 - 8x + 11 \\ B &= -16x + 32 \\ C &= 36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x^2 - 32x - 100$$
$$t = (2x^2 - 8x + 11)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 - 8x + 11)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x-2+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$2+\frac{i\sqrt{6}}{2}$	2	0	3	-2
$2-\frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= 2-(1)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6} + 2x - 4}{2x^2 - 8x + 11}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}}\right)(1) + \left(\left(\frac{2}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5i\sqrt{6}}{2} - 2\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2 - \frac{5i\sqrt{6}}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}}+\frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right)dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\frac{\ln\left(4x^2-16x+22\right)}{2}-5i\arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x+32}{2x^2-8x+11}\,dx} \\
&= z_1 e^{2\ln\left(2x^2-8x+11\right)} \\
&= z_1\left(\left(2x^2-8x+11\right)^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln\left(2x^2-8x+11\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{-\frac{16}{3}x^3+32x^2-\frac{296}{5}x+\frac{496}{15}}{\left(5i\sqrt{6}-2x+4\right)\left(2x-4+i\sqrt{6}\right)^5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}\right) \\
&\quad + c_2\left(\frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}\left(\frac{-\frac{16}{3}x^3+32x^2-\frac{296}{5}x+\frac{496}{15}}{\left(5i\sqrt{6}-2x+4\right)\left(2x-4+i\sqrt{6}\right)^5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\frac{9c_1\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \\
&- \frac{12c_2\left(2x^2-8x+11\right)^5\sqrt{6}\left(10x^3-60x^2+111x-62\right)}{5\left(2ix-4i-\sqrt{6}\right)^5\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = &\frac{9c_1\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \\
&- \frac{12c_2\left(2x^2-8x+11\right)^5\sqrt{6}\left(10x^3-60x^2+111x-62\right)}{5\left(2ix-4i-\sqrt{6}\right)^5\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned}
$$

Verified OK.

2.59.1 Maple step by step solution

Let's solve

$\left(2x^2-8x+11\right)y''+\left(-16x+32\right)y'+36y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{36y}{2x^2-8x+11}+\frac{16(x-2)y'}{2x^2-8x+11}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{16(x-2)y'}{2x^2-8x+11}+\frac{36y}{2x^2-8x+11}=0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{16(x-2)}{2x^2-8x+11}, P_3(x) = \frac{36}{2x^2-8x+11}\right]$

- $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)$ is analytic at $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right) \cdot P_2(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

- $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$

- $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 2-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $(2x^2-8x+11)\, y'' + (-16x+32)\, y' + 36y = 0$

- Change variables using $x = u+2-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2-2\,\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right) + (-16u+8\,\mathrm{I}\sqrt{6})\left(\frac{d}{du}y(u)\right) + 36y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2\,\mathrm{I}\sqrt{6}\,r(r-5)\,a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1} + 2a_k(k+r-3)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\,\mathrm{I}\sqrt{6}\,r(r-5) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\,\mathrm{I}\sqrt{6}\,(k+1+r)\,(k-4+r)\,a_{k+1} + 2a_k(k+r-3)\,(k+r-6) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k\,(k^2+2kr+r^2-9k-9r+18)\sqrt{6}}{k^2+2kr+r^2-3k-3r-4}$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k\,(k^2-9k+18)\sqrt{6}}{k^2-3k-4}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{3\,\mathrm{I}}{4}a_0\sqrt{6}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{5\,\mathrm{I}}{18}a_1\sqrt{6}$

- Express in terms of a_0

  $a_2 = -\frac{5a_0}{4}$

- Apply recursion relation for $k = 2$

  $a_3 = \frac{\mathrm{I}}{9}a_2\sqrt{6}$

- Express in terms of a_0

  $a_3 = -\frac{5\,\mathrm{I}}{36}a_0\sqrt{6}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{3\,\mathrm{I}\sqrt{6}\,u}{4} - \frac{5u^2}{4} - \frac{5\,\mathrm{I}\sqrt{6}\,u^3}{36}\right)$

- Revert the change of variables $u = x - 2 + \frac{\mathrm{I}\sqrt{6}}{2}$

  $\left[y = -\frac{\mathrm{I}}{72}a_0\sqrt{6}\,(10x^3 - 60x^2 + 111x - 62)\right]$

- Recursion relation for $r=5$; series terminates at $k=1$

  $a_{k+1}=\frac{-\frac{I}{6}a_k(k^2+k-2)\sqrt{6}}{k^2+7k+6}$

- Apply recursion relation for $k=0$

  $a_1=\frac{I}{18}a_0\sqrt{6}$

- Terminating series solution of the ODE for $r=5$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1+\frac{I\sqrt{6}\,u}{18}\right)$

- Revert the change of variables $u=x-2+\frac{I\sqrt{6}}{2}$

  $\left[y=a_0\left(\frac{5}{6}+\frac{I(x-2)\sqrt{6}}{18}\right)\right]$

- Combine solutions and rename parameters

  $\left[y=-\frac{Ia_0\sqrt{6}\,(10x^3-60x^2+111x-62)}{72}+b_0\left(\frac{5}{6}+\frac{I(x-2)\sqrt{6}}{18}\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((11-8*x+2*x^2)*diff(y(x),x$2)-16*(x-2)*diff(y(x),x)+36*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\left(-\frac{31}{5}+x^3-6x^2+\frac{111}{10}x\right)+c_2\left(x^6-12x^5+\frac{165}{2}x^4-\frac{16577}{8}-\frac{5445}{4}x^2+3267x\right)$$

✓ Solution by Mathematica

Time used: 0.998 (sec). Leaf size: 91

```
DSolve[(11-8*x+2*x^2)*y''[x]-16*(x-2)*y'[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{15} i c_2 \left(10x^3 - 60x^2 + 111x - 62\right) + \frac{c_1 \left(2x + 5i\sqrt{6} - 4\right) (2(x-4)x + 11)^2 \left(2ix + \sqrt{6} - 4i\right)^3}{2\left(-2ix + \sqrt{6} + 4i\right)^2}$$

2.60 problem 62

2.60.1 Maple step by step solution 545

Internal problem ID [7550]
Internal file name [`OUTPUT/6483_Sunday_June_05_2022_04_54_27_PM_3595735/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 62.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (-3 + x)\, y' + 3y = 0}$$

Writing the ode as

$$\begin{align*} y'' + (-3 + x)\, y' + 3y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= -3 + x \tag{3} \\ C &= 3 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= -\frac{3}{2} + \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4} - \frac{3}{2}x + \frac{1}{4}x^2$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6x - 1}{4} \\
&= Q + \frac{R}{4} \\
&= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\
&= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{3}{2} + \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{3}{2} + \frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(-\frac{3}{2} + \frac{x}{2}\right) \\ &= \frac{3}{2} - \frac{x}{2} \\ &= \frac{3}{2} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2} - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(\frac{3}{2} - \frac{x}{2}\right)^2 - \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right)\right) = 0$$

$$(x + 3)\, a_1 + 6x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6x + 8\right) e^{\int \left(\frac{3}{2} - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 6x + 8\right) e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= \left(x^2 - 6x + 8\right) e^{-\frac{x(x-6)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3+x}{1} \, dx} \\
&= z_1 e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= z_1 \left(e^{-\frac{x(x-6)}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2 - 6x + 8\right) e^{-\frac{x(x-6)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3+x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3x - \frac{1}{2}x^2}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x(x-6)}{2}}}{\left(x^2 - 6x + 8\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 6x + 8\right) e^{-\frac{x(x-6)}{2}}\right) + c_2 \left(\left(x^2 - 6x + 8\right) e^{-\frac{x(x-6)}{2}} \left(\int \frac{e^{\frac{x(x-6)}{2}}}{\left(x^2 - 6x + 8\right)^2} dx\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-6x+8)\,\mathrm{e}^{-\frac{x(x-6)}{2}} + c_2(x^2-6x+8)\,\mathrm{e}^{-\frac{x(x-6)}{2}}\left(\int \frac{\mathrm{e}^{\frac{x(x-6)}{2}}}{(x^2-6x+8)^2}dx\right) \quad (1)$$

Verification of solutions

$$y = c_1(x^2-6x+8)\,\mathrm{e}^{-\frac{x(x-6)}{2}} + c_2(x^2-6x+8)\,\mathrm{e}^{-\frac{x(x-6)}{2}}\left(\int \frac{\mathrm{e}^{\frac{x(x-6)}{2}}}{(x^2-6x+8)^2}dx\right)$$

Verified OK.

2.60.1 Maple step by step solution

Let's solve

$y'' + (-3+x)\,y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\,x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

  ◦ Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$

  Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\left(k+1\right)-3a_{k+1}(k+1)+a_{k}(k+3)\right)x^{k}=0$$

- Each term in the series must be 0, giving the recursion relation

  $$k^{2}a_{k+2}+\left(a_{k}-3a_{k+1}+3a_{k+2}\right)k+3a_{k}-3a_{k+1}+2a_{k+2}=0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+2}=-\frac{a_{k}k-3a_{k+1}k+3a_{k}-3a_{k+1}}{k^{2}+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 68

```
dsolve(diff(y(x),x$2)+(x-3)*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{1}{2}x^2+3x}(x^2-6x+8) + c_2 e^{-\frac{1}{2}x^2+3x}(x^2-6x+8)\left(\int \frac{e^{\frac{1}{2}x^2-3x}}{(x-2)^2(x-4)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.588 (sec). Leaf size: 90

```
DSolve[y''[x]+(x-3)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{1}{2}(x-6)x-8}\left(e^{7/2}\sqrt{2\pi}c_2(x^2-6x+8)\operatorname{erfi}\left(\frac{x-3}{\sqrt{2}}\right) + 4e^8 c_1(x^2-6x+8) - 2c_2 e^{\frac{1}{2}(x-4)^2+x}(x-3)\right)$$

2.61 problem 63

2.61.1 Maple step by step solution 554

Internal problem ID [7551]
Internal file name [`OUTPUT/6484_Sunday_June_05_2022_04_54_31_PM_53716205/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 63.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2 - 8x + 14)\, y'' - 8(x - 4)\, y' + 20y = 0}$$

Writing the ode as

$$(x^2 - 8x + 14)\, y'' + (-8x + 32)\, y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 8x + 14 \\ B &= -8x + 32 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48}{(x^2 - 8x + 14)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48$$
$$t = (x^2 - 8x + 14)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48}{(x^2 - 8x + 14)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 4 + \sqrt{2}\right)^2} + \frac{6}{\left(x - 4 - \sqrt{2}\right)^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-4+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48}{(x^2 - 8x + 14)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$4+\sqrt{2}$	2	0	3	-2
$4-\sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\ &= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\ &= \frac{x-4-5\sqrt{2}}{x^2-8x+14} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right)(0) + \left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^2} - \frac{3}{\left(x-4+\sqrt{2}\right)^2}\right) + \left(-\frac{2}{x-4-\sqrt{2}} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right) dx} \\ &= \frac{\left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x+32}{x^2-8x+14}\, dx} \\ &= z_1 e^{2\ln\left(x^2-8x+14\right)} \\ &= z_1\left(\left(x^2-8x+14\right)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x^2-8x+14)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-5x^4+80x^3-500x^2+1440x-1604}{5\left(x-4+\sqrt{2}\right)^5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x-4+\sqrt{2}\right)^3 (x^2-8x+14)^2}{\left(-x+4+\sqrt{2}\right)^2}\right) \\
&\quad + c_2 \left(\frac{\left(x-4+\sqrt{2}\right)^3 (x^2-8x+14)^2}{\left(-x+4+\sqrt{2}\right)^2} \left(\frac{-5x^4+80x^3-500x^2+1440x-1604}{5\left(x-4+\sqrt{2}\right)^5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\left(x-4+\sqrt{2}\right)^3 (x^2-8x+14)^2}{\left(-x+4+\sqrt{2}\right)^2} \\
& - \frac{c_2(x^2-8x+14)^2 \left(x^4-16x^3+100x^2-288x+\frac{1604}{5}\right)}{\left(x-4-\sqrt{2}\right)^2 \left(x-4+\sqrt{2}\right)^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1\left(x-4+\sqrt{2}\right)^3 (x^2-8x+14)^2}{\left(-x+4+\sqrt{2}\right)^2} \\
& - \frac{c_2(x^2-8x+14)^2 \left(x^4-16x^3+100x^2-288x+\frac{1604}{5}\right)}{\left(x-4-\sqrt{2}\right)^2 \left(x-4+\sqrt{2}\right)^2}
\end{aligned}
$$

Verified OK.

2.61.1 Maple step by step solution

Let's solve

$\left(x^2-8x+14\right)y''+\left(-8x+32\right)y'+20y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{20y}{x^2-8x+14}+\frac{8(x-4)y'}{x^2-8x+14}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{8(x-4)y'}{x^2-8x+14}+\frac{20y}{x^2-8x+14}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{8(x-4)}{x^2-8x+14}, P_3(x)=\frac{20}{x^2-8x+14}\right]$

  ◦ $\left(x-4+\sqrt{2}\right)\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

  $\left(\left(x-4+\sqrt{2}\right)\cdot P_2(x)\right)\bigg|_{x=4-\sqrt{2}}=0$

  ◦ $\left(x-4+\sqrt{2}\right)^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

  $\left(\left(x-4+\sqrt{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=4-\sqrt{2}}=0$

  ◦ $x=4-\sqrt{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=4-\sqrt{2}$

- Multiply by denominators

  $\left(x^2-8x+14\right)y''+\left(-8x+32\right)y'+20y=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

  $\left(u^2-2u\sqrt{2}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-8u+8\sqrt{2}\right)\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\, r(r-5)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)\,(k+r-4)\, a_{k+1} + a_k(k+r-4)\,(k+r-5)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\, r(r-5) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-4)\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-5)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{5a_0\sqrt{2}}{4}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{2}$
- Express in terms of a_0

$a_2 = \frac{5a_0}{4}$

- Apply recursion relation for $k = 2$

$a_3 = -\frac{a_2\sqrt{2}}{4}$

- Express in terms of a_0

$a_3 = -\frac{5a_0\sqrt{2}}{16}$

- Apply recursion relation for $k = 3$

$a_4 = -\frac{a_3\sqrt{2}}{8}$

- Express in terms of a_0

$a_4 = \frac{5a_0}{64}$

- Apply recursion relation for $k = 4$

$a_5 = -\frac{a_4\sqrt{2}}{20}$

- Express in terms of a_0

$a_5 = -\frac{a_0\sqrt{2}}{256}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

$\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$

- Recursion relation for $r = 5$

$a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$

- Solution for $r = 5$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

$\left[y = \sum_{k=0}^{\infty} a_k \left(x - 4 + \sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - 4 + \right.\right.\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((x^2-8*x+14)*diff(y(x),x$2)-8*(x-4)*diff(y(x),x)+20*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{1604}{5} + x^4 - 16x^3 + 100x^2 - 288x\right) + c_2(x^5 - 140x^3 + 1120x^2 - 3500x + 4032)$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 77

```
DSolve[(x^2-8*x+14)*y''[x]+8*(x-4)*y'[x]+20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{(x^2 - 8x + 14)^{3/2}}$$

2.62 problem 64

2.62.1 Maple step by step solution 564

Internal problem ID [7552]
Internal file name [`OUTPUT/6485_Sunday_June_05_2022_04_54_33_PM_21506465/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 64.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+4x+5)\,y''-20(1+x)\,y'+60y=0}$$

Writing the ode as

$$(2x^2+4x+5)\,y''+(-20x-20)\,y'+60y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+4x+5 \\ B &= -20x-20 \\ C &= 60 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-210}{(2x^2 + 4x + 5)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -210 \\ t &= \left(2x^2 + 4x + 5\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{210}{(2x^2 + 4x + 5)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 4x + 5)^2$. There is a pole at $x = -1 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x + 1 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x + 1 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x + 1 - \frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x + 1 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = -1 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=-\frac{210}{\left(2x^2+4x+5\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-1+\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1-\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d&=\alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&=1-(1)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{+}}{x-c_2}\right)+(-)[\sqrt{r}]_\infty\\
&=-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)}+\frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}+(-)\,(0)\\
&=-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)}+\frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\\
&=\frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\
&= \frac{27\sqrt{2}\,(2x^2+4x+5)^{\frac{7}{2}}}{\left(3+i\,(1+x)\sqrt{6}\right)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5} \, dx} \\
&= z_1 e^{\frac{5\ln\left(2x^2+4x+5\right)}{2}} \\
&= z_1 \left((2x^2+4x+5)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i - \frac{(1+x)\sqrt{3}\sqrt{2}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5\ln(2x^2+4x+5)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-16x^5-80x^4-80x^3+80x^2+124x+28}{\left(2x+2+i\sqrt{6}\right)^6} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{(2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} \right) \\
&\quad + c_2 \left(-\frac{(2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} \left(\frac{-16x^5-80x^4-80x^3+80x^2+124x+28}{\left(2x+2+i\sqrt{6}\right)^6} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-\frac{c_1 (2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} \\
&+ \frac{108 c_2 (2x^2+4x+5)^6 \sqrt{2}\, (4x^5+20x^4+20x^3-20x^2-31x-7)}{\left(-\sqrt{6}\,x-\sqrt{6}+3i\right)^6 \left(2x+2+i\sqrt{6}\right)^6}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1 (2x^2+4x+5)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\sqrt{2}}{3}\right)^6} + \frac{108 c_2 (2x^2+4x+5)^6 \sqrt{2}\,(4x^5+20x^4+20x^3-20x^2-31x-7)}{(-\sqrt{6}\,x-\sqrt{6}+3i)^6 \,(2x+2+i\sqrt{6})^6}$$

Verified OK.

2.62.1 Maple step by step solution

Let's solve

$(2x^2+4x+5)\,y''+(-20x-20)\,y'+60y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{60y}{2x^2+4x+5}+\frac{20(1+x)y'}{2x^2+4x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{20(1+x)y'}{2x^2+4x+5}+\frac{60y}{2x^2+4x+5}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{20(1+x)}{2x^2+4x+5}, P_3(x)=\frac{60}{2x^2+4x+5}\right]$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

  $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

- ○ $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $\left(2x^2+4x+5\right)y''+\left(-20x-20\right)y'+60y=0$

- Change variables using $x=u-1-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

  $\left(2u^2-2\,\mathrm{I}u\sqrt{6}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-20u+10\,\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right)+60y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2\,\mathrm{I}\sqrt{6}\,(r-6)\,ra_0u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2\,\mathrm{I}\sqrt{6}\,(k+r-5)\,(k+1+r)\,a_{k+1}+2a_k(k+r-5)\,(k+r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\,\mathrm{I}\sqrt{6}\,(r-6)\,r=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,6\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+r-5)\left(\mathrm{I}(k+1+r)\,a_{k+1}\sqrt{6}-a_k(k+r-6)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{-\frac{\mathrm{I}}{6}a_k(k+r-6)\sqrt{6}}{k+1+r}$

- Recursion relation for $r = 0$; series terminates at $k = 6$

$$a_{k+1} = \frac{-\frac{I}{6}a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

$$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{I}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{I\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{5} a_k \left(x + 1 + \frac{I\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{I}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{I}{6}a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1} = \frac{-\frac{I}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u = x + 1 + \frac{I\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + 1 + \frac{I\sqrt{6}}{2}\right)^{k+6}, a_{k+1} = \frac{-\frac{I}{6}a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{5} a_k \left(x + 1 + \frac{I\sqrt{6}}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + 1 + \frac{I\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1} = \frac{-\frac{I}{6}a_k(k-6)\sqrt{6}}{k+1}, b_{k+1} = \frac{-\frac{I}{6}b_k k\sqrt{6}}{k+7}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve((2*x^2+4*x+5)*diff(y(x),x$2)-20*(x+1)*diff(y(x),x)+60*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{7}{4} + x^5 + 5x^4 + 5x^3 - 5x^2 - \frac{31}{4}x\right) + c_2\left(x^6 + \frac{155}{8} - \frac{75}{2}x^4 - 100x^3 - \frac{225}{4}x^2 + 30x\right)$$

✓ Solution by Mathematica

Time used: 1.003 (sec). Leaf size: 83

```
DSolve[(2*x^2+4*x+5)*y''[x]-20*(x+1)*y'[x]+60*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{(2x^2+4x+5)^{5/2}\left(4c_2(4x^5+20x^4+20x^3-20x^2-31x-7)+c_1\left(2ix+\sqrt{6}+2i\right)^6\right)}{(4x^2+8x+10)^{5/2}}$$

2.63 problem 65

2.63.1 Maple step by step solution 574

Internal problem ID [7553]
Internal file name [OUTPUT/6486_Sunday_June_05_2022_04_54_37_PM_38989325/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 65.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^3+1)\,y''+7x^2y'+9yx=0}$$

Writing the ode as

$$(x^3+1)\,y''+7x^2y'+9yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^3+1\\B&=7x^2\\C&=9x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x(x^3 + 8)}{4(x^3 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x(x^3 + 8) \\ t &= 4(x^3 + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{x(x^3 + 8)}{4(x^3 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{7}{36\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{7}{36\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36} + \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{5}{36} - \frac{5i\sqrt{3}}{36}}{x - \frac{1}{2} + \frac{i\sqrt{3}}{2}} + \frac{5}{18(1+x)} + \frac{7}{36(1+x)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{x(x^3+8)}{4\,(x^3+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{x(x^3+8)}{4\,(x^3+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)(0) \\ &= -\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\ &= -\frac{x^2}{2x^3+2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)(1) + \left(\left(\frac{1}{6(1+x)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln(1+x)}{6} - \frac{\ln\left(4x^2-4x+4\right)}{6}} \\
&= \frac{x 2^{\frac{2}{3}}}{2\,(1+x)^{\frac{1}{6}}\,(x^2-x+1)^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2}{x^3+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(x^3+1\right)}{6}} \\
&= z_1 \left(\frac{1}{(x^3+1)^{\frac{7}{6}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x^3+1)}{3}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{(x^3+1)^{\frac{1}{3}} 2^{\frac{2}{3}}}{x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{\frac{2}{3}} x}{2 (x^3+1)^{\frac{4}{3}}}\right) + c_2 \left(\frac{2^{\frac{2}{3}} x}{2 (x^3+1)^{\frac{4}{3}}} \left(\int \frac{(x^3+1)^{\frac{1}{3}} 2^{\frac{2}{3}}}{x^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 2^{\frac{2}{3}} x}{2 (x^3+1)^{\frac{4}{3}}} + \frac{c_2 2^{\frac{1}{3}} x \left(\int \frac{(x^3+1)^{\frac{1}{3}}}{x^2} dx\right)}{(x^3+1)^{\frac{4}{3}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 2^{\frac{2}{3}} x}{2 (x^3+1)^{\frac{4}{3}}} + \frac{c_2 2^{\frac{1}{3}} x \left(\int \frac{(x^3+1)^{\frac{1}{3}}}{x^2} dx\right)}{(x^3+1)^{\frac{4}{3}}}
$$

Verified OK.

2.63.1 Maple step by step solution

Let's solve

$(x^3+1) y'' + 7x^2 y' + 9yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{7x^2 y'}{x^3+1} - \frac{9xy}{x^3+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7x^2 y'}{x^3+1} + \frac{9xy}{x^3+1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(x^3+1)\, y'' + 7x^2 y' + 9yx = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 3u^2 + 3u)\left(\frac{d^2}{du^2} y(u)\right) + (7u^2 - 14u + 7)\left(\frac{d}{du} y(u)\right) + (9u - 9)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(4+3r)\, u^{-1+r} + \left(a_1(1+r)\,(7+3r) - a_0(3r^2+11r+9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+7\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(4+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{4}{3}\right\}$$

- Each term must be 0

$$a_1(1+r)\,(7+3r) - a_0(3r^2+11r+9) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(3k+7+3r) - a_k(3k^2+6kr+3r^2+11k+11r+9) + a_{k-1}(k+2+r)^2 = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(3k+10+3r) - a_{k+1}\left(3(k+1)^2+6(k+1)\,r+3r^2+11k+20+11r\right) + a_k(k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 2kra_k - 6kra_{k+1} + r^2a_k - 3r^2a_{k+1} + 6ka_k - 17ka_{k+1} + 6ra_k - 17ra_{k+1} + 9a_k - 23a_{k+1}}{(k+2+r)(3k+10+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 3k^2a_{k+1} + 6ka_k - 17ka_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 6k a_k - 17k a_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 6k a_k - 17k a_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((1+x^3)*diff(y(x),x$2)+7*x^2*diff(y(x),x)+9*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(x^3+1)^{\frac{4}{3}}} + \frac{c_2 x \left(\int \frac{((x+1)(x^2-x+1))^{\frac{1}{3}}}{x^2} dx\right)}{(x^3+1)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 1.109 (sec). Leaf size: 118

```
DSolve[(1+x^3)*y''[x]+7*x^2*y'[x]+9*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2\sqrt{3}c_2 x \arctan\left(\frac{\sqrt{3}x}{2\sqrt[3]{x^3+1}+x}\right) - 6c_2\sqrt[3]{x^3+1} - 2c_2 x \log\left(\sqrt[3]{x^3+1}-x\right) + c_2 x \log\left(\sqrt[3]{x^3+1}x + (x^3 -\right.}{6\left(x^3+1\right)^{4/3}}$$

2.64 problem 66

2.64.1 Maple step by step solution 588

Internal problem ID [7554]
Internal file name [`OUTPUT/6487_Sunday_June_05_2022_04_54_41_PM_89049387/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 66.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^5+1)\,y''+14y'x^4+10yx^3=0}$$

Writing the ode as

$$(2x^5+1)\,y''+14y'x^4+10yx^3=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^5+1 \\ B &= 14x^4 \\ C &= 10x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^3(5x^5+6)$$
$$t = (2x^5+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^3(5x^5+6)}{(2x^5+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = -\frac{2^{\frac{4}{5}}}{2}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{2^{\frac{4}{5}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}$$

For the pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}$$

For the pole at $x = -\frac{2^{\frac{4}{5}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{\frac{4}{5}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

For the pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{2^{\frac{4}{5}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{2^{\frac{4}{5}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}}{8}-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{\frac{4}{5}}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}}{8}-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^{-}}{x-c_4}\right) \\
&= \frac{3}{10\left(x-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x+\frac{2^{\frac{4}{5}}}{2}\cdots\right.} \\
&= \frac{3}{10\left(x-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x+\frac{2^{\frac{4}{5}}}{2}\cdots\right.} \\
&= \frac{3x^4}{2x^5+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{3}{10\left(x-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x+\frac{2^{\frac{4}{5}}}{2}\right)} + \frac{\cdots}{10\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\cdots\right.}\cdots\right.} \\
&= (x)\, e^{\frac{3\ln\left(2^{\frac{4}{5}}+2x\right)}{10} + \frac{3\ln\left(32\,2^{\frac{3}{5}}-16x2^{\frac{4}{5}}\sqrt{5}-16x2^{\frac{4}{5}}+64x^2\right)}{10} + \frac{3\ln\left(32\,2^{\frac{3}{5}}+64x^2-16x2^{\frac{4}{5}}+16x2^{\frac{4}{5}}\sqrt{5}\right)}{10} - \frac{3i\arctan\left(\frac{8x-2^{\frac{4}{5}}+2^{\frac{4}{5}}\sqrt{5}}{-\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5}-\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\cdots}\right.}{10}\cdots} \\
&= 4x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}}\left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}\left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1 \left(\frac{1}{\left(2x^5+1\right)^{\frac{7}{10}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{4x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}}\left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}\left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{1}{5}}}{32x^2\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}}\left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{5}}\left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{5}}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{4x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} \right)$$

$$+ c_2 \left(\frac{4x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} \left(\int \frac{}{32x^2 (} \right.\right.$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{4c_1 x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} + \frac{c_2 x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{3}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(\int \frac{}{x^2\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}} (} \right.}{8\left(2x^5+1\right)^{\frac{7}{10}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{4c_1 x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} + \frac{c_2 x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{3}{5}} \left(-x\left(\sqrt{5}+1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right) 2^{\frac{4}{5}} + 4x^2 + 2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(\int \frac{}{x^2\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}} (} \right.}{8\left(2x^5+1\right)^{\frac{7}{10}}}$$

Verified OK.

2.64.1 Maple step by step solution

Let's solve

$(2x^5+1)\,y''+14y'x^4+10yx^3=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{14x^4y'}{2x^5+1}-\frac{10x^3y}{2x^5+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{14x^4y'}{2x^5+1}+\frac{10x^3y}{2x^5+1}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{14x^4}{2x^5+1},P_3(x)=\frac{10x^3}{2x^5+1}\right]$

- ○ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

$\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

- ○ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

$\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

- ○ $x=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$ is

Check to see if x_0 is a regular singular point

$x_0=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$

- Multiply by denominators

$(2x^5+1)\,y''+14y'x^4+10yx^3=0$

- Change variables using $x = u + \left(\left(\left(-\frac{\sqrt{5}}{8} - \frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right) - \frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}} + \mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}\right.\right.$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..3$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..4$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..5$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3\cos\left(\frac{\pi}{5}\right) - 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2\sin\left(\frac{\pi}{5}\right) + \mathrm{I}\sin\left(\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3\cos\left(\frac{\pi}{5}\right) - 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2\sin\left(\frac{\pi}{5}\right) + \mathrm{I}\sin\left(\right.\right.$
- Values of r that satisfy the indicial equation

  $r = r$
- The coefficients of each power of u must be 0

$$\left[\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right)+\cos\left(\frac{\pi}{5}\right)+1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^{3}-4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^{3}\cos\left(\frac{\pi}{5}\right)-3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^{2}\sin\left(\frac{\pi}{5}\right)+\mathrm{I}\sin\right.\right.$$

- Solve for the dependent coefficient(s)
- Each term in the series must be 0, giving the recursion relation

$$\frac{5\,\mathrm{I}\left(-a_{k+1}\left(k+r+\frac{7}{5}\right)\left(\sqrt{5}+1\right)(k+r+1)2^{\frac{1}{5}}-\left(k^{2}+\left(2r+\frac{3}{5}\right)k+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k-2}\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4a_{k}2^{\frac{2}{5}}\left(k^{2}+\left(2r+\frac{9}{5}\right)k+r^{2}+\frac{9r}{5}+\frac{1}{2}\right)+4\,2^{\frac{3}{5}}a\right.}{4}$$

- Shift index using $k->k+3$

$$\frac{5\,\mathrm{I}\left(-a_{k+4}\left(k+\frac{22}{5}+r\right)\left(\sqrt{5}+1\right)(k+4+r)2^{\frac{1}{5}}-\left((k+3)^{2}+\left(2r+\frac{3}{5}\right)(k+3)+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k+1}\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4a_{k+3}2^{\frac{2}{5}}\left((k+3)^{2}+\left(2r+\frac{9}{5}\right)(k\right.\right.}{4}$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{2\left(-40a_{k+1}2^{\frac{4}{5}}k^{2}-40a_{k+1}2^{\frac{4}{5}}r^{2}-264a_{k+1}2^{\frac{4}{5}}k-264a_{k+1}2^{\frac{4}{5}}r+80\,2^{\frac{3}{5}}a_{k+2}k^{2}+80\,2^{\frac{3}{5}}a_{k+2}r^{2}+576\,2^{\frac{3}{5}}a_{k+2}k+576\,2^{\frac{3}{5}}a_{k+2}\right.}{}$$

- Recursion relation for $r=r$

$$a_{k+4}=-\frac{2\left(-40a_{k+1}2^{\frac{4}{5}}k^{2}-40a_{k+1}2^{\frac{4}{5}}r^{2}-264a_{k+1}2^{\frac{4}{5}}k-264a_{k+1}2^{\frac{4}{5}}r+80\,2^{\frac{3}{5}}a_{k+2}k^{2}+80\,2^{\frac{3}{5}}a_{k+2}r^{2}+576\,2^{\frac{3}{5}}a_{k+2}k+576\,2^{\frac{3}{5}}a_{k+2}\right.}{}$$

- Solution for $r=r$
- Revert the change of variables $u=x-\left(\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-}}{}\right.\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve((1+2*x^5)*diff(y(x),x$2)+14*x^4*diff(y(x),x)+10*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^5+1)^{\frac{2}{5}}} + \frac{c_2 x \left(\int \frac{1}{(2x^5+1)^{\frac{3}{5}} x^2} dx\right)}{(2x^5+1)^{\frac{2}{5}}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(1+2*x^5)*y''[x]+14*x^4*y'[x]+10*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.65 problem 67

2.65.1 Maple step by step solution 599

Internal problem ID [7555]
Internal file name [`OUTPUT/6488_Sunday_June_05_2022_04_54_44_PM_45095827/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 67.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + y'x^6 + 7yx^5 = 0}$$

Writing the ode as

$$y'' + y'x^6 + 7yx^5 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^5(x^7 - 16) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{6} a_i x^i \\ &= \frac{x^6}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^5(x^7 - 16)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\ &= \frac{1}{4}x^{12} - 4x^5 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^6}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &= -7 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^6}{2}\right) \\ &= -\frac{x^6}{2} \\ &= -\frac{x^6}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7-16)}{4}\right)\right) = 0$$
$$x^5 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int -\frac{x^6}{2} dx} \\
&= (x) \, e^{-\frac{x^7}{14}} \\
&= x \, e^{-\frac{x^7}{14}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^6}{1} \, dx} \\
&= z_1 e^{-\frac{x^7}{14}} \\
&= z_1 \left(e^{-\frac{x^7}{14}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x \, e^{-\frac{x^7}{7}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-7 \, e^{\frac{x^7}{7}} \left(-x^7\right)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}} \left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right) \right)}{7 \left(-x^7\right)^{\frac{6}{7}} x} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^7}{7}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^7}{7}}\left(\frac{-7\,\mathrm{e}^{\frac{x^7}{7}}\left(-x^7\right)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}\,x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^7}{7}} + \frac{c_2\left(-7(-x^7)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^7}{7}} + \frac{c_2\left(-7(-x^7)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}}
$$

Verified OK.

2.65.1 Maple step by step solution

Let's solve

$y'' + y'x^6 + 7yx^5 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^5 \cdot y$ to series expansion

    $x^5 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+5}$

  ◦ Shift index using $k- >k-5$

    $x^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} x^k$

- Convert $x^6 \cdot y'$ to series expansion

  $$x^6 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+5}$$

- Shift index using $k- >k - 5$

  $$x^6 \cdot y' = \sum_{k=5}^{\infty} a_{k-5}(k-5) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k + 2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-5}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2} + a_{k-5} + a_{k+2}) = 0$$

- Shift index using $k- >k + 5$

  $$(k+7)((k+5)a_{k+7} + a_k + a_{k+7}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve(diff(y(x),x$2)+x^6*diff(y(x),x)+7*x^5*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^7}{7}} x + \frac{7c_2(-1)^{\frac{6}{7}} \mathrm{e}^{-\frac{x^7}{7}} \left(-\Gamma\left(\frac{6}{7}\right) x^7 + (-x^7)^{\frac{6}{7}} 7^{\frac{1}{7}} \mathrm{e}^{\frac{x^7}{7}} + \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right) x^7\right)}{(-x^7)^{\frac{6}{7}}}$$

✓ Solution by Mathematica

Time used: 0.201 (sec). Leaf size: 53

```
DSolve[y''[x]+x^6*y'[x]+7*x^5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{49} e^{-\frac{x^7}{7}} \left(49 c_1 x - 7^{6/7} c_2 \sqrt[7]{-x^7} \Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

2.66 problem 68

Internal problem ID [7556]
Internal file name [`OUTPUT/6489_Sunday_June_05_2022_04_54_48_PM_30415200/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 68.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^8+1)\,y''-16y'x^7+72yx^6=0}$$

Writing the ode as

$$(x^8+1)\,y''-16y'x^7+72yx^6=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^8+1 \\ B &= -16x^7 \\ C &= 72x^6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-128x^6}{(x^8+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -128x^6$$
$$t = (x^8+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{128x^6}{(x^8+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8+1)^2$. There is a pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned}
r = {} & \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\
& + \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}} \\
& + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}}
\end{aligned}$$

For the pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$

in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x=-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $10>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{128x^{6}}{\left(x^{8}+1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right) \\ &= 1 - (-5) \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right)$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= \frac{\left((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3(x^2+1)\left((-1+i)x^4 + 1 + i\right)\right)\sqrt{2-\sqrt{2}} - 3\left(\left((-1+i)x^4 + 1 + i\right)\right.}{2\left(-x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x^2 - x\sqrt{2-\sqrt{2}} + 1\right)\left(x\left(1+\sqrt{2}\right)\right.}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i\sqrt{2} - 1 + i}{1 + i + i\sqrt{2}}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}}, a_2 = -\frac{15\left(-1 + i - \sqrt{2}\right)}{7\left(1 + i + i\sqrt{2}\right)}, a_3 = \frac{32}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-}}\right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} - \frac{15\left(-1 + i - \right.}{7\left(1 + i + i\right.}$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega\, dx}$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} - \frac{15(-1 + i -}{7(1 + i +}\right.$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} - \frac{15(-1 + i -}{7(1 + i +}\right.$$

$$= -\frac{\left(\left(-1 + i - \sqrt{2}\right)\sqrt{2-\sqrt{2}} + 2x\right)^3\left(\left(\left(ix^6 + \frac{15}{7}x^4 + \frac{15}{7}x^2 - i\right)\sqrt{2} + (1+i)x^6 + \left(\frac{15}{7} + \frac{15i}{7}\right)x^4 + \right.\right.}{256\sqrt{2-\sqrt{2}}\left(-x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\left(1+\sqrt{2}\right)\sqrt{2}\right.}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\
&= z_1 e^{\ln(x^8+1)} \\
&= z_1 (x^8+1)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{-7i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x^9 - 7i\sqrt{2-\sqrt{2}}\,x^9 + 9i\sqrt{2}\,x^8 - 7\sqrt{2-\sqrt{2}}\,x^9 + 9\sqrt{2}\,x^8 + 9i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x + 9i\sqrt{2}}{224i\sqrt{2-\sqrt{2}} - 224\sqrt{2}\,\sqrt{2-\sqrt{2}} - 224\sqrt{2-\sqrt{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-1032192x^8 + 802816}{108\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3 \left(\left(\left(-x^4+\frac{4}{3}ix^2+i\right)\sqrt{2}+(-1+i)\,x^4+\left(-\frac{4}{3}+\frac{4i}{3}\right)x^2+1\right.\right.} \right.
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{-7i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x^9 - 7i\sqrt{2-\sqrt{2}}\,x^9 + 9i\sqrt{2}\,x^8 - 7\sqrt{2-\sqrt{2}}\,x^9 + 9\sqrt{2}\,x^8 + 9i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x + 9}{224i\sqrt{2-\sqrt{2}} - 224\sqrt{2}\,\sqrt{2-\sqrt{2}} - 224\sqrt{2-\sqrt{2}}} \right. \\
&\quad + c_2 \left(\frac{-7i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x^9 - 7i\sqrt{2-\sqrt{2}}\,x^9 + 9i\sqrt{2}\,x^8 - 7\sqrt{2-\sqrt{2}}\,x^9 + 9\sqrt{2}\,x^8 + 9i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x +}{224i\sqrt{2-\sqrt{2}} - 224\sqrt{2}\,\sqrt{2-\sqrt{2}} - 224\sqrt{2-\sqrt{2}}} \right.
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\left(-7i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x^9 - 7i\sqrt{2-\sqrt{2}}\,x^9 + 9i\sqrt{2}\,x^8 - 7\sqrt{2-\sqrt{2}}\,x^9 + 9\sqrt{2}\,x^8 + 9i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x + 9i \cdots\right.}{224i\sqrt{2-\sqrt{2}} - 224\sqrt{2}\,\sqrt{2-\sqrt{2}} - 224\sqrt{2-\sqrt{2}}} - \frac{256c_2\left(x^8-\frac{7}{9}\right)\left(x\left(x^8-\frac{9}{7}\right)\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}} + \left(\frac{9}{7}-\frac{9i}{7}\right) \cdots\right.}{\sqrt{2-\sqrt{2}}\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3\left(-1+i-\sqrt{2}\right)\left(\frac{6x\left(1-i+\sqrt{2}\left(ix^4+\frac{4}{3}x^2+1\right)+(1+i)x^4+\left(\frac{4}{3}+\frac{4i}{3}\right)x^2\right)\cdots}{7}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(-7i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x^9 - 7i\sqrt{2-\sqrt{2}}\,x^9 + 9i\sqrt{2}\,x^8 - 7\sqrt{2-\sqrt{2}}\,x^9 + 9\sqrt{2}\,x^8 + 9i\sqrt{2}\,\sqrt{2-\sqrt{2}}\,x + 9i \cdots\right.}{224i\sqrt{2-\sqrt{2}} - 224\sqrt{2}\,\sqrt{2-\sqrt{2}} - 224\sqrt{2-\sqrt{2}}} - \frac{256c_2\left(x^8-\frac{7}{9}\right)\left(x\left(x^8-\frac{9}{7}\right)\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}} + \left(\frac{9}{7}-\frac{9i}{7}\right) \cdots\right.}{\sqrt{2-\sqrt{2}}\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3\left(-1+i-\sqrt{2}\right)\left(\frac{6x\left(1-i+\sqrt{2}\left(ix^4+\frac{4}{3}x^2+1\right)+(1+i)x^4+\left(\frac{4}{3}+\frac{4i}{3}\right)x^2\right)\cdots}{7}\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1+x^8)*diff(y(x),x$2)-16*x^7*diff(y(x),x)+72*x^6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{7}{9} + x^8\right) + c_2\left(x^9 - \frac{9}{7}x\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(1+x^8)*y''[x]-16*x^7*y'[x]+72*x^6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.67 problem 69

2.67.1 Maple step by step solution 618

Internal problem ID [7557]
Internal file name [OUTPUT/6490_Sunday_June_05_2022_04_55_08_PM_57486201/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 69.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + y'x^5 + 6yx^4 = 0}$$

Writing the ode as

$$y'' + y'x^5 + 6yx^4 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4(x^6 - 14)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4(x^6 - 14) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 120: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 10 \\ &= -10 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{5} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 5$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\ &= \frac{x^5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4(x^6 - 14)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\ &= \frac{1}{4}x^{10} - \frac{7}{2}x^4 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^5}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= -6 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4(x^6 - 14)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^5}{2}\right) \\ &= -\frac{x^5}{2} \\ &= -\frac{x^5}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) &= 0 \\ x^4 a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x^5}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^6}{12}} \\
&= x\, \mathrm{e}^{-\frac{x^6}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^5}{1}\, dx} \\
&= z_1 e^{-\frac{x^6}{12}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^6}{12}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^6}{6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-6\,\mathrm{e}^{\frac{x^6}{6}} \left(-x^6\right)^{\frac{5}{6}} + 6^{\frac{5}{6}} x^6 \left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}}\, x} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^6}{6}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^6}{6}}\left(\frac{-6\,\mathrm{e}^{\frac{x^6}{6}}\left(-x^6\right)^{\frac{5}{6}} + 6^{\frac{5}{6}} x^6\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}} x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^6}{6}} + \frac{c_2\left(-6\left(-x^6\right)^{\frac{5}{6}} + x^6 6^{\frac{5}{6}}\,\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^6}{6}} + \frac{c_2\left(-6\left(-x^6\right)^{\frac{5}{6}} + x^6 6^{\frac{5}{6}}\,\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}}}
$$

Verified OK.

2.67.1 Maple step by step solution

Let's solve

$y'' + y' x^5 + 6 y x^4 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^4 \cdot y$ to series expansion

  $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  ◦ Shift index using $k- >k-4$

  $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

- Convert $x^5 \cdot y'$ to series expansion

  $$x^5 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+4}$$

- Shift index using $k- >k - 4$

  $$x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4}(k-4) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k + 2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$$

- Shift index using $k- >k + 4$

  $$(k+6)((k+4)\, a_{k+6} + a_k + a_{k+6}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve(diff(y(x),x$2)+x^5*diff(y(x),x)+6*x^4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^6}{6}} x - \frac{2c_2 \mathrm{e}^{-\frac{x^6}{6}} \left((-x^6)^{\frac{5}{6}} 6^{\frac{2}{3}} \sqrt{3}\,\sqrt{2}\,\mathrm{e}^{\frac{x^6}{6}} - 6\Gamma\left(\frac{5}{6}\right) x^6 + 6\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right) x^6\right)}{(-x^6)^{\frac{5}{6}} \left(\sqrt{3} + i\right)}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 53

```
DSolve[y''[x]+x^5*y'[x]+6*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36} e^{-\frac{x^6}{6}} \left(36c_1 x - 6^{5/6} c_2 \sqrt[6]{-x^6}\,\Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

2.68 problem 70

2.68.1 Maple step by step solution 628

Internal problem ID [7558]
Internal file name [OUTPUT/6491_Sunday_June_05_2022_04_55_12_PM_50245545/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 70.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(3x+1)\,y''+xy'+2y=0}$$

Writing the ode as

$$(3x+1)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x+1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 24x - 6}{4\,(3x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 24x - 6 \\ t &= 4(3x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 24x - 6}{4\,(3x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 122: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x+1)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x+\frac{1}{3}\right)^2} - \frac{37}{54\left(x+\frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{18} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{18}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\
&= Q + \frac{R}{36x^2 + 24x + 4} \\
&= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\
&= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{37}{54}\right) - (0) \\
&= -\frac{37}{54}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= -\frac{37}{18} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= \frac{37}{18}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 24x - 6}{4(3x+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{37}{18}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{37}{18} - \left(\frac{19}{18}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{x-6}{2\left(3x+1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)(1) + \left(\left(-\frac{19}{18\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)^2 - \left(\frac{x^2 - 24x - 6}{4\left(3x+1\right)^2}\right)\right) = 0
$$

$$
\frac{a_0 + 6}{3x + 1} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 6
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-6)\,\mathrm{e}^{\int \left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right) dx} \\
&= (x-6)\,\mathrm{e}^{-\frac{x}{6} + \frac{19\ln(3x+1)}{18}} \\
&= (x-6)\left(3x+1\right)^{\frac{19}{18}} \mathrm{e}^{-\frac{x}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{3x+1}\,dx} \\
&= z_1 e^{-\frac{x}{6}+\frac{\ln(3x+1)}{18}} \\
&= z_1\left((3x+1)^{\frac{1}{18}}\,\mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-6)\,(3x+1)^{\frac{10}{9}}\,\mathrm{e}^{-\frac{x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}+\frac{\ln(3x+1)}{9}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-6)\,(3x+1)^{\frac{10}{9}}\,\mathrm{e}^{-\frac{x}{3}}\right) + c_2\left((x-6)\,(3x+1)^{\frac{10}{9}}\,\mathrm{e}^{-\frac{x}{3}}\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1(x-6)\,(3x+1)^{\frac{10}{9}}\,\mathrm{e}^{-\frac{x}{3}} + c_2(x-6)\,(3x+1)^{\frac{10}{9}}\,\mathrm{e}^{-\frac{x}{3}}\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1(x-6)(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}} + c_2(x-6)(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}} \left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2 (3x+1)^{\frac{19}{9}}} dx\right)$$

Verified OK.

2.68.1 Maple step by step solution

Let's solve

$(3x+1)\, y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{3x+1} - \frac{xy'}{3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{3x+1} + \frac{2y}{3x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x}{3x+1}, P_3(x) = \frac{2}{3x+1}\right]$

  ◦ $\left(x+\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right) \cdot P_2(x)\right)\Big|_{x=-\frac{1}{3}} = -\frac{1}{9}$

  ◦ $\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)^2 \cdot P_3(x)\right)\Big|_{x=-\frac{1}{3}} = 0$

  ◦ $x = -\frac{1}{3}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

  $x_0 = -\frac{1}{3}$

- Multiply by denominators

  $(3x+1)\, y'' + xy' + 2y = 0$

- Change variables using $x = u - \frac{1}{3}$ so that the regular singular point is at $u = 0$

$3u\left(\frac{d^2}{du^2}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

  - Shift index using $k->k+1$

    $u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{a_0 r(-10+9r)u^{-1+r}}{3}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(9k-1+9r)}{3}+a_k(k+r+2)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{r(-10+9r)}{3}=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{10}{9}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+1+r)\left(k-\frac{1}{9}+r\right)a_{k+1}+a_k(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k(k+r+2)}{(k+1+r)(9k-1+9r)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{3a_k(k+2)}{(k+1)(9k-1)}$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+1}=-\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$$

- Revert the change of variables $u=x+\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{1}{3}\right)^k, a_{k+1}=-\frac{3a_k(k+2)}{(k+1)(9k-1)}\right]$$

- Recursion relation for $r=\frac{10}{9}$

$$a_{k+1}=-\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}$$

- Solution for $r=\frac{10}{9}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{10}{9}}, a_{k+1}=-\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$$

- Revert the change of variables $u=x+\frac{1}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1}=-\frac{3a_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x+\frac{1}{3}\right)^k\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+\frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1}=-\frac{3a_k(k+2)}{(k+1)(9k-1)}, b_{k+1}=-\frac{3b_k\left(k+\frac{28}{9}\right)}{\left(k+\frac{19}{9}\right)(9k+9)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve((1+3*x)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}(x-6) + c_2(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}(x-6)\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2 (3x+1)^{\frac{19}{9}}}dx\right)$$

✓ Solution by Mathematica

Time used: 3.176 (sec). Leaf size: 124

```
DSolve[(1+3*x)*y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{x}{3}-\frac{1}{9}}\left(1520c_1\sqrt[9]{3x+1}(3x^2-17x-6)-2^{8/9}c_2e^{\frac{x}{3}+\frac{1}{9}}(9x^2-48x-26)+2^{8/9}3^{7/9}c_2\sqrt[9]{-3x-1}(3x^2-17}{380\ 2^{17/18}}$$

2.69 problem 71

2.69.1 Maple step by step solution 639

Internal problem ID [7559]
Internal file name [`OUTPUT/6492_Sunday_June_05_2022_04_55_17_PM_24050482/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 71.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0}$$

Writing the ode as

$$(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2+x+1 \\ B &= 2+15x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^2 - 12x - 18$$
$$t = 4\left(3x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 124: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = \frac{i\sqrt{11}}{6} - \frac{1}{6}$ of order 2. There is a pole at $x = -\frac{i\sqrt{11}}{6} - \frac{1}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{3i\sqrt{11}}{88} + \frac{27}{88}}{\left(-\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)^2} + \frac{-\frac{3i\sqrt{11}}{88} + \frac{27}{88}}{\left(\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(-\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)} - \frac{57i\sqrt{11}}{242\left(\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)}$$

For the pole at $x = \frac{i\sqrt{11}}{6} - \frac{1}{6}$ let b be the coefficient of $\frac{1}{\left(-\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3i\sqrt{11}}{88} + \frac{27}{88}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{11}}{6} - \frac{1}{6}$ let b be the coefficient of $\frac{1}{\left(\frac{i\sqrt{11}}{6} + x + \frac{1}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3i\sqrt{11}}{88} + \frac{27}{88}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 - 66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 - 66i\sqrt{11}}}{44}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{11}}{6}-\frac{1}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{i\sqrt{11}}{6}-\frac{1}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{-\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{-\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} \\
&= -\frac{3x}{6x^2+2x+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{-\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\frac{i\sqrt{11}}{6}+x+\frac{1}{6}}\right)(1) + \left(\left(-\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(-\frac{i\sqrt{11}}{6}+x+\frac{1}{6}\right)^2} - \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(\frac{i\sqrt{11}}{6}+x+\frac{1}{6}\right)^2}\right) + \left(\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}}{-\frac{i\sqrt{11}}{6}+x+\frac{1}{6}} + \frac{\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\frac{i\sqrt{11}}{6}+x+\frac{1}{6}}\right) dx} \\
&= (x)\, e^{\frac{\ln\left(36x^2+12x+12\right)}{2} - \frac{\sqrt{1078-66i\sqrt{11}}\,\ln\left(36x^2+12x+12\right)}{88} + \frac{i\sqrt{1078-66i\sqrt{11}}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{44} - \frac{\sqrt{1078+66i\sqrt{11}}\,\ln\left(36x^2+12x+12\right)}{88} - \frac{i\sqrt{1\cdots}}{}} \\
&= \frac{x\sqrt{2}\, 3^{\frac{3}{4}} e^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}}}{6\left(3x^2+x+1\right)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+15x}{3x^2+x+1}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(3x^2+x+1\right)}{4}+\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}}}{\left(3x^2+x+1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{2}\,3^{\frac{3}{4}}\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{2\sqrt{3}\,\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{2}\,3^{\frac{3}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\,(3x^2+x+1)^{\frac{3}{2}}}\right) \\
&\quad + c_2 \left(\frac{x\sqrt{2}\,3^{\frac{3}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\,(3x^2+x+1)^{\frac{3}{2}}} \left(\int \frac{2\sqrt{3}\,\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1 x\sqrt{2}\,3^{\frac{3}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\,(3x^2+x+1)^{\frac{3}{2}}} \\
& + \frac{c_2 x\sqrt{2}\,3^{\frac{1}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}} \left(\int \frac{\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2} dx\right)}{(3x^2+x+1)^{\frac{3}{2}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1 x\sqrt{2}\,3^{\frac{3}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\,(3x^2+x+1)^{\frac{3}{2}}} \\
& + \frac{c_2 x\sqrt{2}\,3^{\frac{1}{4}} \mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}} \left(\int \frac{\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2} dx\right)}{(3x^2+x+1)^{\frac{3}{2}}}
\end{aligned}
$$

Verified OK.

2.69.1 Maple step by step solution

Let's solve

$(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{12y}{3x^2+x+1} - \frac{(2+15x)y'}{3x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2+15x)y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

- ○ $\left(\frac{\mathrm{I}\sqrt{11}}{6} + x + \frac{1}{6}\right) \cdot P_2(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(\frac{\mathrm{I}\sqrt{11}}{6} + x + \frac{1}{6}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ○ $\left(\frac{\mathrm{I}\sqrt{11}}{6} + x + \frac{1}{6}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(\frac{\mathrm{I}\sqrt{11}}{6} + x + \frac{1}{6}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ○ $x = -\frac{\mathrm{I}\sqrt{11}}{6} - \frac{1}{6}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{\mathrm{I}\sqrt{11}}{6} - \frac{1}{6}$

- Multiply by denominators

  $(3x^2 + x + 1)\, y'' + (2 + 15x)\, y' + 12y = 0$

- Change variables using $x = u - \frac{\mathrm{I}\sqrt{11}}{6} - \frac{1}{6}$ so that the regular singular point is at $u = 0$

  $(3u^2 - \mathrm{I}u\sqrt{11}) \left(\frac{d^2}{du^2} y(u)\right) + \left(-\frac{1}{2} + 15u - \frac{5\,\mathrm{I}\sqrt{11}}{2}\right) \left(\frac{d}{du} y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k + r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{\mathrm{I}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right)a_0u^{-1+r}}{22} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\,(k+1+r)\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)a_{k+1}}{22} + 3a_k(k+r+2)^2\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{\mathrm{I}}{22}\sqrt{11}\,r\left(\mathrm{I}\sqrt{11}-33-22r\right) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3a_k(k+r+2)^2 - a_{k+1}(k+1+r)\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Recursion relation for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+\frac{111\,\mathrm{I}\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\mathrm{I}\sqrt{11}}{11}\right)}{2\mathrm{I}\sqrt{11}\,k^2+4\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\mathrm{I}k\sqrt{11}+7\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}\right.$$

- Revert the change of variables $u=\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1}=\frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2\right.}{2\mathrm{I}\sqrt{11}\,k^2+4\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\mathrm{I}k\sqrt{11}+7\mathrm{I}\left(\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(\frac{\mathrm{I}\sqrt{11}}{6}+x+\frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}\right), a_{k+1}=\frac{6a_k\left(k^2+4k+4\right)}{2\mathrm{I}\sqrt{11}\,k^2+1+7\mathrm{I}k\sqrt{11}+5\mathrm{I}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 143

```
dsolve((1+x+3*x^2)*diff(y(x),x$2)+(2+15*x)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \left(\frac{i\sqrt{11}-6x-1}{i\sqrt{11}+6x+1}\right)^{-\frac{i\sqrt{11}}{22}} x}{\left(3x^2+x+1\right)^{\frac{3}{2}}} + \frac{c_2 \left(\frac{i\sqrt{11}-6x-1}{i\sqrt{11}+6x+1}\right)^{-\frac{i\sqrt{11}}{22}} x \left(\int \frac{\sqrt{3x^2+x+1}\left(\frac{i\sqrt{11}+6x+1}{i\sqrt{11}-6x-1}\right)^{-\frac{i\sqrt{11}}{22}}}{x^2} dx\right)}{\left(3x^2+x+1\right)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 3.347 (sec). Leaf size: 93

```
DSolve[(1+x+3*x^2)*y''[x]+(2+15*x)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x e^{\frac{\arctan\left(\frac{6x+1}{\sqrt{11}}\right)}{\sqrt{11}}} \left(c_2 \int_1^x \frac{e^{-\frac{\arctan\left(\frac{6K[1]+1}{\sqrt{11}}\right)}{\sqrt{11}}} \sqrt{3K[1]^2+K[1]+1}}{K[1]^2} dK[1] + c_1\right)}{(3x^2+x+1)^{3/2}}$$

2.70 problem 72

2.70.1 Maple step by step solution 652

Internal problem ID [7560]
Internal file name [`OUTPUT/6493_Sunday_June_05_2022_04_55_23_PM_24509136/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 72.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+2)\, y''+(1+x)\, y'+3y=0}$$

Writing the ode as

$$(x+2)\, y''+(1+x)\, y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= 1+x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 21}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x - 21$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 21}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 126: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{7}{2(x+2)} + \frac{3}{4(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-10x-21}{4x^2+16x+16} \\ &= Q + \frac{R}{4x^2+16x+16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-14x-25}{4x^2+16x+16}\right) \\ &= \frac{1}{4} + \frac{-14x-25}{4x^2+16x+16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 21}{4\left(x+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2\,(x+2)} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2\,(x+2)} - \frac{1}{2} \\
&= -\frac{x-1}{2\,(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2\,(x+2)} - \frac{1}{2}\right)(2x+a_1) + \left(\left(-\frac{3}{2\,(x+2)^2}\right) + \left(\frac{3}{2\,(x+2)} - \frac{1}{2}\right)^2 - \left(\frac{x^2-10x-21}{4\,(x+2)^2}\right)\right) = 0$$

$$\frac{(a_1+4)\,x + 2a_0 + a_1 + 4}{x+2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 - 4x\right) \mathrm{e}^{\int \left(\frac{3}{2(x+2)} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x\right) \mathrm{e}^{-\frac{x}{2} + \frac{3\ln(x+2)}{2}} \\
&= x(x-4)\,(x+2)^{\frac{3}{2}}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{\ln(x+2)}{2}} \\
&= z_1 \left(\sqrt{x+2}\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(x-4)\,(x+2)^2\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-x(x-4)\,(x+2)^2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^3-x^2-10x-6)}{48x\,(x-4)\,(x+2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x(x-4)\,(x+2)^2\,\mathrm{e}^{-x}\right) + c_2\Bigg(x(x-4)\,(x \\
&\quad +2)^2\,\mathrm{e}^{-x}\left(\frac{-x(x-4)\,(x+2)^2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^3-x^2-10x-6)}{48x\,(x-4)\,(x+2)^2}\right)\Bigg)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x(x-4)(x+2)^2 e^{-x} + c_2 \left(-\frac{x(x-4)(x+2)^2 e^{-x-2} \operatorname{expIntegral}_1(-x-2)}{48} - \frac{x^3}{48} + \frac{x^2}{48} + \frac{5x}{24} + \frac{1}{8}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(x-4)(x+2)^2 e^{-x} + c_2 \left(-\frac{x(x-4)(x+2)^2 e^{-x-2} \operatorname{expIntegral}_1(-x-2)}{48} - \frac{x^3}{48} + \frac{x^2}{48} + \frac{5x}{24} + \frac{1}{8}\right)$$

Verified OK.

2.70.1 Maple step by step solution

Let's solve

$(x+2)\, y'' + (1+x)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x+2} - \frac{(1+x)y'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x+2} + \frac{3y}{x+2} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{x+2}, P_3(x) = \frac{3}{x+2}\right]$

  ◦ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = -1$

  ◦ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

  ◦ $x = -2$is a regular singular point

Check to see if $x_0 = -2$ is a regular singular point

$x_0 = -2$

- Multiply by denominators

$(x+2)\, y'' + (1+x)\, y' + 3y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-1+u)\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

  - Shift index using $k->k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+2)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve((2+x)*diff(y(x),x$2)+(1+x)*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x(x^3 - 12x - 16) - \frac{c_2(e^{-2}\,\mathrm{expIntegral}_1(-2-x)\,x^4 + e^x x^3 - 12\,e^{-2}\,\mathrm{expIntegral}_1(-2-x)\,x^2 - e^x x^2 - 16\,e^{-2}\,\mathrm{expIntegral}}{48}$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 99

```
DSolve[(2+x)*y''[x]+(1+x)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-1}(c_2(x-4)(x+2)^2 x\,\mathrm{ExpIntegralEi}(x+2) + 384c_1x^4 - c_2e^{x+2}x^3 + x^2(c_2e^{x+2} - 4608c_1) + x(10c_2e}{96\sqrt{2}}$$

2.71 problem 73

2.71.1 Maple step by step solution 663

Internal problem ID [7561]
Internal file name [`OUTPUT/6494_Sunday_June_05_2022_04_55_26_PM_43237432/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 73.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4+x)\,y''+(x+2)\,y'+2y=0}$$

Writing the ode as

$$(4+x)\,y''+(x+2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4+x \\ B &= x+2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 24}{4\left(4+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x - 24$$
$$t = 4(4+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x - 24}{4\left(4+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 128: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{4+x} + \frac{2}{(4+x)^2}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\ &= Q + \frac{R}{4x^2 + 32x + 64} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\ &= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &&= 3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x - 24}{4\left(4 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{4+x} - \frac{1}{2} \\
&= -\frac{x}{2\,(4+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{(4+x)^2}\right) + \left(\frac{2}{4+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-4x-24}{4\,(4+x)^2}\right)\right) = 0
$$

$$
\frac{a_0}{4+x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{2} + 2\ln(4+x)} \\
&= x(4+x)^2\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+2}{4+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(4+x)} \\
&= z_1\left((4+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(4+x)^3\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+2}{4+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(4+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-x(4+x)^3\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1(-x-4)-\mathrm{e}^{x}(x^3+9x^2+22x+6)}{24x\,(4+x)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x(4+x)^3\,\mathrm{e}^{-x}\right) + c_2\left(x(4 \right. \\
&\quad \left. +x)^3\,\mathrm{e}^{-x}\left(\frac{-x(4+x)^3\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1(-x-4)-\mathrm{e}^{x}(x^3+9x^2+22x+6)}{24x\,(4+x)^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x(4+x)^3 \mathrm{e}^{-x} + c_2 \left(-\frac{x(4+x)^3 \mathrm{e}^{-x-4} \operatorname{expIntegral}_1(-x-4)}{24} - \frac{x^3}{24} - \frac{3x^2}{8} - \frac{11x}{12} - \frac{1}{4}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(4+x)^3 \mathrm{e}^{-x} + c_2 \left(-\frac{x(4+x)^3 \mathrm{e}^{-x-4} \operatorname{expIntegral}_1(-x-4)}{24} - \frac{x^3}{24} - \frac{3x^2}{8} - \frac{11x}{12} - \frac{1}{4}\right)$$

Verified OK.

2.71.1 Maple step by step solution

Let's solve

$(4+x)\, y'' + (x+2)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{4+x} - \frac{(x+2)y'}{4+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+2)y'}{4+x} + \frac{2y}{4+x} = 0$

□ Check to see if $x_0 = -4$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x+2}{4+x}, P_3(x) = \frac{2}{4+x}\right]$

  ◦ $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

    $\left((4+x) \cdot P_2(x)\right)\Big|_{x=-4} = -2$

  ◦ $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

    $\left((4+x)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

  ◦ $x = -4$is a regular singular point

    Check to see if $x_0 = -4$ is a regular singular point

$x_0 = -4$

- Multiply by denominators

  $(4+x)\, y'' + (x+2)\, y' + 2y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (u-2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$

  - Shift index using $k- >k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = 4 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (4+x)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 108

```
dsolve((4+x)*diff(y(x),x$2)+(2+x)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x\left(x^3 + 12x^2 + 48x + 64\right) - \frac{c_2(e^{-4}\,\mathrm{expIntegral}_1\,(-x-4)\,x^4 + 12\,e^{-4}\,\mathrm{expIntegral}_1\,(-x-4)\,x^3 + e^x x^3 + 48\,e^{-4}\,\mathrm{expIntegral}_1\,(-x -}{24}$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 97

```
DSolve[(4+x)*y''[x]+(2+x)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24} e^{-x-4}\left(c_2 x(x+4)^3 \operatorname{ExpIntegralEi}(x+4) + e^4\left(24c_1x^4 + x^3(288c_1 - c_2e^x) + 9x^2(128c_1 - c_2e^x) + 2x(768c_1 - 11c_2e^x) - 6c_2e^x\right)\right)$$

2.72 problem 74

2.72.1 Maple step by step solution 673

Internal problem ID [7562]
Internal file name [OUTPUT/6495_Sunday_June_05_2022_04_55_29_PM_26378490/index.tex]

Book: Collection of Kovacic problems
Section: section 1
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ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+3x)\,y''+10(1+x)\,y'+8y=0}$$

Writing the ode as

$$(2x^2+3x)\,y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+3x \\ B &= 10x+10 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6x + 10$$
$$t = (2x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 130: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{22}{27x} + \frac{10}{9x^2} - \frac{5}{36\left(x+\frac{3}{2}\right)^2} + \frac{22}{27\left(x+\frac{3}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3}
\end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6x+10}{(2x^2+3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2+6x+10}{(2x^2+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{3x} + \frac{1}{6x+9} + (-)\,(0) \\
&= -\frac{2}{3x} + \frac{1}{6x+9} \\
&= -\frac{x+2}{x\,(3+2x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)\right) = 0
$$

$$
\frac{-4+2a_0}{x\,(3+2x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\
&= (x+2)\,\mathrm{e}^{\frac{\ln(3+2x)}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{(x+2)\,(3+2x)^{\frac{1}{6}}}{x^{\frac{2}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{3}-\frac{5\ln(3+2x)}{6}} \\
&= z_1\left(\frac{1}{x^{\frac{5}{3}}\,(3+2x)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{10\ln(x)}{3}-\frac{5\ln(3+2x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}\,(x+2)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}\right) + c_2\left(\frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}\,(x+2)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x+2)}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}} + \frac{c_2(x+2)\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}(x+2)^2}dx\right)}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x+2)}{(3+2x)^{\frac{2}{3}} x^{\frac{7}{3}}} + \frac{c_2(x+2)\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}(x+2)^2}dx\right)}{(3+2x)^{\frac{2}{3}} x^{\frac{7}{3}}}$$

Verified OK.

2.72.1 Maple step by step solution

Let's solve

$(2x^2+3x)\,y''+(10x+10)\,y'+8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{x(3+2x)}-\frac{10(1+x)y'}{x(3+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{10(1+x)y'}{x(3+2x)}+\frac{8y}{x(3+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{10(1+x)}{x(3+2x)}, P_3(x)=\frac{8}{x(3+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{10}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(3+2x)+(10x+10)\,y'+8y=0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(7+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k (k+r+2)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(7+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{7}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k (k+r+2)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k (k+r+2)^2}{(k+1+r)(3k+10+3r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k (k+2)^2}{(k+1)(3k+10)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k (k+2)^2}{(k+1)(3k+10)}\right]$

- Recursion relation for $r = -\frac{7}{3}$

$$a_{k+1} = -\frac{2a_k \left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = -\frac{7}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k \left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k (k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k \left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((3*x+2*x^2)*diff(y(x),x$2)+10*(1+x)*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+2)}{x^{\frac{7}{3}}(2x+3)^{\frac{2}{3}}} + \frac{c_2(x+2)\left(\int \frac{x^{\frac{4}{3}}}{(x+2)^2(2x+3)^{\frac{1}{3}}}dx\right)}{x^{\frac{7}{3}}(2x+3)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.887 (sec). Leaf size: 245

```
DSolve[(3*x+2*x^2)*y''[x]+10*(1+x)*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{2\ 2^{2/3}\sqrt{3}c_2(x+2)\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{\sqrt[3]{x}+2^{2/3}\sqrt[3]{2x+3}}\right) + 2^{2/3}c_2x\log\left(2x^{2/3}+2^{2/3}\sqrt[3]{2x+3}\sqrt[3]{x}+\sqrt[3]{2}(2x+3)^{2/3}\right.}{}$$

2.73 problem 75

Internal problem ID [7563]
Internal file name [`OUTPUT/6496_Sunday_June_05_2022_04_55_32_PM_28701391/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 75.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - (6-7x)\,y' + 8y = 0}$$

Writing the ode as

$$x^2y'' + (-6+7x)\,y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6+7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 60x + 36 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 132: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{15}{x^3} + \frac{9}{x^4} + \frac{3}{4x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$\begin{aligned} b &= (-15) - (0) \\ &= -15 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{3}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-15}{3} + 2\right) &&= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-15}{3} + 2\right) &&= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 60x + 36}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{x^2} - \frac{3}{2x} + (-)(0) \\ &= \frac{3}{x^2} - \frac{3}{2x} \\ &= -\frac{3(x-2)}{2x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{x^2} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{6}{x^3} + \frac{3}{2x^2}\right) + \left(\frac{3}{x^2} - \frac{3}{2x}\right)^2 - \left(\frac{3x^2 - 60x + 36}{4x^4}\right)\right) = 0$$

$$\frac{6 + 3a_0}{x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x-2)\, e^{\int \left(\frac{3}{x^2} - \frac{3}{2x}\right) dx} \\ &= (x-2)\, e^{-\frac{3}{x} - \frac{3\ln(x)}{2}} \\ &= \frac{(x-2)\, e^{-\frac{3}{x}}}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6+7x}{x^2} \, dx} \\ &= z_1 e^{-\frac{3}{x} - \frac{7\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{3}{x}}}{x^{\frac{7}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-2)\, e^{-\frac{6}{x}}}{x^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{6}{x} - 7\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(108x - 216)\, \mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + e^{\frac{6}{x}} x(x^2 + 12x - 36)}{2x - 4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5}\right) \\
&\quad + c_2 \left(\frac{(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} \left(\frac{(108x-216)\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + \mathrm{e}^{\frac{6}{x}} x(x^2+12x-36)}{2x-4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} + \frac{c_2\left(108(x-2)\,\mathrm{e}^{-\frac{6}{x}}\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + x^3 + 12x^2 - 36x\right)}{2x^5} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} + \frac{c_2\left(108(x-2)\,\mathrm{e}^{-\frac{6}{x}}\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + x^3 + 12x^2 - 36x\right)}{2x^5}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 81

```
dsolve(x^2*diff(y(x),x$2)-(6-7*x)*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{6}{x}}(x-2)}{x^5} + \frac{c_2\left(x^3\mathrm{e}^{\frac{6}{x}} + 12x^2\mathrm{e}^{\frac{6}{x}} + 108 \operatorname{expIntegral}_1\left(-\frac{6}{x}\right)x - 36x\,\mathrm{e}^{\frac{6}{x}} - 216 \operatorname{expIntegral}_1\left(-\frac{6}{x}\right)\right)\mathrm{e}^{-\frac{6}{x}}}{2x^5}$$

✓ Solution by Mathematica

Time used: 0.179 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]-(6-7*x)*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6/x}\left(-108c_2(x-2)\operatorname{ExpIntegralEi}\left(\frac{6}{x}\right) + c_2e^{6/x}x(x^2+12x-36) + 2c_1(x-2)\right)}{2x^5}$$

2.74 problem 76

2.74.1 Maple step by step solution 691

Internal problem ID [7564]
Internal file name [`OUTPUT/6497_Sunday_June_05_2022_04_55_35_PM_34154560/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 76.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2 + x + 1) y'' + (1 + 7x) y' + 2y = 0}$$

Writing the ode as

$$(2x^2 + x + 1) y'' + (1 + 7x) y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 + x + 1 \\ B &= 1 + 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 2x + 5$$
$$t = 4\left(2x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+x+1)^2$. There is a pole at $x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ of order 2. There is a pole at $x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{9i\sqrt{7}}{224} - \frac{29}{224}}{\left(-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)^2} + \frac{-\frac{9i\sqrt{7}}{224} - \frac{29}{224}}{\left(\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)} + \frac{8i\sqrt{7}}{49\left(\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)}$$

For the pole at $x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(-\frac{i\sqrt{7}}{4}+x+\frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9i\sqrt{7}}{224} - \frac{29}{224}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42+14i\sqrt{7}}}{56} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(\frac{i\sqrt{7}}{4}+x+\frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{9i\sqrt{7}}{224} - \frac{29}{224}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42-14i\sqrt{7}}}{56} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 2x + 5}{4\left(2x^2+x+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{i\sqrt{7}}{4}-\frac{1}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{i\sqrt{7}}{4}-\frac{1}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{4}$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\ &= \frac{5}{4}-\left(\frac{1}{4}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} + (0) \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} \\
&= \frac{1 + x}{4x^2 + 2x + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\frac{i\sqrt{7}}{4} + x + \frac{1}{4}}\right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)^2} - \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(\frac{i\sqrt{7}}{4} + x + \frac{1}{4}\right)^2}\right) + \left(\frac{\frac{1}{2} - \cdots}{-\frac{i\sqrt{\cdots}}{4}\cdots}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{-\frac{i\sqrt{7}}{4} + x + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\frac{i\sqrt{7}}{4} + x + \frac{1}{4}}\right) dx} \\
&= (1 + x)\, e^{\frac{\ln(16x^2+8x+8)}{2} - \frac{3\sqrt{42-14i\sqrt{7}}\, \ln(16x^2+8x+8)}{112} + \frac{3i\sqrt{42-14i\sqrt{7}}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{56} - \frac{3\sqrt{42+14i\sqrt{7}}\, \ln(16x^2+8x+8)}{112} - \frac{3i\sqrt{42+14i\sqrt{\cdots}}}{\cdots}} \\
&= 2^{\frac{3}{8}} (1 + x) \left(2x^2 + x + 1\right)^{\frac{1}{8}} e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+x+1\right)}{8}+\frac{3\sqrt{7}\,\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}}}{\left(2x^2+x+1\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{\frac{3}{8}}(1+x)\,\mathrm{e}^{\frac{3\sqrt{7}\,\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{\left(2x^2+x+1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+x+1\right)}{4}+\frac{3\sqrt{7}\,\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{1}{4}}\mathrm{e}^{-\frac{3\sqrt{7}\,\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{2(1+x)^2\left(2x^2+x+1\right)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{\frac{3}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}}\right) \\
&\quad + c_2 \left(\frac{2^{\frac{3}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}}\left(\int \frac{2^{\frac{1}{4}}\mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{2\,(1+x)^2\,(2x^2+x+1)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1 2^{\frac{3}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \\
& + \frac{c_2 2^{\frac{5}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}\left(\int \frac{\mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(1+x)^2(2x^2+x+1)^{\frac{1}{4}}}dx\right)}{2\,(2x^2+x+1)^{\frac{3}{4}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1 2^{\frac{3}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \\
& + \frac{c_2 2^{\frac{5}{8}}(1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}\left(\int \frac{\mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(1+x)^2(2x^2+x+1)^{\frac{1}{4}}}dx\right)}{2\,(2x^2+x+1)^{\frac{3}{4}}}
\end{aligned}
$$

Verified OK.

2.74.1 Maple step by step solution

Let's solve

$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y}{2x^2+x+1} - \frac{(1+7x)y'}{2x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(1+7x)y'}{2x^2+x+1} + \frac{2y}{2x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

- ○ $\left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

$\left(\left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}} = 0$

- ○ $\left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

$\left(\left(\frac{\mathrm{I}\sqrt{7}}{4} + x + \frac{1}{4}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}} = 0$

- ○ $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

- Multiply by denominators

$(2x^2 + x + 1)\, y'' + (1 + 7x)\, y' + 2y = 0$

- Change variables using $x = u - \frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$ so that the regular singular point is at $u = 0$

$(2u^2 - \mathrm{I}u\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right) + \left(-\frac{3}{4} + 7u - \frac{7\,\mathrm{I}\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$\frac{\mathrm{I}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)r a_0 u^{-1+r}}{28} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-28k-49-28r\right)(k+1+r)a_{k+1}}{28} + a_k(k+r+2)(2k+2r+1)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$\frac{\mathrm{I}}{28}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-\mathrm{I}(k+1+r)\left(k+r+\frac{7}{4}\right)a_{k+1}\sqrt{7} + \frac{(-3k-3r-3)a_{k+1}}{4} + 2\left(k+r+\frac{1}{2}\right)a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{4a_k\left(2k^2+4kr+2r^2+5k+5r+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,kr+4\,\mathrm{I}\sqrt{7}\,r^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\,r+7\,\mathrm{I}\sqrt{7}+3k+3r}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Revert the change of variables $u = \frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}$

  $$\left[y = \sum_{k=0}^{\infty} a_k\left(\frac{\mathrm{I}\sqrt{7}}{4}+x+\frac{1}{4}\right)^k, a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

  $$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,\mathrm{I}\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3 I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15 I\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4 I\sqrt{7}\, k^2+8 I\sqrt{7}\, k\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)+4 I\sqrt{7}\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11 I\sqrt{7}\, k+11 I\sqrt{7}\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)+$$

- Revert the change of variables $u = \frac{I\sqrt{7}}{4} + x + \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3 I\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15 I\sqrt{7}}{28}\right.}{\frac{3}{4}+4 I\sqrt{7}\, k^2+8 I\sqrt{7}\, k\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)+4 I\sqrt{7}\left(\frac{3 I\sqrt{7}}{28}-\frac{3}{4}\right)^2+11 I\sqrt{7}\, k+11 I\sqrt{7}}$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(\frac{I\sqrt{7}}{4} + x + \frac{1}{4}\right)^{k+\frac{3 I\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4 I\sqrt{7}\, k^2+11 I\sqrt{7}\, k+7 I\sqrt{7}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 149

```
dsolve((1+x+2*x^2)*diff(y(x),x$2)+(1+7*x)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \left(\frac{i\sqrt{7}-4x-1}{i\sqrt{7}+4x+1}\right)^{-\frac{3i\sqrt{7}}{28}} (x+1)}{(2x^2+x+1)^{\frac{3}{4}}} + \frac{c_2 \left(\frac{i\sqrt{7}-4x-1}{i\sqrt{7}+4x+1}\right)^{-\frac{3i\sqrt{7}}{28}} (x+1) \left(\int \frac{\left(\frac{i\sqrt{7}+4x+1}{i\sqrt{7}-4x-1}\right)^{-\frac{3i\sqrt{7}}{28}}}{(x+1)^2 (2x^2+x+1)^{\frac{1}{4}}} dx\right)}{(2x^2+x+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 2.389 (sec). Leaf size: 102

```
DSolve[(1+x+2*x^2)*y''[x]+(1+7*x)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x+1)e^{\frac{3\arctan\left(\frac{4x+1}{\sqrt{7}}\right)}{2\sqrt{7}}}\left(c_2\int_1^x \frac{e^{-\frac{3\arctan\left(\frac{4K[1]+1}{\sqrt{7}}\right)}{2\sqrt{7}}}}{(K[1]+1)^2\sqrt[4]{2K[1]^2+K[1]+1}}dK[1]+c_1\right)}{(2x^2+x+1)^{3/4}}$$

2.75 problem 77

2.75.1 Maple step by step solution 702

Internal problem ID [7565]
Internal file name [OUTPUT/6498_Sunday_June_05_2022_04_55_41_PM_58819639/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 77.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+3)\,y''+(2x+1)\,y'-(-x+2)\,y=0}$$

Writing the ode as

$$
\begin{align*}
(x+3)\,y''+(2x+1)\,y'+(x-2)\,y &= 0 \tag{1}\\
Ay''+By'+Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x+3 \\
B &= 2x+1 \tag{3}\\
C &= x-2
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4(x+3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35$$
$$t = 4(x+3)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4(x+3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+3)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x+3)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4(x+3)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4(x+3)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2(x+3)} + (-)(0) \\
&= -\frac{5}{2(x+3)} \\
&= -\frac{5}{2(x+3)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2(x+3)}\right)(0) + \left(\left(\frac{5}{2(x+3)^2}\right) + \left(-\frac{5}{2(x+3)}\right)^2 - \left(\frac{35}{4(x+3)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int -\frac{5}{2(x+3)} dx} \\ &= \frac{1}{(x+3)^{\frac{5}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x+1}{x+3}\, dx} \\ &= z_1 e^{-x+\frac{5\ln(x+3)}{2}} \\ &= z_1\left((x+3)^{\frac{5}{2}}\, e^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x+3}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x+5\ln(x+3)}}{(y_1)^2}\, dx \\ &= y_1\left(\frac{x(x+6)(x^2+9x+27)(x^2+3x+9)}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{x(x+6)\,(x^2+9x+27)\,(x^2+3x+9)}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{-x}x(x+6)\,(x^2+9x+27)\,(x^2+3x+9)}{6} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{-x}x(x+6)\,(x^2+9x+27)\,(x^2+3x+9)}{6}
$$

Verified OK.

2.75.1 Maple step by step solution

Let's solve

$(x+3)\,y'' + (2x+1)\,y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-2)y}{x+3} - \frac{(2x+1)y'}{x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x+3} + \frac{(x-2)y}{x+3} = 0$

□ Check to see if $x_0 = -3$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{x+3}, P_3(x) = \frac{x-2}{x+3}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x = -3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = -5$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x = -3$

$$\left((x+3)^2 \cdot P_3(x)\right)\Bigg|_{x=-3} = 0$$

- $x = -3$is a regular singular point

  Check to see if $x_0 = -3$ is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $(x+3)\,y'' + (2x+1)\,y' + (x-2)\,y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (2u-5)\left(\frac{d}{du}y(u)\right) + (u-5)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- > k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-6+r)\,u^{-1+r} + (a_1(1+r)\,(-5+r) + a_0(-5+2r))\,u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-6+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 6\}$$

- Each term must be 0

  $$a_1(1+r)\,(-5+r) + a_0(-5+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+2+r)\,(k-4+r) + 2a_{k+1}(k+1) + 2ra_{k+1} - 5a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$$

- Recursion relation for $r=0$

  $$a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=4$

  $$a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$$

- Recursion relation for $r=6$

  $$a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$$

- Solution for $r=6$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve((3+x)*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)-(2-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{-x}\left(x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x\right)$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 29

```
DSolve[(3+x)*y''[x]+(1+2*x)*y'[x]-(2-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-x-3}\left(c_2(x+3)^6 + 6c_1\right)$$

2.76 problem 78

2.76.1 Maple step by step solution 712

Internal problem ID [7566]
Internal file name [`OUTPUT/6499_Sunday_June_05_2022_04_55_44_PM_71325356/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 78.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 3xy' + (2x^2 + 4) y = 0}$$

Writing the ode as

$$y'' + 3xy' + (2x^2 + 4) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{1} \, dx} \\
&= z_1 e^{-\frac{3x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-1\right) \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2-1\right) \mathrm{e}^{-x^2}\right) + c_2 \left(\left(x^2-1\right) \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-1)\,\mathrm{e}^{-x^2} + c_2(x^2-1)\,\mathrm{e}^{-x^2}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1)\,\mathrm{e}^{-x^2} + c_2(x^2-1)\,\mathrm{e}^{-x^2}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.76.1 Maple step by step solution

Let's solve

$y'' + 3xy' + (2x^2+4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + 4a_0 + (6a_3 + 7a_1)x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(3k+4) + 2a_{k-2}\right)x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right)a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2} + 2a_k + 10a_{k+2}}{k^2 + 7k + 12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+3*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2}(x^2-1) + c_2 \mathrm{e}^{-x^2}(x^2-1)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x-1)^2 (x+1)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.427 (sec). Leaf size: 63

```
DSolve[y''[x]+3*x*y'[x]+(4+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x^2}\left(\sqrt{2\pi}c_2(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) + 4c_1(x^2-1) - 2c_2 e^{\frac{x^2}{2}}x\right)$$

2.77 problem 79

2.77.1 Maple step by step solution 721

Internal problem ID [7567]
Internal file name [`OUTPUT/6500_Sunday_June_05_2022_04_55_47_PM_98323513/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 79.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4x+2)\,y''-4y'-(6+4x)\,y=0}$$

Writing the ode as

$$(4x+2)\,y''-4y'+(-4x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x+2 \\ B &= -4 \\ C &= -4x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 8x + 6 \\ t &= (2x + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t=(2x+1)^2$. There is a pole at $x=-\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L=[1,2]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r=1+\frac{3}{4\left(x+\frac{1}{2}\right)^2}+\frac{1}{x+\frac{1}{2}}$$

For the pole at $x=-\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty)=0$ then

$$v=\frac{-O_r(\infty)}{2}=\frac{0}{2}=0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v=x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r}\approx 1+\frac{1}{2x}-\frac{1}{4x^3}+\frac{11}{32x^4}-\frac{21}{64x^5}+\frac{15}{64x^6}-\frac{3}{32x^7}-\frac{117}{2048x^8}+\dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a=1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)\,(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(1+x)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{4x+2}\, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1\left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{4x+2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x\, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x$$

Verified OK.

2.77.1 Maple step by step solution

Let's solve

$(4x+2)\, y'' - 4y' + (-4x-6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3+2x)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{2x+1} - \frac{(3+2x)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x)=-\frac{2}{2x+1}, P_3(x)=-\frac{3+2x}{2x+1}\right]$

- ○ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}}=-1$

- ○ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}}=0$

- ○ $x=-\frac{1}{2}$is a regular singular point

  Check to see if $x_0=-\frac{1}{2}$ is a regular singular point

  $x_0=-\frac{1}{2}$

- • Multiply by denominators

  $(2x+1)\,y''-2y'+(-2x-3)\,y=0$

- • Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+(-2u-2)\,y(u)=0$

- • Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - ○ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - ○ Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

  - ○ Shift index using $k->k+1$

    $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

  - ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

◦ Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)(k+r-1) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$2a_1(1+r)(-1+r) - 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2+4*x)*diff(y(x),x$2)-4*diff(y(x),x)-(6+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2 x\, e^{x}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 29

```
DSolve[(2+4*x)*y''[x]-4*y'[x]-(6+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1}x + c_1\right)$$

2.78 problem 80

2.78.1 Maple step by step solution 731

Internal problem ID [7568]
Internal file name [`OUTPUT/6501_Sunday_June_05_2022_04_55_50_PM_77127150/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 80.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 3xy' + (2x^2 + 5)\, y = 0}$$

Writing the ode as

$$y'' - 3xy' + (2x^2 + 5)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 26 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 26}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{13}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{13}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{13}{2}\right) - (0) \\
&= -\frac{13}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= -7 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 1\right) &&= 6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2\right) + 2\left(-\frac{x}{2}\right)\left(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{2}\right.\right.$$

$$a_5x^5 + 2(15 + a_4)x^4 + (3a_3 + 20a_5)x^3 + 4(a_2 + 3a_4)x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{1} \, dx} \\
&= z_1 e^{\frac{3x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x^2}{2}} \left(x^6 - 15x^4 + 45x^2 - 15\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right) \\
&\quad + c_2\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right) \\
&\quad + c_2\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)
\end{aligned}
$$

Verified OK.

2.78.1 Maple step by step solution

Let's solve

$y'' - 3xy' + (2x^2+5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 5a_0 + (6a_3 + 2a_1) x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(3k-5) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 5a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} - 3a_k k + 5a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2}(k+2) + 5a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3k a_{k+2} - 2a_k + a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-3*x*diff(y(x),x)+(5+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^2}{2}} (x^6 - 15x^4 + 45x^2 - 15) + c_2 e^{\frac{x^2}{2}} (x^6 - 15x^4 + 45x^2 - 15) \left(\int \frac{e^{\frac{x^2}{2}}}{(x^6 - 15x^4 + 45x^2 - 15)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.901 (sec). Leaf size: 95

```
DSolve[y''[x]-3*x*y'[x]+(5+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^6-15x^4+45x^2-15)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)-2c_2e^{\frac{x^2}{2}}x(x^4-14x^2+33)+1440c_1(x^6-15x^4+45x^2-}{1440}$$

2.79 problem 81

2.79.1 Maple step by step solution 741

Internal problem ID [7569]
Internal file name [`OUTPUT/6502_Sunday_June_05_2022_04_55_54_PM_44832796/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 81.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2y'' + 5xy' + (2x^2 + 4)\, y = 0}$$

Writing the ode as

$$2y'' + 5xy' + (2x^2 + 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 - 12 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{3x}{4}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^2 - 12}{16} \\
&= Q + \frac{R}{16} \\
&= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\
&= \frac{9x^2}{16} - \frac{3}{4}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{4}\right) - (0) \\
&= -\frac{3}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2}{16} - \frac{3}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{3x}{4}\right) \\
&= -\frac{3x}{4} \\
&= -\frac{3x}{4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3x}{4} dx} \\
&= e^{-\frac{3x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2}\,dx} \\
&= z_1 e^{-\frac{5x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5x^2}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} - \frac{i c_2 \mathrm{e}^{-x^2} \sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \tag{1}
$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^2} - \frac{i c_2 \mathrm{e}^{-x^2} \sqrt{\pi}\, \sqrt{3}\, \mathrm{erf}\left(\frac{i\sqrt{3}\, x}{2}\right)}{3}$$

Verified OK.

2.79.1 Maple step by step solution

Let's solve

$2y'' + 5xy' + (2x^2 + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x^2 - 2)\, y - \frac{5xy'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5xy'}{2} + (x^2 + 2)\, y = 0$

- Multiply by denominators

  $2y'' + 5xy' + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$4a_2 + 4a_0 + (12a_3 + 9a_1)\, x + \left(\sum_{k=2}^{\infty} \left(2a_{k+2}(k+2)\,(k+1) + a_k(5k+4) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(2k^2 + 6k + 4)\, a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2} + 2a_k + 14a_{k+2}}{2(k^2+7k+12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(2*diff(y(x),x$2)+5*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \operatorname{erf}\left(\frac{i\sqrt{3}\, x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 42

```
DSolve[2*y''[x]+5*x*y'[x]+(4+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} e^{-x^2} \left(\sqrt{3\pi} c_2 \operatorname{erfi}\left(\frac{\sqrt{3}x}{2}\right) + 3c_1\right)$$

2.80 problem 82

2.80.1 Maple step by step solution 747

Internal problem ID [7570]
Internal file name [`OUTPUT/6503_Sunday_June_05_2022_04_55_57_PM_3128761/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 82.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2}
$$

Verified OK.

2.80.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$

- Solve for the dependent coefficient(s)

$\{a_2 = -a_0, a_3 = -a_1\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-x^2}(c_2 x + c_1)$$

2.81 problem 83

2.81.1 Maple step by step solution 753

Internal problem ID [7571]
Internal file name [`OUTPUT/6504_Sunday_June_05_2022_04_55_59_PM_37228156/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 83.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2}
$$

Verified OK.

2.81.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^{k}\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

2.82 problem 84

2.82.1 Maple step by step solution 762

Internal problem ID [7572]
Internal file name [OUTPUT/6505_Sunday_June_05_2022_04_56_01_PM_15673183/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 84.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0}$$

Writing the ode as

$$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+2x^3+2x^2\\ B&=11x^3+11x^2+9x\\ C&=7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^4 + 18x^3 + 27x^2 - 2x - 3 \\ t &= 16\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2+3x+1}{4x\left(x^2+x+1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right) dx} \\
&= 2\sqrt{2}\, x^{\frac{1}{4}} \left(x^2 + x + 1\right)^{\frac{3}{4}} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + x + 1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{\frac{9}{4}} \left(x^2 + x + 1\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{2}\, \sqrt{x^2 + x + 1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3 + 11x^2 + 9x}{2x^4 + 2x^3 + 2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2} - \frac{\ln\left(x^2 + x + 1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\, \left(x^2 + x + 1\right)^{\frac{3}{2}}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}\right) \\
&\quad + c_2\left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}\left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \\
&\quad + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}}dx\right)}{4x^2}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \\
&\quad + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}\left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}}dx\right)}{4x^2}
\end{aligned}
$$

Verified OK.

2.82.1 Maple step by step solution

Let's solve

$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{9}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 3$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2 + x + 1)\, y'' + x(11x^2 + 11x + 9)\, y' + (7x^2 + 10x + 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(3+2r)\, x^r + (a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-2, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(2+r)a_0}{3+r}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r+\frac{3}{2}\right)\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k->k+2$

$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{k b_k + k b_{k+1} - \frac{1}{2} b_k +}{k + \frac{5}{2}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 141

```
dsolve(2*x^2*(1+x+x^2)*diff(y(x),x$2)+x*(9+11*x+11*x^2)*diff(y(x),x)+(6+10*x+7*x^2)*y(x)=0,y
```

$$y(x) = \frac{c_1\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}}{x^2} + \frac{c_2\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}\left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{6}}}{(x^2+x+1)^{\frac{3}{2}}\sqrt{x}}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 1.375 (sec). Leaf size: 93

```
DSolve[2*x^2*(1+x+x^2)*y''[x]+x*(9+11*x+11*x^2)*y'[x]+(6+10*x+7*x^2)*y[x]==0,y[x],x,IncludeS
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

2.83 problem 85

2.83.1 Maple step by step solution 774

Internal problem ID [7573]
Internal file name [OUTPUT/6506_Sunday_June_05_2022_04_56_08_PM_81147961/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 85.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{3x^2y'' + 2x(-2x^2 + x + 1)\,y' + (-8x^2 + 2x)\,y = 0}$$

Writing the ode as

$$3x^2y'' + (-4x^3 + 2x^2 + 2x)\,y' + (-8x^2 + 2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 4x^3 + 15x^2 - 4x - 2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{4}{9x} - \frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{2}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{3} + \frac{2x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\ &= Q + \frac{R}{9x^2} \\ &= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\ &= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\ &= \frac{14}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) &&= -\frac{5}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{2}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{2}{3} - \left(\frac{2}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3}\right) \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) =
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\
&= x^{\frac{2}{3}} \mathrm{e}^{\frac{x(x-1)}{3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^3 + 2x^2 + 2x}{3x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-1)}{3}}}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3+2x^2+2x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}}\right) + c_2 \left(x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} + c_2 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} + c_2 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx\right)$$

Verified OK.

2.83.1 Maple step by step solution

Let's solve

$3x^2y'' + (-4x^3 + 2x^2 + 2x)\, y' + (-8x^2 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{2(4x-1)y}{3x} + \frac{2(2x^2-x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2(2x^2-x-1)y'}{3x} - \frac{2(4x-1)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{2(2x^2-x-1)}{3x}, P_3(x) = -\frac{2(4x-1)}{3x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{2}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3xy'' + (-4x^2 + 2x + 2)\, y' + (-8x + 2)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+3r)\, x^{-1+r} + \left(a_1(1+r)\,(2+3r) + 2a_0(1+r)\right) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(3k+2+3r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(1+r)\,(2+3r) + 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(3k a_{k+1} + 3r a_{k+1} + 2a_k - 4a_{k-1} + 2a_{k+1}) = 0$

- Shift index using $k->k+1$

  $(k+r+2)\,(3(k+1)\, a_{k+2} + 3r a_{k+2} + 2a_{k+1} - 4a_k + 2a_{k+2}) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{k+1}+2b_k)}{3k+6}, 4b_1 +\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(3*x^2*diff(y(x),x$2)+2*x*(1+x-2*x^2)*diff(y(x),x)+(2*x-8*x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = c_1 x^{\frac{1}{3}} e^{\frac{2}{3}x^2-\frac{2}{3}x} + c_2 x^{\frac{1}{3}} e^{\frac{2}{3}x^2-\frac{2}{3}x} \left(\int \frac{e^{-\frac{2}{3}x^2+\frac{2}{3}x}}{x^{\frac{4}{3}}}dx\right)$$

✓ Solution by Mathematica

Time used: 9.303 (sec). Leaf size: 53

```
DSolve[3*x^2*y''[x]+2*x*(1+x-2*x^2)*y'[x]+(2*x-8*x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{\frac{2}{3}(x-1)x} \sqrt[3]{x} \left(c_2 \int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}} dK[1] + c_1 \right)$$

2.84 problem 86

2.84.1 Maple step by step solution 786

Internal problem ID [7574]
Internal file name [`OUTPUT/6507_Sunday_June_05_2022_04_56_11_PM_96253996/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 86.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{12x^2(1+x)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0}$$

Writing the ode as

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^3+12x^2 \\ B &= 3x^3+35x^2+11x \\ C &= 5x^2+10x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 30x^3 - 197x^2 - 190x - 95 \\ t &= 576\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2+x)^2$. There is a pole at $x=0$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{95}{576x^2} - \frac{1}{12\,(1+x)}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \\ &= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2} \\ &= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right) \\ &= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{12}\right) - (0) \\ &= -\frac{1}{12} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= -\frac{1}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &&= \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x + 8} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{8x + 8} + \frac{5}{24x} - \frac{1}{8} \\
&= \frac{1}{8x + 8} + \frac{5}{24x} - \frac{1}{8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x + 8} + \frac{5}{24x} - \frac{1}{8}\right)(0) + \left(\left(-\frac{1}{8(1 + x)^2} - \frac{5}{24x^2}\right) + \left(\frac{1}{8x + 8} + \frac{5}{24x} - \frac{1}{8}\right)^2 - \left(\frac{9x^4 - 30x^3 -}{576}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8}\right) dx} \\
&= x^{\frac{5}{24}} (1 + x)^{\frac{1}{8}} e^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx} \\
&= z_1 e^{-\frac{x}{8}-\frac{11\ln(x)}{24}-\frac{7\ln(1+x)}{8}} \\
&= z_1 \left(\frac{e^{-\frac{x}{8}}}{x^{\frac{11}{24}}(1+x)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}\right) + c_2 \left(\frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}\left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}\,dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}} + \frac{c_2 e^{-\frac{x}{4}}\left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}\,dx\right)}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{4}}}{x^{\frac{1}{4}} (1+x)^{\frac{3}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{4}} \left(\int \frac{\mathrm{e}^{\frac{x}{4}}}{x^{\frac{5}{12}} (1+x)^{\frac{1}{4}}} dx\right)}{x^{\frac{1}{4}} (1+x)^{\frac{3}{4}}}$$

Verified OK.

2.84.1 Maple step by step solution

Let's solve

$(12x^3 + 12x^2) y'' + (3x^3 + 35x^2 + 11x) y' + (5x^2 + 10x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+10x-1)y}{12x^2(1+x)} - \frac{(3x^2+35x+11)y'}{12x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+35x+11)y'}{12x(1+x)} + \frac{(5x^2+10x-1)y}{12x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3x^2+35x+11}{12x(1+x)}, P_3(x) = \frac{5x^2+10x-1}{12x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{4}$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $12x^2(1+x) y'' + x(3x^2 + 35x + 11) y' + (5x^2 + 10x - 1) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(12u^3-24u^2+12u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^3+26u^2-50u+21)\left(\frac{d}{du}y(u)\right)+(5u^2-6)\,y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m=0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(3+4r)\,u^{-1+r}+(3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r))\,u^r+(3a_2(2+r)\,(11+4r)-2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(3+4r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{4}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r)=0, 3a_2(2+r)\,(11+4r)-2a_1(7+4r)\,(4+3r)+2a$$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $12(-2a_k + a_{k-1} + a_{k+1})k^2 + (24(-2a_k + a_{k-1} + a_{k+1})r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1})k +$

- Shift index using $k- >k+2$

  $12(-2a_{k+2} + a_{k+1} + a_{k+3})(k+2)^2 + (24(-2a_{k+2} + a_{k+1} + a_{k+3})r - 26a_{k+2} + 3a_k - 10a_{k+1} + 3$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+24kra_{k+1}-48kra_{k+2}+12r^2a_{k+1}-24r^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+3ra_k+38ra_{k+1}-122r}{3(4k^2+8kr+4r^2+27k+27r+45)}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, a_2 =\right.$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a_k+26a_{k+1}-154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7},\right.$

- Recursion relation for $r = -\frac{3}{4}$

  $a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}$

- Solution for $r = -\frac{3}{4}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, a_2\right.$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+20ka_{k+1}-86ka_{k+2}+\frac{11}{4}a_k+\frac{17}{4}a_{k+1}-76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0,\right.$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{3}{4}}\right), a_{k+3} = -\frac{12k^2a_{k+1}-24k^2a_{k+2}+3ka_k+38ka_{k+1}-122ka_{k+2}+5a}{3(4k^2+27k+45)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve(12*x^2*(1+x)*diff(y(x),x$2)+x*(11+35*x+3*x^2)*diff(y(x),x)-(1-10*x-5*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{x}{4}}}{(x+1)^{\frac{3}{4}} x^{\frac{1}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{4}} \left(\int \frac{\mathrm{e}^{\frac{x}{4}}}{(x+1)^{\frac{1}{4}} x^{\frac{5}{12}}} dx\right)}{(x+1)^{\frac{3}{4}} x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 20.702 (sec). Leaf size: 61

```
DSolve[12*x^2*(1+x)*y''[x]+x*(11+35*x+3*x^2)*y'[x]-(1-10*x-5*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to \frac{e^{-x/4}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{K[1]^{5/12}\sqrt[4]{K[1]+1}}dK[1] + c_1\right)}{\sqrt[4]{x}(x+1)^{3/4}}$$

2.85 problem 87

2.85.1 Maple step by step solution 798

Internal problem ID [7575]
Internal file name [`OUTPUT/6508_Sunday_June_05_2022_04_56_15_PM_36118531/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 87.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x^2(10x^2+x+5)\,y''+x(48x^2+3x+4)\,y'+(36x^2+x)\,y=0}$$

The ODE is

$$(10x^4+x^3+5x^2)\,y''+(48x^3+3x^2+4x)\,y'+(36x^2+x)\,y=0$$

Or

$$x(10x^3y''+48x^2y'+x^2y''+36yx+3xy'+5xy''+y+4y')=0$$

For $x\neq 0$ the above simplifies to

$$(10x^3+x^2+5x)\,y''+(48x^2+3x+4)\,y'+(36x+1)\,y=0$$

Writing the ode as

$$(10x^4+x^3+5x^2)\,y''+(48x^3+3x^2+4x)\,y'+(36x^2+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=10x^4+x^3+5x^2\\B&=48x^3+3x^2+4x\\C&=36x^2+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -96x^4 - 16x^3 - 97x^2 - 12x - 24 \\ t &= 4\left(10x^3 + x^2 + 5x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 6 - 4\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(10x^3 + x^2 + 5x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{199}}{20} - \frac{1}{20}$ of order 2. There is a pole at $x = -\frac{i\sqrt{199}}{20} - \frac{1}{20}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned}r = &-\frac{6}{25x^2} - \frac{3}{125x} + \frac{-\frac{i\sqrt{199}}{1990} - \frac{1}{19900}}{\left(-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}\right)^2}\\ &+ \frac{\frac{i\sqrt{199}}{1990} - \frac{1}{19900}}{\left(\frac{i\sqrt{199}}{20} + x + \frac{1}{20}\right)^2} + \frac{-\frac{647i\sqrt{199}}{9900250} + \frac{3}{250}}{-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + \frac{\frac{647i\sqrt{199}}{9900250} + \frac{3}{250}}{\frac{i\sqrt{199}}{20} + x + \frac{1}{20}}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5}
\end{aligned}
$$

For the pole at $x = \frac{i\sqrt{199}}{20} - \frac{1}{20}$ let b be the coefficient of $\frac{1}{\left(-\frac{i\sqrt{199}}{20}+x+\frac{1}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{i\sqrt{199}}{1990} - \frac{1}{19900}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{989826 - 1990i\sqrt{199}}}{1990}
\end{aligned}
$$

For the pole at $x = -\frac{i\sqrt{199}}{20} - \frac{1}{20}$ let b be the coefficient of $\frac{1}{\left(\frac{i\sqrt{199}}{20}+x+\frac{1}{20}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{i\sqrt{199}}{1990} - \frac{1}{19900}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{989826 + 1990i\sqrt{199}}}{1990} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{989826 + 1990i\sqrt{199}}}{1990}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{6}{25}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{5} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{2}{5}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-96x^4 - 16x^3 - 97x^2 - 12x - 24}{4\left(10x^3 + x^2 + 5x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{5}$	$\frac{2}{5}$
$\frac{i\sqrt{199}}{20} - \frac{1}{20}$	2	0	$\frac{1}{2} + \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$	$\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}$
$-\frac{i\sqrt{199}}{20} - \frac{1}{20}$	2	0	$\frac{1}{2} + \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$	$\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{5}$	$\frac{2}{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{5}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{3}{5} - \left(\frac{3}{5}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + (0) \\
&= \frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} \\
&= \frac{12x^2 + x + 6}{20x^3 + 2x^2 + 10x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{\frac{i\sqrt{199}}{20} + x + \frac{1}{20}}\right)(0) + \left(\left(-\frac{3}{5x^2} - \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{\left(-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}\right)^2} - \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{3}{5x} + \frac{\frac{1}{2} - \frac{\sqrt{989826-1990i\sqrt{199}}}{1990}}{-\frac{i\sqrt{199}}{20} + x + \frac{1}{20}} + \frac{\frac{1}{2} - \frac{\sqrt{989826+1990i\sqrt{199}}}{1990}}{\frac{i\sqrt{199}}{20} + x + \frac{1}{20}}\right) dx} \\
&= x^{\frac{3}{5}} e^{-\frac{\sqrt{199}\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{48x^3+3x^2+4x}{10x^4+x^3+5x^2} \, dx}
\end{aligned}
$$

$$= z_1 e^{-\frac{2\ln(x)}{5} - \ln(10x^2+x+5) - \frac{\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}$$

$$= z_1 \left(\frac{e^{-\frac{\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{x^{\frac{2}{5}}\left(10x^2+x+5\right)}\right)$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{5}} e^{-\frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2+x+5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{48x^3+3x^2+4x}{10x^4+x^3+5x^2}\, dx}}{(y_1)^2}\, dx$$

$$= y_1 \int \frac{e^{-\frac{4\ln(x)}{5} - 2\ln(10x^2+x+5) - \frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{(y_1)^2}\, dx$$

$$= y_1 \left(\int \frac{e^{\frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{x^{\frac{6}{5}}} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{x^{\frac{1}{5}} e^{-\frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2+x+5}\right) + c_2 \left(\frac{x^{\frac{1}{5}} e^{-\frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{10x^2+x+5}\left(\int \frac{e^{\frac{2\sqrt{199}\,\arctan\left(\frac{(20x+1)\sqrt{199}}{199}\right)}{995}}}{x^{\frac{6}{5}}} dx\right)\right)$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= \frac{c_1 x^{\frac{1}{5}} {\mathrm e}^{-\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}}}{10 x^{2}+x +5}\\
&+ \frac{c_2 x^{\frac{1}{5}} {\mathrm e}^{-\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}} \left(\int \frac{{\mathrm e}^{\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}}}{x^{\frac{6}{5}}}d x \right)}{10 x^{2}+x +5} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= \frac{c_1 x^{\frac{1}{5}} {\mathrm e}^{-\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}}}{10 x^{2}+x +5}\\
&+ \frac{c_2 x^{\frac{1}{5}} {\mathrm e}^{-\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}} \left(\int \frac{{\mathrm e}^{\frac{2 \sqrt{199}\, \arctan\left(\frac{(20 x +1) \sqrt{199}}{199}\right)}{995}}}{x^{\frac{6}{5}}}d x \right)}{10 x^{2}+x +5}
\end{align*}
$$

Verified OK. `{x <> 0}`

2.85.1 Maple step by step solution

Let's solve

$\left(10 x^{4}+x^{3}+5 x^{2}\right) y^{\prime \prime}+\left(48 x^{3}+3 x^{2}+4 x \right) y^{\prime}+\left(36 x^{2}+x \right) y =0$

- Highest derivative means the order of the ODE is 2

  $y^{\prime \prime}$

- Isolate 2nd derivative

  $y^{\prime \prime}=-\frac{(36 x +1) y}{x \left(10 x^{2}+x +5\right)}-\frac{\left(48 x^{2}+3 x +4\right) y^{\prime}}{x \left(10 x^{2}+x +5\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y^{\prime \prime}+\frac{\left(48 x^{2}+3 x +4\right) y^{\prime}}{x \left(10 x^{2}+x +5\right)}+\frac{(36 x +1) y}{x \left(10 x^{2}+x +5\right)}=0$

□ Check to see if x_{0} is a regular singular point

  ◦ Define functions

    $\left[P_{2}(x)=\frac{48 x^{2}+3 x +4}{x \left(10 x^{2}+x +5\right)}, P_{3}(x)=\frac{36 x +1}{x \left(10 x^{2}+x +5\right)}\right]$

  ◦ $x \cdot P_{2}(x)$ is analytic at $x =0$

$(x \cdot P_2(x))\Big|_{x=0} = \frac{4}{5}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $(36x + 1)\, y + (48x^2 + 3x + 4)\, y' + x(10x^2 + x + 5)\, y'' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\, (k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-1+5r)\, x^{-1+r} + \left(a_1(1+r)\,(4+5r) + a_0(1+r)^2\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(5k+4+5r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+5r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{5}\right\}$
- Each term must be 0

  $a_1(1+r)\,(4+5r) + a_0(1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r+1)\,((a_k + 10a_{k-1} + 5a_{k+1})\,k + (a_k + 10a_{k-1} + 5a_{k+1})\,r + a_k + 8a_{k-1} + 4a_{k+1}) = 0$
- Shift index using $k- >k+1$

  $(k+r+2)\,((a_{k+1} + 10a_k + 5a_{k+2})\,(k+1) + (a_{k+1} + 10a_k + 5a_{k+2})\,r + a_{k+1} + 8a_k + 4a_{k+2}) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{10ka_k + ka_{k+1} + 10ra_k + ra_{k+1} + 18a_k + 2a_{k+1}}{5k+5r+9}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{10ka_k + ka_{k+1} + 18a_k + 2a_{k+1}}{5k+9}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{10ka_k + ka_{k+1} + 18a_k + 2a_{k+1}}{5k+9}, 4a_1 + a_0 = 0\right]$
- Recursion relation for $r = \frac{1}{5}$

  $a_{k+2} = -\frac{10ka_k + ka_{k+1} + 20a_k + \frac{11}{5}a_{k+1}}{5k+10}$
- Solution for $r = \frac{1}{5}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{5}}, a_{k+2} = -\frac{10ka_k + ka_{k+1} + 20a_k + \frac{11}{5}a_{k+1}}{5k+10}, 6a_1 + \frac{36a_0}{25} = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{5}}\right), a_{k+2} = -\frac{10ka_k + ka_{k+1} + 18a_k + 2a_{k+1}}{5k+9}, 4a_1 + a_0 = 0, b_{k+2} = -\frac{10kb_k +}{}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 137

```
dsolve(x^2*(5+x+10*x^2)*diff(y(x),x$2)+x*(4+3*x+48*x^2)*diff(y(x),x)+(x+36*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{c_1 x^{\frac{1}{5}} \left(\frac{i\sqrt{199}+20x+1}{i\sqrt{199}-20x-1}\right)^{-\frac{i\sqrt{199}}{995}}}{10x^2+x+5} + \frac{c_2 x^{\frac{1}{5}} \left(\frac{i\sqrt{199}+20x+1}{i\sqrt{199}-20x-1}\right)^{-\frac{i\sqrt{199}}{995}} \left(\int \frac{\left(\frac{i\sqrt{199}-20x-1}{i\sqrt{199}+20x+1}\right)^{-\frac{i\sqrt{199}}{995}}}{x^{\frac{6}{5}}}dx\right)}{10x^2+x+5}$$

✓ Solution by Mathematica

Time used: 2.147 (sec). Leaf size: 88

```
DSolve[x^2*(5+x+10*x^2)*y''[x]+x*(4+3*x+48*x^2)*y'[x]+(x+36*x^2)*y[x]==0,y[x],x,IncludeSingu
```

$$y(x) \to \frac{\sqrt[5]{x}e^{-\frac{2\arctan\left(\frac{20x+1}{\sqrt{199}}\right)}{5\sqrt{199}}}\left(c_2\int_1^x \frac{e^{\frac{2\arctan\left(\frac{20K[1]+1}{\sqrt{199}}\right)}{5\sqrt{199}}}}{K[1]^{6/5}}dK[1]+c_1\right)}{10x^2+x+5}$$

2.86 problem 88

2.86.1 Maple step by step solution 810

Internal problem ID [7576]
Internal file name [`OUTPUT/6509_Sunday_June_05_2022_04_56_25_PM_21958520/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 88.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0}$$

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 18x^3+18x^2 \\ B &= 3x^3+33x^2+15x \\ C &= 5x^2+2x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 18x^3 - 45x^2 - 18x - 27$$
$$t = 144\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{4x} - \frac{35}{144\,(1+x)^2} - \frac{3}{16x^2} - \frac{7}{18\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\ &= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\ &= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{36}\right) - (0) \\ &= -\frac{5}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= -\frac{5}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= \frac{5}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12\,(1 + x)} + \frac{1}{4x} + (-)\left(\frac{1}{12}\right) \\
&= \frac{7}{12\,(1 + x)} + \frac{1}{4x} - \frac{1}{12} \\
&= \frac{7}{12 + 12x} + \frac{1}{4x} - \frac{1}{12}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12\,(1 + x)} + \frac{1}{4x} - \frac{1}{12}\right)(0) + \left(\left(-\frac{7}{12\,(1 + x)^2} - \frac{1}{4x^2}\right) + \left(\frac{7}{12\,(1 + x)} + \frac{1}{4x} - \frac{1}{12}\right)^2 - \left(\frac{x^4 - 1}{}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{12(1+x)} + \frac{1}{4x} - \frac{1}{12}\right) dx} \\
&= x^{\frac{1}{4}} (1 + x)^{\frac{7}{12}} \, \mathrm{e}^{-\frac{x}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+33x^2+15x}{18x^3+18x^2}\,dx} \\
&= z_1 e^{-\frac{x}{12}-\frac{5\ln(x)}{12}-\frac{5\ln(1+x)}{12}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{12}}}{x^{\frac{5}{12}}(1+x)^{\frac{5}{12}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{\frac{1}{6}}\,\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{5\ln(1+x)}{6}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)^{\frac{1}{6}}\,\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\right) + c_2\left(\frac{(1+x)^{\frac{1}{6}}\,\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\left(\int \frac{\mathrm{e}^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}}\,dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(1+x)^{\frac{1}{6}}\,\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}} + \frac{c_2(1+x)^{\frac{1}{6}}\,\mathrm{e}^{-\frac{x}{6}}\left(\int \frac{\mathrm{e}^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}}\,dx\right)}{x^{\frac{1}{6}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+x)^{\frac{1}{6}} \mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}} + \frac{c_2(1+x)^{\frac{1}{6}} \mathrm{e}^{-\frac{x}{6}} \left(\int \frac{\mathrm{e}^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}}dx\right)}{x^{\frac{1}{6}}}$$

Verified OK.

2.86.1 Maple step by step solution

Let's solve

$(18x^3+18x^2)\,y''+(3x^3+33x^2+15x)\,y'+(5x^2+2x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5x^2+2x-1)y}{18x^2(1+x)}-\frac{(x^2+11x+5)y'}{6x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2+11x+5)y'}{6x(1+x)}+\frac{(5x^2+2x-1)y}{18x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x^2+11x+5}{6x(1+x)}, P_3(x)=\frac{5x^2+2x-1}{18x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=\frac{5}{6}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $18x^2(1+x)\,y''+3x(x^2+11x+5)\,y'+(5x^2+2x-1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$\left(18u^3-36u^2+18u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(3u^3+24u^2-42u+15\right)\left(\frac{d}{du}y(u)\right)+\left(5u^2-8u+2\right)y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r}+\left(3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)\right)u^r+\left(3a_2(2+r)\,(11+6r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-1+6r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{6}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)=0, 3a_2(2+r)\,(11+6r)-2a_1(4+3r)\,(5+6r)+$$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2 = \frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $18(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 3(12(-2a_k + a_{k-1} + a_{k+1})\, r - 2a_k + a_{k-2} - 10a_{k-1} + 11a_{k+1})\, k +$

- Shift index using $k- >k+2$

  $18(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(12(-2a_{k+2} + a_{k+1} + a_{k+3})\, r - 2a_{k+2} + a_k - 10a_{k+1} + 11$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+36kra_{k+1}-72kra_{k+2}+18r^2a_{k+1}-36r^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+3ra_k+42ra_{k+1}-150}{3(6k^2+12kr+6r^2+35k+35r+51)}$

- Recursion relation for $r = 0$

  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}, a\right.$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a_k+16a_{k+1}-154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}\right.$

- Recursion relation for $r = \frac{1}{6}$

  $a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}$

- Solution for $r = \frac{1}{6}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1 = 0, a\right.$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+48ka_{k+1}-162ka_{k+2}+\frac{11}{2}a_k+\frac{47}{2}a_{k+1}-180a_{k+2}}{3(6k^2+37k+57)}, a_1 = \right.$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+\frac{1}{6}}\right), a_{k+3} = -\frac{18k^2a_{k+1}-36k^2a_{k+2}+3ka_k+42ka_{k+1}-150ka_{k+2}+5a}{3(6k^2+35k+51)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(18*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x+x^2)*diff(y(x),x)-(1-2*x-5*x^2)*y(x)=0,y(x),
```

$$y(x) = c_1 e^{-\frac{x}{6}} \left(\frac{x+1}{x}\right)^{\frac{1}{6}} + c_2 e^{-\frac{x}{6}} \left(\frac{x+1}{x}\right)^{\frac{1}{6}} \left(\int \frac{e^{\frac{x}{6}}}{(x+1)^{\frac{7}{6}} \sqrt{x}} dx\right)$$

✓ Solution by Mathematica

Time used: 3.726 (sec). Leaf size: 73

```
DSolve[18*x^2*(1+x)*y''[x]+3*x*(5+11*x+x^2)*y'[x]-(1-2*x-5*x^2)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to \frac{e^{-x/6}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{6}}\sqrt[3]{\frac{K[1]}{K[1]+1}}}{K[1]^{5/6}(K[1]+1)^{5/6}}dK[1] + c_1\right)}{\sqrt[6]{\frac{x}{x+1}}}$$

2.87 problem 89

2.87.1 Maple step by step solution 822

Internal problem ID [7577]
Internal file name [`OUTPUT/6510_Sunday_June_05_2022_04_56_29_PM_21020949/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 89.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0}$$

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2+4x+5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x+5}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x+5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{1}{4x} - \frac{1}{2} \\ &= -\frac{1}{4x} - \frac{1}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\ &= \frac{e^{-\frac{x}{2}}}{x^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2+3x}{2x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2} - \frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{e^{-\frac{x}{2}}}{x^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\sqrt{x}\,\mathrm{e}^{x} - \frac{\sqrt{\pi}\,\mathrm{erfi}\left(\sqrt{x}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x}\left(\sqrt{x}\,\mathrm{e}^{x} - \frac{\sqrt{\pi}\,\mathrm{erfi}\left(\sqrt{x}\right)}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\mathrm{e}^{-x}}{x} + \frac{c_2\left(-\sqrt{\pi}\,\mathrm{e}^{-x}\,\mathrm{erfi}\left(\sqrt{x}\right) + 2\sqrt{x}\right)}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\mathrm{e}^{-x}}{x} + \frac{c_2\left(-\sqrt{\pi}\,\mathrm{e}^{-x}\,\mathrm{erfi}\left(\sqrt{x}\right) + 2\sqrt{x}\right)}{2x}$$

Verified OK.

2.87.1 Maple step by step solution

Let's solve

$2x^2y'' + (2x^2 + 3x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{2x^2} - \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+2x)y'}{2x} + \frac{(x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3+2x}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(3+2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r-\frac{1}{2}\right)\left(a_k (k+r+1) + a_{k-1}\right) = 0$$

- Shift index using $k- > k+1$

  $$2\left(k+\frac{1}{2}+r\right)\left(a_{k+1}(k+2+r) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+\frac{5}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+x*(3+2*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{-x} \left(\int \sqrt{x}\, e^{x} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 33

```
DSolve[2*x^2*y''[x]+x*(3+2*x)*y'[x]-(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}\left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

2.88 problem 90

2.88.1 Maple step by step solution 833

Internal problem ID [7578]
Internal file name [`OUTPUT/6511_Sunday_June_05_2022_04_56_32_PM_8505700/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 90.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + x(x+5)\,y' - (2-3x)\,y = 0}$$

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2 + 5x \\ C &= 3x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 14x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 14x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 14x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{7}{8x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= \frac{7}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 14x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\
&= \frac{7}{4x} - \frac{1}{4} \\
&= -\frac{x - 7}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 14x + 21}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\
&= x^{\frac{7}{4}} \mathrm{e}^{-\frac{x}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{4} - \frac{5 \ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} + 2\,\mathrm{e}^{\frac{x}{2}}(x^2+x+3)}{15x^{\frac{5}{2}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} + 2\,\mathrm{e}^{\frac{x}{2}}(x^2+x+3)}{15x^{\frac{5}{2}}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + \frac{c_2\left(-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}}\mathrm{e}^{-\frac{x}{2}} - 2x^2 - 2x - 6\right)}{15x^2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + \frac{c_2\left(-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}}\mathrm{e}^{-\frac{x}{2}} - 2x^2 - 2x - 6\right)}{15x^2}$$

Verified OK.

2.88.1 Maple step by step solution

Let's solve

$2x^2y'' + (x^2 + 5x)\,y' + (3x - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{2x^2} - \frac{(x+5)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+5)y'}{2x} + \frac{(3x-2)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+5}{2x}, P_3(x) = \frac{3x-2}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(x + 5)\,y' + (3x - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (2+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+2)\,(2k+2r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-2, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right)(k+r+2) = 0$$

- Shift index using $k->k+1$

  $$2\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right)(k+r+3) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{2k+1+2r}$$

- Recursion relation for $r = -2$

  $$a_{k+1} = -\frac{a_k}{2k-3}$$

- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{2k-3}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = -\frac{a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k-3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(2*x^2*diff(y(x),x$2)+x*(5+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(\int \frac{\mathrm{e}^{\frac{x}{2}}}{x^{\frac{7}{2}}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 70

```
DSolve[2*x^2*y''[x]+x*(5+x)*y'[x]-(2-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{15}\left(-\frac{2c_2(x^2+x+3)}{x^2} + 15c_1e^{-x/2}\sqrt{x} + \sqrt{2}c_2e^{-x/2}\sqrt{-x}\Gamma\left(\frac{1}{2}, -\frac{x}{2}\right)\right)$$

2.89 problem 91

2.89.1 Maple step by step solution 844

Internal problem ID [7579]
Internal file name [`OUTPUT/6512_Sunday_June_05_2022_04_56_35_PM_60104680/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 91.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' + x(1+x)\,y' - y = 0}$$

Writing the ode as

$$3x^2y'' + \left(x^2 + x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{1}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= \frac{1}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\
&= -\frac{1}{6x} - \frac{1}{6} \\
&= -\frac{1+x}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2+2x+7}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6x} - \frac{1}{6}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2+x}{3x^2} \, dx} \\
&= z_1 e^{-\frac{x}{6} - \frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.89.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{3}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(1+x)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

$$x^{2}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(1+3r)\,(-1+r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(\left(k+r+\frac{1}{3}\right) a_{k} + \frac{a_{k-1}}{3}\right)(k+r-1) = 0$$

- Shift index using $k->k+1$

$$3\left(\left(k+\frac{4}{3}+r\right) a_{k+1} + \frac{a_{k}}{3}\right)(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_{k}}{3k+4+3r}$$

- Recursion relation for $r=1$

$$a_{k+1} = -\frac{a_{k}}{3k+7}$$

- Solution for $r=1$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+1} = -\frac{a_{k}}{3k+7}\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+1} = -\frac{a_{k}}{3k+3}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_{k}}{3k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_{k} x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_{k} x^{k-\frac{1}{3}}\right), a_{k+1} = -\frac{a_{k}}{3k+7}, b_{k+1} = -\frac{b_{k}}{3k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 e^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} e^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 50

```
DSolve[3*x^2*y''[x]+x*(1+x)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x/3}\left(c_2x^{2/3} - 3\sqrt[3]{3}c_1(-x)^{2/3}\Gamma\left(\frac{4}{3}, -\frac{x}{3}\right)\right)}{x}$$

2.90 problem 92

2.90.1 Maple step by step solution 853

Internal problem ID [7580]
Internal file name [`OUTPUT/6513_Sunday_June_05_2022_04_56_38_PM_11918926/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 92.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' - xy' + (1-2x)\,y = 0}$$

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\}\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\ &= \frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\
&= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2\sqrt{x}} \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2\sqrt{x}} \sqrt{x}\right) + c_2 \left(\mathrm{e}^{2\sqrt{x}} \sqrt{x} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2\sqrt{x}}\sqrt{x} - \frac{c_2 e^{-2\sqrt{x}}\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2\sqrt{x}}\sqrt{x} - \frac{c_2 e^{-2\sqrt{x}}\sqrt{x}}{2}$$

Verified OK.

2.90.1 Maple step by step solution

Let's solve

$2x^2y'' - xy' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} - \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$2x^2y'' - xy' + (1-2x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2\left(k+r-\frac{1}{2}\right)(k+r-1)\,a_k - 2a_{k-1} = 0$

- Shift index using $k- > k+1$

$2\left(k+\frac{1}{2}+r\right)(k+r)\,a_{k+1} - 2a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$

- Recursion relation for $r = 1$

$a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}, b_{k+1} = \frac{2b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh\left(2\sqrt{x}\right) + c_2\sqrt{x}\,\cosh\left(2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 41

```
DSolve[2*x^2*y''[x]-x*y'[x]+(1-2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2\sqrt{x}}\sqrt{x}\left(2c_1e^{4\sqrt{x}} - c_2\right)$$

2.91 problem 93

2.91.1 Maple step by step solution 864

Internal problem ID [7581]
Internal file name [`OUTPUT/6514_Sunday_June_05_2022_04_56_41_PM_26003100/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 93.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' + x(1+x)\,y' - (3x+1)\,y = 0}$$

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' + (-3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2+x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 38x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 38x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 38x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 38x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{38x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{38x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{19}{18}\right) - (0) \\ &= \frac{19}{18} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= \frac{19}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{19}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 38x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{19}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{19}{6} - \left(\frac{7}{6}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{7}{6x} + \left(\frac{1}{6}\right) \\ &= \frac{7}{6x} + \frac{1}{6} \\ &= \frac{7 + x}{6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 38x + 7}{36x^2}\right)\right) = 0$$

$$\frac{(-a_1 + 20)\, x - 2a_0 + 7a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 70, a_1 = 20\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 20x + 70$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 + 20x + 70\right) e^{\int \left(\frac{7}{6x} + \frac{1}{6}\right) dx} \\ &= \left(x^2 + 20x + 70\right) e^{\frac{x}{6} + \frac{7 \ln(x)}{6}} \\ &= \left(x^2 + 20x + 70\right) x^{\frac{7}{6}} e^{\frac{x}{6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+20x+70\right)x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\left(x^2+20x+70\right)x\right) + c_2\left(\left(x^2+20x+70\right)x\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+20x+70\right)x + c_2\left(x^2+20x+70\right)x\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1 (x^2 + 20x + 70) x + c_2 (x^2 + 20x + 70) x \left(\int \frac{e^{-\frac{x}{3}}}{x^{\frac{7}{3}} (x^2 + 20x + 70)^2} dx \right)$$

Verified OK.

2.91.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2 + x) y' + (-3x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{3x} - \frac{(3x+1)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{3x+1}{3x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{3}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(1 + x) y' + (-3x - 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)(k+r-1) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $3(k+r-1)\left(k+r+\frac{1}{3}\right) a_k + a_{k-1}(k-4+r) = 0$
- Shift index using $k- >k + 1$

  $3(k+r)\left(k+\frac{4}{3}+r\right) a_{k+1} + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+r)(3k+4+3r)}$
- Recursion relation for $r = 1$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(3k+7)}$

- Apply recursion relation for $k=0$

  $a_1 = \frac{2a_0}{7}$
- Apply recursion relation for $k=1$

  $a_2 = \frac{a_1}{20}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{70}$
- Terminating series solution of the ODE for $r=1$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)$
- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}$
- Solution for $r=-\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1+\frac{2}{7}x+\frac{1}{70}x^2\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x(x^2 + 20x + 70) + c_2 x(x^2 + 20x + 70) \left(\int \frac{e^{-\frac{x}{3}}}{x^{\frac{7}{3}} (x^2 + 20x + 70)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 1.549 (sec). Leaf size: 78

```
DSolve[3*x^2*y''[x]+x*(1+x)*y'[x]-(1+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x\left(x^2+20x+70\right) - \frac{c_2 x(x^2+20x+70)\,\Gamma\left(\frac{2}{3},\frac{x}{3}\right)}{1680\sqrt[3]{3}} + \frac{c_2 e^{-x/3}(x^3+19x^2+54x-18)}{1680\sqrt[3]{x}}$$

2.92 problem 94

2.92.1 Maple step by step solution 874

Internal problem ID [7582]
Internal file name [`OUTPUT/6515_Sunday_June_05_2022_04_56_44_PM_21578990/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 94.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x+3)\,y''+x(1+5x)\,y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+6x^2\\B&=5x^2+x\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 30x - 35$$
$$t = 16\left(x^2 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{108x} + \frac{5}{108\,(x+3)} - \frac{35}{144x^2} + \frac{7}{36\,(x+3)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\,(x^2+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 30x - 35}{16 \left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6\left(x+3\right)} + \frac{5}{12x} + (-)\left(0\right) \\ &= -\frac{1}{6\left(x+3\right)} + \frac{5}{12x} \\ &= \frac{x+5}{4x\left(x+3\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(x+3)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6\,(x+3)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6\,(x+3)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2 - 30x - 3}{16\,(x^2+3x)^2}\right.\right.$$

(

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(x+3)} + \frac{5}{12x}\right) dx} \\
&= \frac{x^{\frac{5}{12}}}{(x+3)^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x^2+x}{2x^3+6x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(x+3)}{6} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{1}{(x+3)^{\frac{7}{6}}\, x^{\frac{1}{12}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x+3)}{3}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}\left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}}\left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}}dx\right)}{(x+3)^{\frac{4}{3}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}}\left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}}dx\right)}{(x+3)^{\frac{4}{3}}}
$$

Verified OK.

2.92.1 Maple step by step solution

Let's solve

$\left(2x^3+6x^2\right)y''+\left(5x^2+x\right)y'+(1+x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(1+x)y}{2x^2(x+3)} - \frac{(1+5x)y'}{2x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+5x)y'}{2x(x+3)} + \frac{(1+x)y}{2x^2(x+3)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+5x}{2x(x+3)}, P_3(x) = \frac{1+x}{2x^2(x+3)}\right]$

  - $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

    $\left((x+3) \cdot P_2(x)\right)\Big|_{x=-3} = \frac{7}{3}$

  - $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

    $\left((x+3)^2 \cdot P_3(x)\right)\Big|_{x=-3} = 0$

  - $x = -3$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -3$

- Multiply by denominators

  $2x^2(x+3)\, y'' + x(1+5x)\, y' + (1+x)\, y = 0$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(2u^3 - 12u^2 + 18u)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 29u + 42)\left(\frac{d}{du}y(u)\right) + (-2+u)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$6a_{0}r(4+3r)\,u^{-1+r}+\left(6a_{1}(1+r)\,(7+3r)-a_{0}(12r^{2}+17r+2)\right)u^{r}+\left(\sum_{k=1}^{\infty}(6a_{k+1}(k+r+1)\,(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$6r(4+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{4}{3}\right\}$$

- Each term must be 0

$$6a_{1}(1+r)\,(7+3r)-a_{0}(12r^{2}+17r+2)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-6a_{k}+a_{k-1}+9a_{k+1})\,k^{2}+\left(4(-6a_{k}+a_{k-1}+9a_{k+1})\,r-17a_{k}-a_{k-1}+60a_{k+1}\right)k+2(-6a_{k}+$$

- Shift index using $k->k+1$

$$2(-6a_{k+1}+a_{k}+9a_{k+2})\,(k+1)^{2}+\left(4(-6a_{k+1}+a_{k}+9a_{k+2})\,r-17a_{k+1}-a_{k}+60a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^{2}a_{k}-12k^{2}a_{k+1}+4kra_{k}-24kra_{k+1}+2r^{2}a_{k}-12r^{2}a_{k+1}+3ka_{k}-41ka_{k+1}+3ra_{k}-41ra_{k+1}+a_{k}-31a_{k+1}}{6(3k^{2}+6kr+3r^{2}+16k+16r+20)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^{2}a_{k}-12k^{2}a_{k+1}+3ka_{k}-41ka_{k+1}+a_{k}-31a_{k+1}}{6(3k^{2}+16k+20)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{2k^{2}a_{k}-12k^{2}a_{k+1}+3ka_{k}-41ka_{k+1}+a_{k}-31a_{k+1}}{6(3k^{2}+16k+20)},42a_{1}-2a_{0}=0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31 a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31 a_{k+1}}{6(3k^2+16k+20)}, 42a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(2*x^2*(3+x)*diff(y(x),x$2)+x*(1+5*x)*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}} dx\right)}{(x+3)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 20.076 (sec). Leaf size: 50

```
DSolve[2*x^2*(3+x)*y''[x]+x*(1+5*x)*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6\sqrt[3]{3}c_2\sqrt[6]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{3},\frac{1}{6},\frac{7}{6},-\frac{x}{3}\right)+c_1\right)}{(x+3)^{4/3}}$$

2.93 problem 95

2.93.1 Maple step by step solution 885

Internal problem ID [7583]
Internal file name [`OUTPUT/6516_Sunday_June_05_2022_04_56_47_PM_21704608/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 95.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(4+x)\, y'' - x(-3x+1)\, y' + y = 0}$$

Writing the ode as

$$x^2(4+x)\, y'' + \left(3x^2 - x\right) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2 - x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 6x - 7$$
$$t = 4\left(x^2 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{128x} + \frac{5}{128\,(4+x)} - \frac{7}{64x^2} + \frac{65}{64\,(4+x)^2}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4\,(x^2+4x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{8\,(4 + x)} + \frac{1}{8x} + (-)\,(0) \\ &= -\frac{5}{8\,(4 + x)} + \frac{1}{8x} \\ &= -\frac{x - 1}{2x\,(4 + x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8(4+x)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2 - 6x - 7}{4(x^2+4x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\ &= \frac{x^{\frac{1}{8}}}{(4+x)^{\frac{5}{8}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2 - x}{x^2(4+x)} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \frac{13\ln(4+x)}{8}} \\ &= z_1 \left(\frac{x^{\frac{1}{8}}}{(4+x)^{\frac{13}{8}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{x^2(4+x)}dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-\frac{13\ln(4+x)}{4}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} \right) + c_2 \left(\frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)}{(4+x)^{\frac{9}{4}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)}{(4+x)^{\frac{9}{4}}}
$$

Verified OK.

2.93.1 Maple step by step solution

Let's solve

$x^2(4+x)\, y'' + (3x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x^2(4+x)} - \frac{(3x-1)y'}{x(4+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-1)y'}{x(4+x)} + \frac{y}{x^2(4+x)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3x-1}{x(4+x)}, P_3(x) = \frac{1}{x^2(4+x)}\right]$

- $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

  $\left((4+x) \cdot P_2(x)\right)\Big|_{x=-4} = \frac{13}{4}$

- $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

  $\left((4+x)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

- $x = -4$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -4$

- Multiply by denominators

  $x^2(4+x)\, y'' + x(3x-1)\, y' + y = 0$

- Change variables using $x = u - 4$ so that the regular singular point is at $u = 0$

  $(u^3 - 8u^2 + 16u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 25u + 52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(9+4r)\,u^{-1+r} + (4a_1(1+r)(13+4r) - a_0(8r^2+17r-1))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(4k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(9+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{9}{4}\right\}$$

- Each term must be 0

$$4a_1(1+r)(13+4r) - a_0(8r^2+17r-1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-8a_k + a_{k-1} + 16a_{k+1})\,k^2 + (2(-8a_k + a_{k-1} + 16a_{k+1})\,r - 17a_k + 68a_{k+1})\,k + (-8a_k + a_{k-1} +$$

- Shift index using $k->k+1$

$$(-8a_{k+1} + a_k + 16a_{k+2})\,(k+1)^2 + (2(-8a_{k+1} + a_k + 16a_{k+2})\,r - 17a_{k+1} + 68a_{k+2})\,(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2kra_k - 16kra_{k+1} + r^2a_k - 8r^2a_{k+1} + 2ka_k - 33ka_{k+1} + 2ra_k - 33ra_{k+1} - 24a_{k+1}}{4(4k^2+8kr+4r^2+25k+25r+34)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}, 52a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 4 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(4+x)^k, a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} + 2ka_k - 33ka_{k+1} - 24a_{k+1}}{4(4k^2+25k+34)}, 52a_1 + a_0 = 0\right]$$

- Recursion relation for $r = -\frac{9}{4}$

$$a_{k+2} = -\frac{k^2a_k - 8k^2a_{k+1} - \frac{5}{2}ka_k + 3ka_{k+1} + \frac{9}{16}a_k + \frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}$$

- Solution for $r = -\frac{9}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}-\frac{5}{2}k a_k+3k a_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Revert the change of variables $u=4+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (4+x)^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}-\frac{5}{2}k a_k+3k a_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (4+x)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k (4+x)^{k-\frac{9}{4}}\right), a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}+2k a_k-33k a_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(x^2*(4+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{(x+4)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(x+4)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx\right)}{(x+4)^{\frac{9}{4}}}$$

✓ Solution by Mathematica

Time used: 0.399 (sec). Leaf size: 89

```
DSolve[x^2*(4+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{\sqrt[4]{x}\left(-10c_2 \arctan\left(\sqrt[4]{\frac{x}{x+4}}\right) + 10c_2 \operatorname{arctanh}\left(\sqrt[4]{\frac{x}{x+4}}\right) + c_2\sqrt[4]{x+4}x^{7/4} + 9c_2\sqrt[4]{x+4}x^{3/4} + 2c_1\right)}{2(x+4)^{9/4}}$$

2.94 problem 96

2.94.1 Maple step by step solution 895

Internal problem ID [7584]
Internal file name [`OUTPUT/6517_Sunday_June_05_2022_04_56_50_PM_9484037/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 96.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + 5xy' + (1 + x)\, y = 0}$$

Writing the ode as

$$2x^2y'' + 5xy' + (1 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 - 8x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 - 8x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\}\\ &= \{1, 2, 3\}\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(1 - (1))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\ &= \frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{8x+1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-x}}{4x} dx} \\
&= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\,\sqrt{-x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\sqrt{2}\,\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{-x}}}{x} \left(\frac{\sqrt{2}\,\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{2}\,\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^{\sqrt{2}\sqrt{-x}}}{x} + \frac{c_2 \sqrt{2}\sqrt{-x}\left(-e^{\sqrt{2}\sqrt{-x}} + e^{-\sqrt{2}\sqrt{-x}}\right)}{2x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\sqrt{2}\sqrt{-x}}}{x} + \frac{c_2 \sqrt{2}\sqrt{-x}\left(-e^{\sqrt{2}\sqrt{-x}} + e^{-\sqrt{2}\sqrt{-x}}\right)}{2x^{\frac{3}{2}}}$$

Verified OK.

2.94.1 Maple step by step solution

Let's solve

$2x^2y'' + 5xy' + (1 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{2x} - \frac{(1+x)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} + \frac{(1+x)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1+x}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 5xy' + (1+x)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r+1) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+r+1)\,a_k + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2\left(k+\frac{3}{2}+r\right)(k+2+r)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(2k+3+2r)(k+2+r)}$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k}{(2k+1)(k+1)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(2k+1)(k+1)}, b_{k+1} = -\frac{b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right)}{x} + \frac{c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 60

```
DSolve[2*x^2*y''[x]+5*x*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

2.95 problem 97

2.95.1 Maple step by step solution 906

Internal problem ID [7585]
Internal file name [`OUTPUT/6518_Sunday_June_05_2022_04_56_53_PM_25584178/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 97.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2y'' + x(10 - x)\,y' - (x + 2)\,y = 0}$$

Writing the ode as

$$6x^2y'' + \left(-x^2 + 10x\right)y' + (-x - 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2 + 10x \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 28$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 28}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{36x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\
&= -\frac{1}{6x} - \frac{1}{12} \\
&= -\frac{x+2}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2 + 4x + 28}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{12}}}{x^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + 10x}{6x^2} \, dx} \\
&= z_1 e^{\frac{x}{12} - \frac{5 \ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{12}}}{x^{\frac{5}{6}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}}\, dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}}\, dx \right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}}\, dx \right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}}\, dx \right)}{x}$$

Verified OK.

2.95.1 Maple step by step solution

Let's solve

$6x^2y'' + (-x^2 + 10x)\, y' + (-x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+2)y}{6x^2} + \frac{(x-10)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-10)y'}{6x} - \frac{(x+2)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-10}{6x}, P_3(x) = -\frac{x+2}{6x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{3}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' - x(x - 10)\, y' + (-x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0(1+r)\,(-1+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(2a_k(k+r+1)\,(3k+3r-1) - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2(1+r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$6\left(k+r-\frac{1}{3}\right)(k+r+1)\, a_k - a_{k-1}(k+r) = 0$$

- Shift index using $k- > k+1$

  $$6\left(k+\frac{2}{3}+r\right)(k+2+r)\, a_{k+1} - a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r+1)}{2(3k+2+3r)(k+2+r)}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+1} = \frac{a_k\left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k k}{2(3k-1)(k+1)}, b_{k+1} = \frac{b_k \left(k+\frac{4}{3}\right)}{2(3k+3)\left(k+\frac{7}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(6*x^2*diff(y(x),x$2)+x*(10-x)*diff(y(x),x)-(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + \frac{c_2 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 38

```
DSolve[6*x^2*y''[x]+x*(10-x)*y'[x]-(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6} c_1}{x}$$

2.96 problem 98

2.96.1 Maple step by step solution 916

Internal problem ID [7586]
Internal file name [OUTPUT/6519_Sunday_June_05_2022_04_56_56_PM_62230507/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 98.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0}$$

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 48x^2 + 8x + 91 \\ t &= 4\left(4x^2 + 3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{176}{27x} + \frac{91}{36x^2} + \frac{28}{9\left(\frac{3}{4} + x\right)^2} + \frac{176}{27\left(\frac{3}{4} + x\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{6}
\end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(\frac{3}{4}+x\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{4}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\
&= -\frac{1}{2}-\left(-\frac{5}{2}\right)\\
&= 2
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)} + (-)\,(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)} \\
&= \frac{-7-20x}{8x^2+6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right)(2x+a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(\frac{3}{4}+x\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right)^2 - \left(\frac{48x^2+8x}{4\,(4x^2+3}\right.\right.
$$

$$
\frac{12a_1x - 8x + 32a_0 -}{x\,(3+4x)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) \mathrm{e}^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right)dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) \mathrm{e}^{-\frac{7\ln(x)}{6} - \frac{4\ln(3+4x)}{3}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^{\frac{7}{6}}\,(3+4x)^{\frac{4}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2}\,dx} \\
&= z_1 e^{-\frac{11\ln(x)}{6}+\frac{4\ln(3+4x)}{3}} \\
&= z_1\left(\frac{(3+4x)^{\frac{4}{3}}}{x^{\frac{11}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{2304x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\right) + c_2\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\left(\int \frac{2304x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)}{x^3} + \frac{c_2(2304x^2+1536x+336)\left(\int \frac{x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)}{x^3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)}{x^3} + \frac{c_2(2304x^2+1536x+336)\left(\int \frac{x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)}{x^3}$$

Verified OK.

2.96.1 Maple step by step solution

Let's solve

$(4x^3+3x^2)\,y''+(4x^2+11x)\,y'+(-3-4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x^2}-\frac{(11+4x)y'}{x(3+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(11+4x)y'}{x(3+4x)}-\frac{y}{x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11+4x}{x(3+4x)}, P_3(x)=-\frac{1}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{11}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-1$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2(3+4x)\,y''+x(11+4x)\,y'+(-3-4x)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r) (k + 1 - m + r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (3 + r) (-1 + 3r) x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k + r + 3) (3k + 3r - 1) + 4a_{k-1} (k + r) (k - 2 + r) \right) x^{k+r} \right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3 + r) (-1 + 3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-3, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k + r - \frac{1}{3}\right) (k + r + 3) a_k + 4a_{k-1} (k + r) (k - 2 + r) = 0$$

- Shift index using $k- >k + 1$

$$3\left(k + \frac{2}{3} + r\right) (k + 4 + r) a_{k+1} + 4a_k (k + r + 1) (k + r - 1) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(3k+2+3r)(k+4+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(3k-7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{32a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(x^2*(3+4*x)*diff(y(x),x$2)+x*(11+4*x)*diff(y(x),x)-(3+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(48x^2+32x+7)}{x^3} + \frac{c_2(48x^2+32x+7)\left(\int \frac{(4x+3)^{\frac{8}{3}} x^{\frac{7}{3}}}{(48x^2+32x+7)^2} dx\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 1.197 (sec). Leaf size: 339

```
DSolve[x^2*(3+4*x)*y''[x]+x*(11+4*x)*y'[x]-(3+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \rightarrow \frac{-12\sqrt[3]{2}\sqrt{3}c_2(48x^2+32x+7)\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{8x+6}}\right)+384c_2(4x+3)^{2/3}x^{10/3}+576c_2(4x+3)^{2/3}x^{7/3}}{}$$

2.97 problem 99

2.97.1 Maple step by step solution 926

Internal problem ID [7587]
Internal file name [`OUTPUT/6520_Sunday_June_05_2022_04_56_59_PM_77073462/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 99.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(3x+2)\,y''+x(4+11x)\,y'-(1-x)\,y=0}$$

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 11x^2+4x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{16\,(3x+2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -35$$
$$t = 16(3x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{16\,(3x+2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 179: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(3x+2)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{35}{144\left(x+\frac{2}{3}\right)^2}
$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{35}{16\left(3x+2\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{16 (3x+2)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} + (-)\,(0) \\ &= \frac{5}{12\left(x+\frac{2}{3}\right)} \\ &= \frac{5}{12x+8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{5}{12\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)^2 - \left(-\frac{35}{16\left(3x+2\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{5}{12\left(x+\frac{2}{3}\right)} dx} \\ &= (3x+2)^{\frac{5}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^2+4x}{6x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(3x+2)}{12}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\,(3x+2)^{\frac{5}{12}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(3x+2)}{6}}}{(y_1)^2}\,dx \\
&= y_1\left(2(3x+2)^{\frac{1}{6}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}}\right) + c_2\left(\frac{1}{\sqrt{x}}\left(2(3x+2)^{\frac{1}{6}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}}$$

Verified OK.

2.97.1 Maple step by step solution

Let's solve

$(6x^3+4x^2)\,y'' + (11x^2+4x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{2x^2(3x+2)} - \frac{(4+11x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(4+11x)y'}{2x(3x+2)} + \frac{(x-1)y}{2x^2(3x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{4+11x}{2x(3x+2)}, P_3(x) = \frac{x-1}{2x^2(3x+2)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(3x+2)\, y'' + x(4+11x)\, y' + (x-1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + a_{k-1}(2k+2r-1)(3k-2+3r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(\left(\frac{3k}{2} + \frac{3r}{2} - 1\right) a_{k-1} + a_k\left(k+r+\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$$

- Shift index using $k- >k+1$

$$4\left(\left(\frac{3k}{2} + \frac{1}{2} + \frac{3r}{2}\right) a_k + a_{k+1}\left(k+\frac{3}{2}+r\right)\right)\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{(3k+3r+1)a_k}{2k+3+2r}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 32

```
DSolve[2*x^2*(2+3*x)*y''[x]+x*(4+11*x)*y'[x]-(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{c_2 \sqrt[6]{6x+4} + 2^{5/6} c_1}{\sqrt{x}}$$

2.98 problem 100

2.98.1 Maple step by step solution 936

Internal problem ID [7588]
Internal file name [`OUTPUT/6521_Sunday_June_05_2022_04_57_01_PM_7123217/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 100.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x+2)\,y''+5x(1-x)\,y'-(-8x+2)\,y=0}$$

Writing the ode as

$$x^2(x+2)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x+2) \\ B &= -5x^2+5x \\ C &= 8x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 126x + 21$$
$$t = 4\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 181: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{147}{16x} + \frac{147}{16(x+2)} + \frac{285}{16(x+2)^2} + \frac{21}{16x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{9}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{15}{4\left(x+2\right)} - \frac{3}{4x} + (-)\left(0\right) \\
&= -\frac{15}{4\left(x+2\right)} - \frac{3}{4x} \\
&= -\frac{3(3x+1)}{2x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{15}{4\left(x+2\right)} - \frac{3}{4x}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \left(\left(\frac{15}{4\left(x+2\right)^2} + \frac{3}{4x^2}\right) + \left(-\frac{}{4}\right.\right.$$

$$\frac{3(4 + a_3)x^3 + (8a_2 + }{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{\int \left(-\frac{15}{4(x+2)} - \frac{3}{4x}\right) dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{-\frac{3\ln(x)}{4} - \frac{15\ln(x+2)}{4}} \\
&= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^{\frac{3}{4}}\left(x+2\right)^{\frac{15}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2+5x}{x^2(x+2)} \, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4} + \frac{15\ln(x+2)}{4}} \\
&= z_1 \left(\frac{\left(x+2\right)^{\frac{15}{4}}}{x^{\frac{5}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(x+2)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}+\frac{15\ln(x+2)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{80\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right) + \left(525x^4 - 2100x^3 + \frac{1575}{2}x^2 + 105\right.\right.}{\sqrt{x\,(x+2)}\,(40x^4 - 160x^3 + 60x^2 + 8x + 1} \right. \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\right) \\ &\quad + c_2\left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}\left(\frac{80\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right)\right.}{\sqrt{x\,(x+}}\right.\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{40x^2} \\ &\quad + \frac{2c_2\sqrt{x}\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right) + \left(525x^4 - 2100x^3 + \frac{1575}{2}x^2 + 10\right.\right.}{\sqrt{x\,(x+2)}\,\left(x + \sqrt{x\,(x+2)}\right)} \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{40x^2} + \frac{2c_2\sqrt{x}\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right) + \left(525x^4 - 2100x^3 + \frac{1575}{2}x^2 + 105\right.}{\sqrt{x\,(x+2)}\,\left(x + \sqrt{x\,(x+2)}\right)}$$

Verified OK.

2.98.1 Maple step by step solution

Let's solve

$x^2(x+2)\,y'' + (-5x^2 + 5x)\,y' + (8x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(4x-1)y}{x^2(x+2)} + \frac{5(x-1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5(x-1)y'}{x(x+2)} + \frac{2(4x-1)y}{x^2(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{5(x-1)}{x(x+2)}, P_3(x) = \frac{2(4x-1)}{x^2(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -\frac{15}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

  ◦ $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $x^2(x+2)\,y'' - 5x(x-1)\,y' + (8x-2)\,y = 0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $\left(u^{3}-4u^{2}+4u\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)+\left(-5u^{2}+25u-30\right)\left(\frac{d}{du}y(u)\right)+\left(8u-18\right)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..1$

    $u^{m}\cdot y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^{m}\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_{0}r(-17+2r)\,u^{-1+r}+\left(2a_{1}(1+r)\,(-15+2r)-a_{0}(4r^{2}-29r+18)\right)u^{r}+\left(\sum_{k=1}^{\infty}(2a_{k+1}(k+1+\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $2r(-17+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{17}{2}\right\}$

- Each term must be 0

$2a_1(1+r)(-15+2r)-a_0(4r^2-29r+18)=0$

- Each term in the series must be 0, giving the recursion relation

$(-4a_k+a_{k-1}+4a_{k+1})k^2+((-8a_k+2a_{k-1}+8a_{k+1})r+29a_k-8a_{k-1}-26a_{k+1})k+(-4a_k+a$

- Shift index using $k->k+1$

$(-4a_{k+1}+a_k+4a_{k+2})(k+1)^2+((-8a_{k+1}+2a_k+8a_{k+2})r+29a_{k+1}-8a_k-26a_{k+2})(k+1)$ -

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+2kra_k-8kra_{k+1}+r^2a_k-4r^2a_{k+1}-6ka_k+21ka_{k+1}-6ra_k+21ra_{k+1}+8a_k+7a_{k+1}}{2(2k^2+4kr+2r^2-9k-9r-26)}$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}$

- Solution for $r=0$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30a_1-18a_0=0\right]$

- Revert the change of variables $u=x+2$

$\left[y=\sum_{k=0}^{\infty}a_k(x+2)^k, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30a_1-18a_0=0\right]$

- Recursion relation for $r=\frac{17}{2}$

$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}$

- Solution for $r=\frac{17}{2}$

$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{17}{2}}, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1-\frac{121a_0}{2}=0\right]$

- Revert the change of variables $u=x+2$

$\left[y=\sum_{k=0}^{\infty}a_k(x+2)^{k+\frac{17}{2}}, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1-\frac{121a_0}{2}=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_k(x+2)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+\frac{17}{2}}\right), a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 88

```
dsolve(x^2*(2+x)*diff(y(x),x$2)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{x^2} + \frac{c_2(40x^4 - 160x^3 + 60x^2 + 8x + 1)\left(\int \frac{x^{\frac{3}{2}}(x+2)^{\frac{15}{2}}}{(40x^4-160x^3+60x^2+8x+1)^2}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 48.622 (sec). Leaf size: 1347

```
DSolve[x^2*(2+x)*y''[x]+5*x*(1-x)*y'[x]-(2-8*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

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2.99 problem 101

2.99.1 Maple step by step solution 947

Internal problem ID [7589]
Internal file name [`OUTPUT/6522_Sunday_June_05_2022_04_57_07_PM_8444003/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 101.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^2(1-x^2)\,y''+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0}$$

Writing the ode as

$$(-8x^4+8x^2)\,y''+(-26x^3+2x)\,y'+(-9x^2+1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-8x^4+8x^2\\B&=-26x^3+2x\\C&=-9x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7x^4 - 26x^2 - 15$$
$$t = 64\left(x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 183: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(1+x)^2} - \frac{3}{16\,(x-1)^2} - \frac{15}{64x^2} + \frac{1}{4x-4} - \frac{1}{4\,(1+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \\
&= \frac{7x^2 - 3}{8x^3 - 8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(1+x)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \right) dx} \\
&= x^{\frac{3}{8}} (x - 1)^{\frac{1}{4}} (1 + x)^{\frac{1}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-26x^3 + 2x}{-8x^4 + 8x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{8} - \frac{3\ln(1+x)}{4} - \frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{8}} (1 + x)^{\frac{3}{4}} (x - 1)^{\frac{3}{4}}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{4}-\frac{3\ln(1+x)}{2}-\frac{3\ln(x-1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}\right) + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}$$

Verified OK.

2.99.1 Maple step by step solution

Let's solve

$(-8x^4+8x^2)\,y''+(-26x^3+2x)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x^2-1)y}{8x^2(x^2-1)}-\frac{(13x^2-1)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(13x^2-1)y'}{4x(x^2-1)}+\frac{(9x^2-1)y}{8x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)},P_3(x)=\frac{9x^2-1}{8x^2(x^2-1)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=\frac{3}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $8y''x^2(x^2-1)+2x(13x^2-1)\,y'+y(9x^2-1)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(8u^4-32u^3+40u^2-16u)\left(\frac{d^2}{du^2}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)\,y$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..3$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..4$

  $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-8a_0 r(1+2r)\, u^{-1+r} + (-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r))\, u^r + (-8a_2(2+r)\,(5+2r)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-8r(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$
- The coefficients of each power of u must be 0

  $[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) -$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = \frac{a_0(10r^2+9r+2)}{2(2r^2+5r+3)}, a_2 = \frac{a_0(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 18a_k - 7a_{k-2} + 9a_{k-1}$

- Shift index using $k- >k+2$

  $8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})\left(k+2\right)^2+2(8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})r+18a_{k+2}-7a_k+9$
- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+16kra_k-64kra_{k+1}+80kra_{k+2}+8r^2a_k-32r^2a_{k+1}+40r^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+}}{8(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r=0$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\frac{a_0}{3}\right.$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\right.$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$
- Solution for $r=-\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1=\right.$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka}{8(2k^2+13k+21)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
dsolve(8*x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-13*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x), sings
```

$$y(x) = c_1\sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}} + c_2\sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}}\left(\int \frac{\sqrt{\frac{1}{(x-1)(x+1)}}}{x^{\frac{3}{4}}}dx\right)$$

✓ Solution by Mathematica

Time used: 20.097 (sec). Leaf size: 47

```
DSolve[8*x^2*(1-x^2)*y''[x]+2*x*(1-13*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{\sqrt[4]{x}\left(4c_2\sqrt[4]{x}\,\text{Hypergeometric2F1}\left(\frac{1}{8},\frac{1}{2},\frac{9}{8},x^2\right)+c_1\right)}{\sqrt{1-x^2}}$$

2.100 problem 102

2.100.1 Maple step by step solution 958

Internal problem ID [7590]
Internal file name [OUTPUT/6523_Sunday_June_05_2022_04_57_10_PM_39121908/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 102.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''-2x(-x^2+2)\,y'+4y=0}$$

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2+2}{(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2+2 \\ t &= (x^3+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+2}{(x^3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 185: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{7i}{4(x-i)} - \frac{7i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+2}{(x^3+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} + (-)\,(0) \\
&= \frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2\,(x-i)^2} + \frac{1}{2\,(x+i)^2}\right) + \left(\frac{2}{x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{x^2}{\sqrt{x^2+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-4x}{x^4+x^2}\, dx} \\
&= z_1 e^{2\ln(x) - \frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{x^2}{\left(x^2+1\right)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^4}{(x^2+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)-3\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-3x^2-1}{3x^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^4}{(x^2+1)^2}\right) + c_2 \left(\frac{x^4}{(x^2+1)^2}\left(\frac{-3x^2-1}{3x^3}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 x^4}{(x^2+1)^2} + \frac{c_2(-3x^3-x)}{3(x^2+1)^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 x^4}{(x^2+1)^2} + \frac{c_2(-3x^3-x)}{3(x^2+1)^2}$$

Verified OK.

2.100.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (2x^3 - 4x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{2(x^2-2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x^2-2)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(x^2-2)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -4$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 4$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1) y'' + 2x(x^2 - 2) y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)\,(-4+r)\, x^r + a_1 r(-3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-4) + a_{k-2}(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)\,(-4+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{1, 4\}$$

- Each term must be 0

  $$a_1 r(-3+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-4) + a_{k-2}(k-2+r)) = 0$$

- Shift index using $k->k+2$

  $$(k+r+1)\,(a_{k+2}(k-2+r) + a_k(k+r)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)}{k-2+r}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k(k+1)}{k-1}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0\right]$$

- Recursion relation for $r = 4$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 4$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0, b_{k+2} = -\frac{b_k(k+4)}{k+2}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-2*x*(2-x^2)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x(3x^2 + 1)}{(x^2 + 1)^2} + \frac{c_2 x^4}{(x^2 + 1)^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 35

```
DSolve[x^2*(1+x^2)*y''[x]-2*x*(2-x^2)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{-3c_1x^4 + 3c_2x^3 + c_2x}{3(x^2+1)^2}$$

2.101 problem 103

2.101.1 Maple step by step solution 968

Internal problem ID [7591]
Internal file name [`OUTPUT/6524_Sunday_June_05_2022_04_57_13_PM_52086247/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 103.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+3)\,y''+(-x^2+2)\,y'-8yx=0}$$

Writing the ode as

$$(x^3+3x)\,y''+(-x^2+2)\,y'-8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+3x \\ B &= -x^2+2 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 74x^2 - 8$$
$$t = 4\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 187: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2} + \frac{85}{144 \left(x - i\sqrt{3}\right)^2} + \frac{85}{144 \left(x + i\sqrt{3}\right)^2} - \frac{187 i\sqrt{3}}{144 \left(x - i\sqrt{3}\right)} + \frac{187 i\sqrt{3}}{144 \left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3} \end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{17}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{12} \end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)} + (0) \\
&= \frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)} \\
&= \frac{2}{3x} + \frac{17x}{6x^2 + 18}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)}\right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x - i\sqrt{3}\right)^2} - \frac{17}{12\left(x + i\sqrt{3}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)}\right) dx} \\
&= x^{\frac{2}{3}} \left(x^2 + 3\right)^{\frac{17}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^3+3x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{3} + \frac{5\ln\left(x^2+3\right)}{12}} \\
&= z_1 \left(\frac{\left(x^2+3\right)^{\frac{5}{12}}}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{1}{3}} \left(x^2 + 3\right)^{\frac{11}{6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{2\ln(x)}{3} + \frac{5\ln\left(x^2+3\right)}{6}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{-8x^4 - 44x^2 - 55}{55x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{3}}\left(x^2+3\right)^{\frac{11}{6}}\right) + c_2\left(x^{\frac{1}{3}}\left(x^2+3\right)^{\frac{11}{6}}\left(\frac{-8x^4-44x^2-55}{55x^{\frac{1}{3}}\left(x^2+3\right)^{\frac{11}{6}}}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}}\left(x^2+3\right)^{\frac{11}{6}} + c_2\left(-\frac{8}{55}x^4 - \frac{4}{5}x^2 - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}}\left(x^2+3\right)^{\frac{11}{6}} + c_2\left(-\frac{8}{55}x^4 - \frac{4}{5}x^2 - 1\right)
$$

Verified OK.

2.101.1 Maple step by step solution

Let's solve

$\left(x^3+3x\right)y'' + \left(-x^2+2\right)y' - 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y'}{x(x^2+3)} + \frac{8y}{x^2+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-2)y'}{x(x^2+3)} - \frac{8y}{x^2+3} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-2}{x(x^2+3)}, P_3(x) = -\frac{8}{x^2+3}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+3)\, y'' + (-x^2+2)\, y' - 8yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+3r)\, x^{-1+r} + a_1(1+r)\,(2+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(3k+2+3r) + a_{k-1}(k+r-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, \frac{1}{3}\}$

- Each term must be 0

  $a_1(1+r)(2+3r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $\left(a_{k-1}(k-5+r) + 3a_{k+1}\left(k+\frac{2}{3}+r\right)\right)(k+r+1) = 0$

- Shift index using $k- >k+1$

  $\left(a_k(k+r-4) + 3a_{k+2}\left(k+\frac{5}{3}+r\right)\right)(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r-4)}{3k+5+3r}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+2} = -\frac{a_k(k-4)}{3k+5}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0\right]$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}$

- Solution for $r=\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k-\frac{11}{3}\right)}{3k+6}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k(k-4)}{3k+5}, 2a_1 = 0, b_{k+2} = -\frac{b_k\left(k-\frac{11}{3}\right)}{3k+6}, 4b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*(3+x^2)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^4 + \frac{11}{2}x^2 + \frac{55}{8}\right) + c_2(x^2+3)^{\frac{11}{6}} x^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 1.51 (sec). Leaf size: 41

```
DSolve[x*(3+x^2)*y''[x]+(2-x^2)*y'[x]-8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt[3]{x}(x^2+3)^{11/6} - \frac{1}{55}c_2(8x^4+44x^2+55)$$

2.102 problem 104

2.102.1 Maple step by step solution 978

Internal problem ID [7592]
Internal file name [OUTPUT/6525_Sunday_June_05_2022_04_57_15_PM_9005042/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 104.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1-x^2)\,y''+x(-19x^2+7)\,y'-(14x^2+1)\,y=0}$$

Writing the ode as

$$(-4x^4+4x^2)\,y''+(-19x^3+7x)\,y'+(-14x^2-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^4+4x^2 \\ B &= -19x^3+7x \\ C &= -14x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -15x^4 - 42x^2 + 9$$
$$t = 64\left(x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 189: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(1+x)^2} - \frac{3}{16(x-1)^2} + \frac{9}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{8} - \left(\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} + (-)(0) \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \\
&= \frac{3x^2 + 1}{8x^3 - 8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4(x - 1)^2} - \frac{1}{4(1 + x)^2} \right) + \left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x +}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \right) dx} \\
&= \frac{(1 + x)^{\frac{1}{4}} (x - 1)^{\frac{1}{4}}}{x^{\frac{1}{8}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-19x^3+7x}{-4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{8}-\frac{3\ln(1+x)}{4}-\frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{7}{8}}\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3+7x}{-4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x)}{4}-\frac{3\ln(1+x)}{2}-\frac{3\ln(x-1)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}} + \frac{c_2\left(x^2-1\right)^{\frac{1}{4}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2-1)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}} + \frac{c_2(x^2-1)^{\frac{1}{4}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}$$

Verified OK.

2.102.1 Maple step by step solution

Let's solve

$(-4x^4+4x^2)\,y''+(-19x^3+7x)\,y'+(-14x^2-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(14x^2+1)y}{4x^2(x^2-1)}-\frac{(19x^2-7)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(19x^2-7)y'}{4x(x^2-1)}+\frac{(14x^2+1)y}{4x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{19x^2-7}{4x(x^2-1)}, P_3(x)=\frac{14x^2+1}{4x^2(x^2-1)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{3}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4y''x^2(x^2-1)+x(19x^2-7)\,y'+(14x^2+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$\left(4u^4-16u^3+20u^2-8u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(19u^3-57u^2+50u-12\right)\left(\frac{d}{du}y(u)\right)+\left(14u^2-28u+15\right)$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0r(1+2r)\,u^{-1+r}+\left(-4a_1(1+r)\,(3+2r)+5a_0(4r^2+6r+3)\right)u^r+\left(-4a_2(2+r)\,(5+2r)+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

$$\left[-4a_1(1+r)\,(3+2r)+5a_0(4r^2+6r+3)=0,-4a_2(2+r)\,(5+2r)+5a_1(4r^2+14r+13)-a_0\right.$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})r + 30a_k - a_{k-2} - 9a_{k-1} - 2$$

- Shift index using $k- >k+2$

$$4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})r + 30a_{k+2} - a_k - 9a_k$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 8kra_k - 32kra_{k+1} + 40kra_{k+2} + 4r^2a_k - 16r^2a_{k+1} + 20r^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}, a_1 = \frac{5a}{4}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_{k+2} + 14a_k - 85a_{k+1} + 155a_{k+2}}{4(2k^2+13k+21)}, a_1 =\right.$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}, a_1 =\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 11ka_k - 57ka_{k+1} + 90ka_{k+2} + \frac{15}{2}a_k - \frac{105}{2}a_{k+1} + 105a_{k+2}}{4(2k^2+11k+15)}, a\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{4k^2a_k - 16k^2a_{k+1} + 20k^2a_{k+2} + 15ka_k - 73ka_{k+1} + 110ka_k}{4(2k^2+13k+21)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(4*x^2*(1-x^2)*diff(y(x),x$2)+x*(7-19*x^2)*diff(y(x),x)-(1+14*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1\sqrt{\frac{1}{(x-1)(x+1)}}}{x} + \frac{c_2\sqrt{\frac{1}{(x-1)(x+1)}}\left(\int \sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 20.113 (sec). Leaf size: 50

```
DSolve[4*x^2*(1-x^2)*y''[x]+x*(7-19*x^2)*y'[x]-(1+14*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{4c_2 x^{5/4} \text{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{8}, \frac{13}{8}, x^2\right) + 5c_1}{5x\sqrt{1-x^2}}$$

2.103 problem 105

2.103.1 Maple step by step solution 989

Internal problem ID [7593]
Internal file name [`OUTPUT/6526_Sunday_June_05_2022_04_57_18_PM_37518853/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 105.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2(-x^2+2)\,y''+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0}$$

Writing the ode as

$$\left(-3x^4+6x^2\right)y''+\left(-11x^3+x\right)y'+\left(-5x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-3x^4+6x^2\\B&=-11x^3+x\\C&=-5x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5x^4 - 4x^2 - 35 \\ t &= 36\left(x^3 - 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 191: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2} - \frac{7}{64\left(x - \sqrt{2}\right)^2} - \frac{7}{64\left(x + \sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x - \sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} + (0) \\
&= \frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} \\
&= \frac{5x^2 - 7}{6x^3 - 12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} \right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x - \sqrt{2}\right)^2} - \frac{1}{8\left(x + \sqrt{2}\right)^2} \right) + \left(\frac{7}{12x} + \frac{}{8x} \right. \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} + \frac{1}{8x - 8\sqrt{2}} + \frac{1}{8x + 8\sqrt{2}} \right) dx} \\
&= x^{\frac{7}{12}} \left(x - \sqrt{2} \right)^{\frac{1}{8}} \left(x + \sqrt{2} \right)^{\frac{1}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^3+x}{-3x^4+6x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{12}-\frac{7\ln\left(x^2-2\right)}{8}} \\
&= z_1\left(\frac{1}{x^{\frac{1}{12}}\left(x^2-2\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln\left(x^2-2\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{1}{x^{\frac{7}{6}}\left(x^2-2\right)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}\left(\int \frac{1}{x^{\frac{7}{6}}\left(x^2-2\right)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}}{(x^2-2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{x^{\frac{7}{6}}(x^2-2)^{\frac{1}{4}}}dx\right)}{(x^2-2)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{(x^2-2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{x^{\frac{7}{6}}(x^2-2)^{\frac{1}{4}}}dx\right)}{(x^2-2)^{\frac{3}{4}}}$$

Verified OK.

2.103.1 Maple step by step solution

Let's solve

$(-3x^4+6x^2)\,y''+(-11x^3+x)\,y'+(-5x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5x^2-1)y}{3x^2(x^2-2)}-\frac{(11x^2-1)y'}{3x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(11x^2-1)y'}{3x(x^2-2)}+\frac{(5x^2-1)y}{3x^2(x^2-2)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11x^2-1}{3x(x^2-2)}, P_3(x)=\frac{5x^2-1}{3x^2(x^2-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{1}{6}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{1}{6}$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3y''x^2(x^2-2) + x(11x^2-1)\,y' + (5x^2-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-a_0(-1+3r)\,(-1+2r)\,x^r - a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(3k+3r-1)\,(2k+2r-1) +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(-1+3r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $-a_1(2+3r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-6\left(k+r-\frac{1}{3}\right)\left(\frac{(-k-r+1)a_{k-2}}{2}+a_k\left(k+r-\frac{1}{2}\right)\right) = 0$

- Shift index using $k->k+2$

  $-6\left(k+\frac{5}{3}+r\right)\left(\frac{(-k-1-r)a_k}{2}+a_{k+2}\left(k+\frac{3}{2}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{(k+r+1)a_k}{2k+3+2r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1 = 0, b_{k+2} = \frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(3*x^2*(2-x^2)*diff(y(x),x$2)+x*(1-11*x^2)*diff(y(x),x)+(1-5*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2-2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{(x^2-2)^{\frac{1}{4}} x^{\frac{7}{6}}}dx\right)}{(x^2-2)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 20.112 (sec). Leaf size: 57

```
DSolve[3*x^2*(2-x^2)*y''[x]+x*(1-11*x^2)*y'[x]+(1-5*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{c_1\sqrt{x} - 3\ 2^{3/4} c_2 \sqrt[3]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{12}, \frac{1}{4}, \frac{11}{12}, \frac{x^2}{2}\right)}{(2-x^2)^{3/4}}$$

2.104 problem 106

2.104.1 Maple step by step solution 1000

Internal problem ID [7594]
Internal file name [`OUTPUT/6527_Sunday_June_05_2022_04_57_21_PM_86068181/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 106.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+2)\,y''-x(-7x^2+12)\,y'+(3x^2+7)\,y=0}$$

Writing the ode as

$$(2x^4+4x^2)\,y''+(7x^3-12x)\,y'+(3x^2+7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 72x^2 + 128$$
$$t = 16\left(x^3 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{65}{64\left(x - i\sqrt{2}\right)^2} + \frac{65}{64\left(x + i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{8} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{2}{x} - \frac{5x}{4x^2+8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x - i\sqrt{2}\right)^2} + \frac{5}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \frac{x^2}{\left(x^2+2\right)^{\frac{5}{8}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3-12x}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{13\ln\left(x^2+2\right)}{8}+\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{x^{\frac{3}{2}}}{\left(x^2+2\right)^{\frac{13}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3-12x}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{13\ln\left(x^2+2\right)}{4}+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\left(x^2+2\right)^{\frac{5}{4}}}{x^4}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}\right) + c_2\left(\frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}\left(\int \frac{\left(x^2+2\right)^{\frac{5}{4}}}{x^4}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}}$$

Verified OK.

2.104.1 Maple step by step solution

Let's solve

$(2x^4+4x^2)\, y'' + (7x^3-12x)\, y' + (3x^2+7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2+7)y}{2x^2(x^2+2)} - \frac{(7x^2-12)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(7x^2-12)y'}{2x(x^2+2)} + \frac{(3x^2+7)y}{2x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2-12}{2(x^2+2)x}, P_3(x) = \frac{3x^2+7}{2x^2(x^2+2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{7}{4}$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $2x^2(x^2+2)\,y'' + x(7x^2-12)\,y' + (3x^2+7)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-7+2r)\,x^r + a_1(1+2r)\,(-5+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7) + a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-7+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)(-5+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2}+a_k\left(k+r-\frac{7}{2}\right)\right)=0$
- Shift index using $k->k+2$

  $4\left(k+\frac{3}{2}+r\right)\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k-\frac{3}{2}+r\right)\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+1)}{2k-3+2r}$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0\right]$
- Recursion relation for $r=\frac{7}{2}$

  $a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$
- Solution for $r=\frac{7}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{7}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)-x*(12-7*x^2)*diff(y(x),x)+(7+3*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}}$$

✓ Solution by Mathematica

Time used: 20.114 (sec). Leaf size: 57

```
DSolve[2*x^2*(2+x^2)*y''[x]-x*(12-7*x^2)*y'[x]+(7+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{\sqrt{x}\left(3c_1x^3 - 2\sqrt[4]{2}c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}, -\frac{x^2}{2}\right)\right)}{3\left(x^2+2\right)^{9/4}}$$

2.105 problem 107

2.105.1 Maple step by step solution 1011

Internal problem ID [7595]
Internal file name [`OUTPUT/6528_Sunday_June_05_2022_04_57_24_PM_45001831/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 107.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+2)\,y''+x(7x^2+4)\,y'-(-3x^2+1)\,y=0}$$

Writing the ode as

$$\left(2x^4+4x^2\right)y''+\left(7x^3+4x\right)y'+\left(3x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3+4x \\ C &= 3x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 + 24 \\ t &= 16\left(x^2 + 2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 24}{16\left(x^2 + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64\left(x-i\sqrt{2}\right)^2} - \frac{15}{64\left(x+i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x-i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x+i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2+24}{16\left(x^2+2\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2+24}{16\left(x^2+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{3}{4}-\left(\frac{3}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)} + (0) \\ &= \frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)} \\ &= \frac{3x}{4x^2+8}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{3}{8\left(x-i\sqrt{2}\right)^2} - \frac{3}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{3}{8\left(x-i\sqrt{2}\right)} + \frac{3}{8\left(x+i\sqrt{2}\right)}\right) dx} \\ &= \left(x^2+2\right)^{\frac{3}{8}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^3+4x}{2x^4+4x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln\left(x^2+2\right)}{8} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\left(x^2+2\right)^{\frac{5}{8}} \sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{5\ln(x^2+2)}{4}-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{1}{(x^2+2)^{\frac{3}{4}}}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}}\right) + c_2 \left(\frac{1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}} \left(\int \frac{1}{(x^2+2)^{\frac{3}{4}}}dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}} + \frac{c_2\left(\int \frac{1}{(x^2+2)^{\frac{3}{4}}}dx\right)}{(x^2+2)^{\frac{1}{4}}\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}} + \frac{c_2\left(\int \frac{1}{(x^2+2)^{\frac{3}{4}}}dx\right)}{(x^2+2)^{\frac{1}{4}}\sqrt{x}}$$

Verified OK.

2.105.1 Maple step by step solution

Let's solve

$(2x^4 + 4x^2) y'' + (7x^3 + 4x) y' + (3x^2 - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2-1)y}{2x^2(x^2+2)} - \frac{(7x^2+4)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(7x^2+4)y'}{2x(x^2+2)} + \frac{(3x^2-1)y}{2x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x^2+4}{2(x^2+2)x}, P_3(x) = \frac{3x^2-1}{2x^2(x^2+2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2(x^2 + 2) y'' + x(7x^2 + 4) y' + (3x^2 - 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$
- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $$x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$$
- Shift index using $k->k+2-m$

  $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(1+2r)\,(-1+2r)\, x^{r}+a_{1}(3+2r)\,(1+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(2k+2r+1)\,(2k+2r-1)+a_{k-2}\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-1+2r)=0$$
- Values of r that satisfy the indicial equation

  $$r\in\left\{-\frac{1}{2}, \frac{1}{2}\right\}$$
- Each term must be 0

  $$a_{1}(3+2r)\,(1+2r)=0$$
- Solve for the dependent coefficient(s)

  $$a_{1}=0$$
- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{1}{2}\right)\left(\frac{a_{k-2}(k+r-1)}{2}+a_{k}\left(k+r+\frac{1}{2}\right)\right)=0$$
- Shift index using $k->k+2$

  $$4\left(k+\frac{3}{2}+r\right)\left(\frac{a_{k}(k+r+1)}{2}+a_{k+2}\left(k+\frac{5}{2}+r\right)\right)=0$$
- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+x*(4+7*x^2)*diff(y(x),x)-(1-3*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1}{(x^2+2)^{\frac{1}{4}}\sqrt{x}} + \frac{c_2\left(\int \frac{1}{(x^2+2)^{\frac{3}{4}}}dx\right)}{(x^2+2)^{\frac{1}{4}}\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 68

```
DSolve[2*x^2*(2+x^2)*y''[x]+x*(4+7*x^2)*y'[x]-(1-3*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{c_2 \sqrt[8]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1}{\sqrt{x}\sqrt[4]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right)}$$

2.106 problem 108

2.106.1 Maple step by step solution 1022

Internal problem ID [7596]
Internal file name [`OUTPUT/6529_Sunday_June_05_2022_04_57_27_PM_57797146/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 108.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'-(-40x^2+2)\,y=0}$$

Writing the ode as

$$(4x^4+2x^2)\,y''+(30x^3+5x)\,y'+(40x^2-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+2x^2 \\ B &= 30x^3+5x \\ C &= 40x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 20x^4 + 12x^2 + 21 \\ t &= 16(2x^3 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20x^4 + 12x^2 + 21}{16\,(2x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} + (0) \\
&= \frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \\
&= \frac{10x^2 + 7}{8x^3 + 4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{7}{4x} - \right. \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{x^{\frac{7}{4}} 2^{\frac{3}{4}}}{2\left(2x^2 + 1\right)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{30x^3+5x}{4x^4+2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x\left(2x^2+1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3+x\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(2x^3+x\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\sqrt{2}\,\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}}\right) + c_2 \left(\frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}}\left(\int \frac{\sqrt{2}\,\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2 (2x^2+1)^{\frac{5}{4}} (2x^3+x)^{\frac{1}{4}}} + \frac{c_2 x^{\frac{3}{4}} 2^{\frac{1}{4}} \left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}} dx\right)}{(2x^2+1)^{\frac{5}{4}} (2x^3+x)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2 (2x^2+1)^{\frac{5}{4}} (2x^3+x)^{\frac{1}{4}}} + \frac{c_2 x^{\frac{3}{4}} 2^{\frac{1}{4}} \left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}} dx\right)}{(2x^2+1)^{\frac{5}{4}} (2x^3+x)^{\frac{1}{4}}}$$

Verified OK.

2.106.1 Maple step by step solution

Let's solve

$(4x^4+2x^2) y'' + (30x^3+5x) y' + (40x^2-2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(20x^2-1)y}{x^2(2x^2+1)} - \frac{5(6x^2+1)y'}{2x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5(6x^2+1)y'}{2x(2x^2+1)} + \frac{(20x^2-1)y}{x^2(2x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{2}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'+(40x^2-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..4$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+2r)\,x^r+a_1(3+r)\,(1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1)+2a_{k-2}(k -$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3+r)(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(a_{k-2}(2k+1+2r)+a_k\left(k+r-\frac{1}{2}\right)\right)(k+r+2)=0$

- Shift index using $k->k+2$

  $2\left(a_k(2k+2r+5)+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)(k+r+4)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_k(2k+2r+5)}{2k+3+2r}$

- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0, b_{k+2}=-\frac{2b_k(2k+6)}{2k+4}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(2*x^2*(1+2*x^2)*diff(y(x),x$2)+5*x*(1+6*x^2)*diff(y(x),x)-(2-40*x^2)*y(x)=0,y(x), sin
```

$$y(x) = \frac{c_1\sqrt{x}}{(2x^2+1)^{\frac{3}{2}}} + \frac{c_2\sqrt{x}\left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)}{(2x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 20.118 (sec). Leaf size: 52

```
DSolve[2*x^2*(1+2*x^2)*y''[x]+5*x*(1+6*x^2)*y'[x]-(2-40*x^2)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \rightarrow \frac{5c_1x^{5/2} - 2c_2\,\text{Hypergeometric2F1}\left(-\frac{5}{4}, -\frac{1}{2}, -\frac{1}{4}, -2x^2\right)}{5x^2\left(2x^2+1\right)^{3/2}}$$

2.107 problem 109

2.107.1 Maple step by step solution 1033

Internal problem ID [7597]
Internal file name [OUTPUT/6530_Sunday_June_05_2022_04_57_29_PM_54050474/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 109.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+1)\,y''+(7x^2+4)\,y'+8yx=0}$$

Writing the ode as

$$(x^3+x)\,y''+(7x^2+4)\,y'+8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^3+x\\ B&=7x^2+4\\ C&=8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 14x^2 + 8}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 + 14x^2 + 8$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 14x^2 + 8}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 199: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)\,(0) \\ &= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\ &= -\frac{1}{x} + \frac{x}{2x^2+2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\ &= \frac{\left(x^2+1\right)^{\frac{1}{4}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\,dx} \\
&= z_1 e^{-2\ln(x)-\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^2\left(x^2+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x^2+1}\,x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{x^3-\operatorname{arcsinh}\left(x\right)\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x^2+1}\,x^3}\right) + c_2\left(\frac{1}{\sqrt{x^2+1}\,x^3}\left(\frac{x^3-\operatorname{arcsinh}\left(x\right)\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(x^3-\operatorname{arcsinh}\left(x\right)\sqrt{x^2+1}+x\right)}{2\left(x^2+1\right)x^3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(x^3 - \operatorname{arcsinh}(x)\sqrt{x^2+1} + x\right)}{2(x^2+1)\,x^3}$$

Verified OK.

2.107.1 Maple step by step solution

Let's solve

$(x^3+x)\,y'' + (7x^2+4)\,y' + 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8y}{x^2+1} - \frac{(7x^2+4)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(7x^2+4)y'}{x(x^2+1)} + \frac{8y}{x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{7x^2+4}{x(x^2+1)}, P_3(x) = \frac{8}{x^2+1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\,y'' + (7x^2+4)\,y' + 8yx = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k->k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+r+4) + a_{k-1}(k+r+3)(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$a_1(1+r)(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+1)\left(a_{k+1}(k+r+4) + a_{k-1}(k+r+3)\right) = 0$$

- Shift index using $k->k+1$

$$(k+r+2)\left(a_{k+2}(k+5+r)+a_k(k+r+4)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+4)}{k+5+r}$$

- Recursion relation for $r=-3$

$$a_{k+2}=-\frac{a_k(k+1)}{k+2}$$

- Solution for $r=-3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{a_k(k+4)}{k+5}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{a_k(k+4)}{k+5}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0, b_{k+2}=-\frac{b_k(k+4)}{k+5}, 4b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(x*(1+x^2)*diff(y(x),x$2)+(4+7*x^2)*diff(y(x),x)+8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(\frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2}\right)}{\sqrt{x^2+1}\,x^3}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 56

```
DSolve[x*(1+x^2)*y''[x]+(4+7*x^2)*y'[x]+8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x\sqrt{x^2+1} + c_2\log\left(\sqrt{x^2+1} - x\right) + 2c_1}{2x^3\sqrt{x^2+1}}$$

2.108 problem 110

2.108.1 Maple step by step solution 1043

Internal problem ID [7598]
Internal file name [`OUTPUT/6531_Sunday_June_05_2022_04_57_32_PM_14255660/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 110.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0}$$

Writing the ode as

$$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^2 \\ B &= 8x^3+3x \\ C &= 4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^2 + 21}{16(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^2 + 21$$
$$t = 16(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^2 + 21}{16(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} - \frac{15}{64(x-i)^2} - \frac{15}{64(x+i)^2} + \frac{27i}{64(x-i)} - \frac{27i}{64(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} + (0) \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} \\
&= -\frac{3}{4x(x^2+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8(x-i)^2} - \frac{3}{8(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{}{8}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{8}}}{x^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^3 + 3x}{2x^4 + 2x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{5\ln(x^2+1)}{8}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}} (x^2+1)^{\frac{5}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}\right) + c_2 \left(\frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

Verified OK.

2.108.1 Maple step by step solution

Let's solve

$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{\left(4x^2-3\right)y}{2x^2\left(x^2+1\right)}-\frac{\left(8x^2+3\right)y'}{2x\left(x^2+1\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{\left(8x^2+3\right)y'}{2x\left(x^2+1\right)}+\frac{\left(4x^2-3\right)y}{2x^2\left(x^2+1\right)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{8x^2+3}{2x\left(x^2+1\right)}, P_3(x)=\frac{4x^2-3}{2x^2\left(x^2+1\right)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{3}{2}$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2x^2\left(x^2+1\right)y''+x\left(8x^2+3\right)y'+\left(4x^2-3\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2r+3)\,(-1+r)\, x^r + a_1(5+2r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)\,(k+r-1) + 2a_{k-2}(k+r)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2r+3)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(5+2r)\, r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r+\frac{3}{2}\right) a_k + a_{k-2}(k+r)\right)(k+r-1) = 0$

- Shift index using $k->k+2$

  $2\left(\left(k+\frac{7}{2}+r\right) a_{k+2} + a_k(k+r+2)\right)(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(k+r+2)}{2k+7+2r}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k\left(k+\frac{1}{2}\right)}{2k+4}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(2*x^2*(1+x^2)*diff(y(x),x$2)+x*(3+8*x^2)*diff(y(x),x)-(3-4*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1}{(x^2+1)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{(x^2+1)^{\frac{3}{4}}} dx\right)}{(x^2+1)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 20.041 (sec). Leaf size: 60

```
DSolve[2*x^2*(1+x^2)*y''[x]+x*(3+8*x^2)*y'[x]-(3-4*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to -\frac{c_2 \,\text{Hypergeometric2F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, -x^2\right)}{x\sqrt[4]{x^2+1}} + \frac{c_1}{x^{3/2}\sqrt[4]{x^2+1}} + \frac{c_2}{x}$$

2.109 problem 111

2.109.1 Maple step by step solution 1054

Internal problem ID [7599]
Internal file name [OUTPUT/6532_Sunday_June_05_2022_04_57_35_PM_67751673/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 111.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + 3x(x^2+3)\,y' - (-5x^2+1)\,y = 0}$$

Writing the ode as

$$9x^2y'' + (3x^3+9x)\,y' + (5x^2-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^2 - 5}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\
&= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{2}{9}\right) - (0) \\
&= -\frac{2}{9}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= -\frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &&= \frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\
&= \frac{1}{6x} - \frac{x}{6} \\
&= \frac{1}{6x} - \frac{x}{6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\ &= x^{\frac{1}{6}} \mathrm{e}^{-\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x^3 + 9x}{9x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{12} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{12}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2}dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x^2}{6}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x^2}{6}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.109.1 Maple step by step solution

Let's solve

$9x^2y'' + (3x^3 + 9x)\, y' + (5x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2+3)y'}{3x} - \frac{(5x^2-1)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x^2+3)y'}{3x} + \frac{(5x^2-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{9}$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$9x^2 y'' + 3x(x^2+3)\, y' + (5x^2-1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(4+3r)\,(2+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)\,(3a_k k + 3a_k r + a_k + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(3k+3r+5)\,(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{3k+7+3r}$

- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+2} = -\frac{a_k}{3k+6}$

- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k}{3k+8}$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+8}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{b_k}{3k+8}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(3+x^2)*diff(y(x),x)-(1-5*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x^2}{6}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 61

```
DSolve[9*x^2*y''[x]+3*x*(3+x^2)*y'[x]-(1-5*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}} \left(2c_1 x^{4/3} + \sqrt[3]{6} c_2 (-x^2)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

2.110 problem 112

2.110.1 Maple step by step solution 1065

Internal problem ID [7600]
Internal file name [OUTPUT/6533_Sunday_June_05_2022_04_57_38_PM_40007480/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 112.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2y'' + x(6x^2+1)\,y' + (9x^2+1)\,y = 0}$$

Writing the ode as

$$6x^2y'' + (6x^3+x)\,y' + (9x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3 + x \\ C &= 9x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^4 - 132x^2 - 35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= -\frac{17}{12} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{12x} - \frac{x}{2} \\ &= \frac{5}{12x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\
&= x^{\frac{5}{12}} e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6x^3 + x}{6x^2}\, dx} \\
&= z_1 e^{-\frac{x^2}{4} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{4}}}{x^{\frac{1}{12}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}} e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx$$
$$= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}}\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right)$$

Verified OK.

2.110.1 Maple step by step solution

Let's solve

$6x^2y'' + (6x^3 + x)\,y' + (9x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+1)y}{6x^2} - \frac{(6x^2+1)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(6x^2+1)y'}{6x}+\frac{(9x^2+1)y}{6x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{6x^2+1}{6x}, P_3(x)=\frac{9x^2+1}{6x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Bigg|_{x=0}=\frac{1}{6}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Bigg|_{x=0}=\frac{1}{6}$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $6x^2y''+x(6x^2+1)\,y'+(9x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)(-1+2r) x^r + a_1(2+3r)(1+2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(2k+2r-1)+3a$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $6\left(k+r-\frac{1}{2}\right)\left(\left(k+r-\frac{1}{3}\right) a_k + a_{k-2}\right) = 0$

- Shift index using $k- > k+2$

  $6\left(k+\frac{3}{2}+r\right)\left(\left(k+\frac{5}{3}+r\right) a_{k+2} + a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3a_k}{3k+5+3r}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{3a_k}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(6*x^2*diff(y(x),x$2)+x*(1+6*x^2)*diff(y(x),x)+(1+9*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right)$$

✓ Solution by Mathematica

Time used: 0.351 (sec). Leaf size: 61

```
DSolve[6*x^2*y''[x]+x*(1+6*x^2)*y'[x]+(1+9*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}}\left(2c_1x^{11/6}+\sqrt[12]{2}c_2(-x^2)^{11/12}\,\Gamma\left(\frac{1}{12},-\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

2.111 problem 113

2.111.1 Maple step by step solution 1076

Internal problem ID [7601]
Internal file name [`OUTPUT/6534_Sunday_June_05_2022_04_57_41_PM_97334503/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 113.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0}$$

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=9x^4+9x^2\\B&=39x^3+9x\\C&=25x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^4 + 6x^2 - 5}{36 (x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9x^4 + 6x^2 - 5 \\ t &= 36(x^3 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^4 + 6x^2 - 5}{36 (x^3 + x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 207: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2} - \frac{5}{36(x-i)^2} - \frac{5}{36(x+i)^2} - \frac{i}{12(x-i)} + \frac{i}{12x + 12i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} + (-)(0) \\
&= \frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2+3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6(x-i)^2} - \frac{1}{6(x+i)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{6x} + \frac{1}{6x-6i} + \frac{1}{6x+6i}\right) dx} \\
&= x^{\frac{1}{6}} (x^2+1)^{\frac{1}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{6}} \\
&= z_1 \left(\frac{1}{\sqrt{x}\,\left(x^2+1\right)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+1\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{1}{3}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}\right) + c_2 \left(\frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}\left(\int \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{1}{3}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{1}{3}} (x^2+1)^{\frac{2}{3}}} + \frac{c_2 \left(\int \frac{1}{x^{\frac{1}{3}} (x^2+1)^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}} (x^2+1)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^{\frac{1}{3}} (x^2+1)^{\frac{2}{3}}} + \frac{c_2 \left(\int \frac{1}{x^{\frac{1}{3}} (x^2+1)^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}} (x^2+1)^{\frac{2}{3}}}$$

Verified OK.

2.111.1 Maple step by step solution

Let's solve

$(9x^4+9x^2)\, y'' + (39x^3+9x)\, y' + (25x^2-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2-1)y}{9x^2(x^2+1)} - \frac{(13x^2+3)y'}{3x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(13x^2+3)y'}{3x(x^2+1)} + \frac{(25x^2-1)y}{9x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{13x^2+3}{3x(x^2+1)}, P_3(x) = \frac{25x^2-1}{9x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'+(25x^2-1)\,y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\,x^r+a_1(4+3r)\,(2+3r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1)+a_{k-2}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-\frac{1}{3},\frac{1}{3}\right\}$$

- Each term must be 0

  $a_1(4+3r)(2+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(\left(k+r-\frac{1}{3}\right)a_{k-2}+a_k\left(k+r+\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right) = 0$

- Shift index using $k->k+2$

  $9\left(\left(k+\frac{5}{3}+r\right)a_k+a_{k+2}\left(k+\frac{7}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(3k+3r+5)a_k}{3k+7+3r}$

- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+2} = -\frac{(3k+4)a_k}{3k+6}$

- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{(3k+6)a_k}{3k+8}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{(3k+6)a_k}{3k+8}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{(3k+4)a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{(3k+6)b_k}{3k+8}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(9*x^2*(1+x^2)*diff(y(x),x$2)+3*x*(3+13*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1}{(x^2+1)^{\frac{2}{3}} x^{\frac{1}{3}}} + \frac{c_2 \left(\int \frac{1}{(x^3+x)^{\frac{1}{3}}} dx\right)}{(x^2+1)^{\frac{2}{3}} x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.878 (sec). Leaf size: 124

```
DSolve[9*x^2*(1+x^2)*y''[x]+3*x*(3+13*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{2\sqrt{3}c_2 \arctan\left(\frac{\sqrt{3}x^{2/3}}{x^{2/3}+2\sqrt[3]{x^2+1}}\right) - 2c_2 \log\left(\sqrt[3]{x^2+1} - x^{2/3}\right) + c_2 \log\left(x^{4/3} + (x^2+1)^{2/3} + \sqrt[3]{x^2+1}x^{2/3}\right.}{4\sqrt[3]{x}\,(x^2+1)^{2/3}}$$

2.112 problem 114

2.112.1 Maple step by step solution 1087

Internal problem ID [7602]
Internal file name [`OUTPUT/6535_Sunday_June_05_2022_04_57_45_PM_35760596/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 114.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 24x^3+4x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4\left(x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 6$$
$$t = 4\left(x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4\left(x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4(x^2+1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right)\\ &= \frac{1}{2}-\left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-i\right)^2} + \frac{1}{4\left(x+i\right)^2}\right) + \left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)^2 - \left(\frac{-}{4(\cdots}\right.\right.
$$

$$
\frac{(x^2+1\cdots}{(x+i)^2(-\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x)\, \frac{1}{\left(x^2+1\right)^{\frac{1}{4}}} \\
&= \frac{x}{\left(x^2+1\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{24x^3+4x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,\left(x^2+1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}\left(\frac{\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}} + \frac{c_2\left(\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}\right)}{\sqrt{x}\,\left(x^2+1\right)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2\left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2+1}\right)}{\sqrt{x}\,(x^2+1)^{\frac{3}{2}}}$$

Verified OK.

2.112.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\, y'' + (24x^3+4x)\, y' + (25x^2-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2-1)y}{4x^2(x^2+1)} - \frac{(6x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(6x^2+1)y'}{x(x^2+1)} + \frac{(25x^2-1)y}{4x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{6x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2-1}{4x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\, y'' + 4x(6x^2+1)\, y' + (25x^2-1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- ○ Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + a_{k-2}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{5}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_k+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r+5)a_k}{2k+3+2r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+4)a_k}{2k+2}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}},a_{k+2}=-\frac{(2k+4)a_k}{2k+2},a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+6)a_k}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=-\frac{(2k+6)a_k}{2k+4},a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+2}=-\frac{(2k+4)a_k}{2k+2},a_1=0,b_{k+2}=-\frac{(2k+6)b_k}{2k+4},b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(1+6*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x), sings
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2\left(\operatorname{arcsinh}(x)\,x - \sqrt{x^2+1}\right)}{\sqrt{x}\,(x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 57

```
DSolve[4*x^2*(1+x^2)*y''[x]+4*x*(1+6*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{-c_2\sqrt{x^2+1} - c_2 x \log\left(\sqrt{x^2+1} - x\right) + c_1 x}{\sqrt{x}\,(x^2+1)^{3/2}}$$

2.113 problem 115

2.113.1 Maple step by step solution 1097

Internal problem ID [7603]
Internal file name [`OUTPUT/6536_Sunday_June_05_2022_04_57_48_PM_73830340/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 115.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^2(2x^2+1)\,y''+2x(34x^2+5)\,y'-(-30x^2+1)\,y=0}$$

Writing the ode as

$$\left(16x^4+8x^2\right)y''+\left(68x^3+10x\right)y'+\left(30x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=16x^4+8x^2\\B&=68x^3+10x\\C&=30x^2-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 132x^4 + 148x^2 - 7$$
$$t = 64\left(2x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} - \frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{11}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{11}{8} - \left(\frac{11}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}} \\
&= \frac{22x^2 + 7}{16x^3 + 8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}} \right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x + \frac{i\sqrt{2}}{2}\right)^2} \right) + \left(\frac{7}{8x} + \frac{}{4x} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{8x} + \frac{1}{4x - 2i\sqrt{2}} + \frac{1}{4x + 2i\sqrt{2}} \right) dx} \\
&= x^{\frac{7}{8}} 2^{\frac{1}{4}} (2x^2 + 1)^{\frac{1}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{68x^3+10x}{16x^4+8x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{8}-\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{8}}\left(2x^2+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3+10x}{16x^4+8x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln\left(2x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\sqrt{2}}{2x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}\right) + c_2 \left(\frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}\left(\int \frac{\sqrt{2}}{2x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}}2^{\frac{3}{4}}\left(\int \frac{1}{x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)}{2\sqrt{2x^2+1}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}} 2^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}} 2^{\frac{3}{4}} \left(\int \frac{1}{x^{\frac{7}{4}} \sqrt{2x^2+1}} dx\right)}{2\sqrt{2x^2+1}}$$

Verified OK.

2.113.1 Maple step by step solution

Let's solve

$(16x^4 + 8x^2) y'' + (68x^3 + 10x) y' + (30x^2 - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(30x^2-1)y}{8x^2(2x^2+1)} - \frac{(34x^2+5)y'}{4x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(34x^2+5)y'}{4x(2x^2+1)} + \frac{(30x^2-1)y}{8x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{34x^2+5}{4x(2x^2+1)}, P_3(x) = \frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{8}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $8x^2(2x^2+1) y'' + 2x(34x^2+5) y' + (30x^2-1) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-1+4r)\, x^r + a_1(3+2r)\,(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(4k+4r-1) + 2a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

$$a_1(3+2r)\,(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(\left(2k+2r-\frac{5}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{4}\right)\right)\left(k+r+\frac{1}{2}\right)=0$$

- Shift index using $k- >k+2$

$$8\left(\left(2k+\frac{3}{2}+2r\right)a_k+a_{k+2}\left(k+\frac{7}{4}+r\right)\right)\left(k+\frac{5}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2(4k+4r+3)a_k}{4k+7+4r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}$$

- Solution for $r=\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0, b_{k+2}=-\frac{2(4k+4)b_k}{4k+8}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(8*x^2*(1+2*x^2)*diff(y(x),x$2)+2*x*(5+34*x^2)*diff(y(x),x)-(1-30*x^2)*y(x)=0,y(x), si
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{1}{\sqrt{2x^2+1}\, x^{\frac{7}{4}}}dx\right)}{\sqrt{2x^2+1}}$$

✓ Solution by Mathematica

Time used: 20.109 (sec). Leaf size: 54

```
DSolve[8*x^2*(1+2*x^2)*y''[x]+2*x*(5+34*x^2)*y'[x]-(1-30*x^2)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to \frac{3c_1x^{3/4} - 4c_2 \operatorname{Hypergeometric2F1}\left(-\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, -2x^2\right)}{3\sqrt{x}\sqrt{2x^2+1}}$$

2.114 problem 116

2.114.1 Maple step by step solution 1107

Internal problem ID [7604]
Internal file name [`OUTPUT/6537_Sunday_June_05_2022_04_57_51_PM_46937118/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 116.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(1+x)\,y''-x(-3x+1)\,y'+y=0}$$

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= 3x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + (-)(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{4x} dx} \\ &= x^{\frac{1}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-x}{2x^3+2x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}-\ln(1+x)} \\ &= z_1\left(\frac{x^{\frac{1}{4}}}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}-2\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1\left(2\sqrt{x}\right) \end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{\sqrt{x}}{1+x}\right) + c_2\left(\frac{\sqrt{x}}{1+x}(2\sqrt{x})\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}}{1+x} + \frac{2c_2x}{1+x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{1+x} + \frac{2c_2x}{1+x}$$

Verified OK.

2.114.1 Maple step by step solution

Let's solve

$(2x^3 + 2x^2)\, y'' + (3x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x^2(1+x)} - \frac{(3x-1)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-1)y'}{2x(1+x)} + \frac{y}{2x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x-1}{2x(1+x)}, P_3(x) = \frac{1}{2x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = 2$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(1+x)\,y'' + x(3x-1)\,y' + y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 7u + 4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0r(1+r)\,u^{-1+r} + (2a_1(1+r)(2+r) - a_0(1+r)(-1+4r))\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$$2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1})k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})r - 3a_k - 3a_{k-1} + 6a_{k+1})k + (-4a_k + 2a$$

- Shift index using $k- > k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2})(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})r - 3a_{k+1} - 3a_k + 6a_{k+2})(k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2 + 2kr + r^2 + 5k + 5r + 6)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x+1} + \frac{c_2\sqrt{x}}{x+1}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 25

```
DSolve[2*x^2*(1+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1\sqrt{x} + 2c_2 x}{x+1}$$

2.115 problem 117

2.115.1 Maple step by step solution 1116

Internal problem ID [7605]
Internal file name [`OUTPUT/6538_Sunday_June_05_2022_04_57_53_PM_84912216/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 117.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0}$$

Writing the ode as

$$(12x^4+6x^2)\,y''+(50x^3+x)\,y'+(30x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^4+6x^2 \\ B &= 50x^3+x \\ C &= 30x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)(0) \\
&= \frac{5}{12x} \\
&= \frac{5}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12x}\right)(0) + \left(\left(-\frac{5}{12x^2}\right) + \left(\frac{5}{12x}\right)^2 - \left(-\frac{35}{144x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{5}{12x} dx} \\ &= x^{\frac{5}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{50x^3+x}{12x^4+6x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{12} - \ln(2x^2+1)} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{12}} (2x^2+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{2x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{50x^3+x}{12x^4+6x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{6} - 2\ln(2x^2+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(6x^{\frac{1}{6}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{2x^2+1} \right) + c_2 \left(\frac{x^{\frac{1}{3}}}{2x^2+1} \left(6x^{\frac{1}{6}} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{2x^2+1} + \frac{6c_2\sqrt{x}}{2x^2+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{2x^2+1} + \frac{6c_2\sqrt{x}}{2x^2+1}
$$

Verified OK.

2.115.1 Maple step by step solution

Let's solve

$(12x^4 + 6x^2) y'' + (50x^3 + x) y' + (30x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(30x^2+1)y}{6x^2(2x^2+1)} - \frac{(50x^2+1)y'}{6x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(50x^2+1)y'}{6x(2x^2+1)} + \frac{(30x^2+1)y}{6x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{50x^2+1}{6x(2x^2+1)}, P_3(x) = \frac{30x^2+1}{6x^2(2x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{6}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$6x^2(2x^2+1)\, y'' + x(50x^2+1)\, y' + (30x^2+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)\,(-1+2r)\, x^r + a_1(2+3r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)\,(2k+2r-1) + 2a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{1}{3}\right\}$
- Each term must be 0

  $a_1(2+3r)(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2})=0$
- Shift index using $k->k+2$

  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-2a_k$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-2a_k$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-2a_k, a_1=0\right]$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-2a_k$
- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+2}=-2a_k, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-2a_k, a_1=0, b_{k+2}=-2b_k, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 33

```
dsolve(6*x^2*(1+2*x^2)*diff(y(x),x$2)+x*(1+50*x^2)*diff(y(x),x)+(1+30*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1\sqrt{x}}{2x^2+1} + \frac{c_2 x^{\frac{1}{3}}}{2x^2+1}$$

✓ Solution by Mathematica
Time used: 0.046 (sec). Leaf size: 32

```
DSolve[6*x^2*(1+2*x^2)*y''[x]+x*(1+50*x^2)*y'[x]+(1+30*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6c_2\sqrt[6]{x}+c_1\right)}{2x^2+1}$$

2.116 problem 118

2.116.1 Maple step by step solution 1125

Internal problem ID [7606]
Internal file name [`OUTPUT/6539_Sunday_June_05_2022_04_57_56_PM_21711049/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 118.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{28x^2(-3x+1)\, y''-7x(5+9x)\, y'+7(2+9x)\, y=0}$$

Writing the ode as

$$\left(-84x^3+28x^2\right) y''+\left(-63x^2-35x\right) y'+(63x+14)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -84x^3+28x^2 \\ B &= -63x^2-35x \\ C &= 63x+14 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 33$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{8x} + (-)(0) \\ &= -\frac{3}{8x} \\ &= -\frac{3}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8x}\right)(0) + \left(\left(\frac{3}{8x^2}\right) + \left(-\frac{3}{8x}\right)^2 - \left(\frac{33}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{8x} dx} \\ &= \frac{1}{x^{\frac{3}{8}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-63x^2-35x}{-84x^3+28x^2} \, dx} \\ &= z_1 e^{-\ln(3x-1)+\frac{5\ln(x)}{8}} \\ &= z_1 \left(\frac{x^{\frac{5}{8}}}{3x-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{3x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-63x^2-35x}{-84x^3+28x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(3x-1)+\frac{5\ln(x)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4x^{\frac{7}{4}}}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{3x-1}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{3x-1}\left(\frac{4x^{\frac{7}{4}}}{7}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}}{3x-1} + \frac{4c_2 x^2}{21x-7} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}}}{3x-1} + \frac{4c_2 x^2}{21x-7}
$$

Verified OK.

2.116.1 Maple step by step solution

Let's solve

$(-84x^3+28x^2)\, y'' + (-63x^2-35x)\, y' + (63x+14)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+9x)y}{4x^2(3x-1)} - \frac{(5+9x)y'}{4x(3x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(5+9x)y'}{4x(3x-1)} - \frac{(2+9x)y}{4x^2(3x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5+9x}{4x(3x-1)}, P_3(x) = -\frac{2+9x}{4x^2(3x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -\frac{5}{4}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $4y''x^2(3x-1) + x(5+9x)\,y' + (-9x-2)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+4r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(4k+4r-1)(k+r-2) + 3a_{k-1}(4k+4r-1)(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+4r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{2, \frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k+r-\frac{1}{4}\right)(k+r-2)(a_k - 3a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$-4\left(k+\frac{3}{4}+r\right)(k+r-1)(a_{k+1} - 3a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = 3a_k$$

- Recursion relation for $r = 2$

$$a_{k+1} = 3a_k$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = 3a_k$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(28*x^2*(1-3*x)*diff(y(x),x$2)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^2}{3x - 1} + \frac{c_2 x^{\frac{1}{4}}}{3x - 1}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 30

```
DSolve[28*x^2*(1-3*x)*y''[x]-7*x*(5+9*x)*y'[x]+7*(2+9*x)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \rightarrow \frac{4c_2 x^2 + 7c_1 \sqrt[4]{x}}{7 - 21x}$$

2.117 problem 119

2.117.1 Maple step by step solution 1134

Internal problem ID [7607]
Internal file name [`OUTPUT/6540_Sunday_June_05_2022_04_57_58_PM_48646030/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 119.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^2(-x^2+2)\,y''+2x(-21x^2+10)\,y'-(35x^2+2)\,y=0}$$

Writing the ode as

$$(-8x^4+16x^2)\,y''+(-42x^3+20x)\,y'+(-35x^2-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -8x^4+16x^2 \\ B &= -42x^3+20x \\ C &= -35x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{8} - \left(\frac{1}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{8x} + (-)\,(0) \\ &= \frac{1}{8x} \\ &= \frac{1}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{8x} dx} \\ &= x^{\frac{1}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-42x^3+20x}{-8x^4+16x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{8} - \ln(x^2-2)} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{8}} (x^2-2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}\,(x^2-2)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-2\ln(x^2-2)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4x^{\frac{3}{4}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}\,(x^2-2)}\right) + c_2\left(\frac{1}{\sqrt{x}\,(x^2-2)}\left(\frac{4x^{\frac{3}{4}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x}\,(x^2-2)} + \frac{4c_2 x^{\frac{1}{4}}}{3x^2-6} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{\sqrt{x}\,(x^2-2)} + \frac{4c_2 x^{\frac{1}{4}}}{3x^2-6}
$$

Verified OK.

2.117.1 Maple step by step solution

Let's solve

$(-8x^4+16x^2)\, y'' + (-42x^3+20x)\, y' + (-35x^2-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(35x^2+2)y}{8x^2(x^2-2)} - \frac{(21x^2-10)y'}{4x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(21x^2-10)y'}{4x(x^2-2)}+\frac{(35x^2+2)y}{8x^2(x^2-2)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{21x^2-10}{4x(x^2-2)}, P_3(x)=\frac{35x^2+2}{8x^2(x^2-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0}=\frac{5}{4}$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0}=-\frac{1}{8}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $8y''x^2(x^2-2)+2x(21x^2-10)\,y'+(35x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0(1+2r)(-1+4r)\, x^r - 2a_1(3+2r)(3+4r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)(4k+4r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2(1+2r)(-1+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

  $$-2a_1(3+2r)(3+4r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$$

- Shift index using $k->k+2$

  $$-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{a_k}{2}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+2} = \frac{a_k}{2}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+2} = \frac{a_k}{2}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2} = \frac{a_k}{2}, a_1 = 0, b_{k+2} = \frac{b_k}{2}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(8*x^2*(2-x^2)*diff(y(x),x$2)+2*x*(10-21*x^2)*diff(y(x),x)-(2+35*x^2)*y(x)=0,y(x), sin
```

$$y(x) = \frac{c_1}{(x^2-2)\sqrt{x}} + \frac{c_2 x^{\frac{1}{4}}}{x^2-2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 34

```
DSolve[8*x^2*(2-x^2)*y''[x]+2*x*(10-21*x^2)*y'[x]-(2+35*x^2)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \to \frac{\frac{3c_1}{\sqrt{x}} + 4c_2\sqrt[4]{x}}{6-3x^2}$$

2.118 problem 120

2.118.1 Maple step by step solution 1141

Internal problem ID [7608]
Internal file name [`OUTPUT/6541_Sunday_June_05_2022_04_58_01_PM_99523383/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 120.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+3x+1)\,y''-4x(-3x^2-3x+1)\,y'+3(x^2-x+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 12x^3+12x^2-4x \\ C &= 3x^2-3x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx} \\ &= z_1 e^{-\ln(x^2+3x+1)+\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x^2+3x+1)+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2 \left(\frac{\sqrt{x}}{x^2+3x+1}(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1}$$

Verified OK.

2.118.1 Maple step by step solution

Let's solve
$(4x^4+12x^3+4x^2)\,y''+(12x^3+12x^2-4x)\,y'+(3x^2-3x+3)\,y=0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y''=-\frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)}-\frac{(3x^2+3x-1)y'}{x(x^2+3x+1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y''+\frac{(3x^2+3x-1)y'}{x(x^2+3x+1)}+\frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)}=0$

☐ Check to see if x_0 is a regular singular point
  - Define functions
    $\left[P_2(x)=\frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x)=\frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$
    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\, y'' + 4x(3x^2+3x-1)\, y' + (3x^2-3x+3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + (a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$

- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)(2k+2r-3)(a_k + 3a_{k-1} + a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(2k+2r+3)(2k+2r+1)(a_{k+2} + 3a_{k+1} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{k+1} - b_k, b_1 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 35

```
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)-4*x*(1-3*x-3*x^2)*diff(y(x),x)+3*(1-x+x^2)*y(x)=0,y(
```

$$y(x) = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1}$$

✓ Solution by Mathematica
Time used: 0.056 (sec). Leaf size: 28

```
DSolve[4*x^2*(1+3*x+x^2)*y''[x]-4*x*(1-3*x-3*x^2)*y'[x]+3*(1-x+x^2)*y[x]==0,y[x],x,IncludeSi
```

$$y(x) \to \frac{\sqrt{x}(c_2 x + c_1)}{x^2+3x+1}$$

2.119 problem 121

2.119.1 Maple step by step solution 1150

Internal problem ID [7609]
Internal file name [`OUTPUT/6542_Sunday_June_05_2022_04_58_03_PM_7899117/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 121.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2(1+x)^2 y'' - x\left(-11x^2-10x+1\right) y' + \left(5x^2+1\right) y = 0}$$

Writing the ode as

$$3x^2(1+x)^2 y'' + \left(11x^3+10x^2-x\right) y' + \left(5x^2+1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3+10x^2-x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + (-)\,(0) \\
&= \frac{1}{6x} \\
&= \frac{1}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x}\right)(0) + \left(\left(-\frac{1}{6x^2}\right) + \left(\frac{1}{6x}\right)^2 - \left(-\frac{5}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \frac{1}{6x} dx} \\
&= x^{\frac{1}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{6} - 2\ln(1+x)} \\
&= z_1 \left(\frac{x^{\frac{1}{6}}}{(1+x)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{3}-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3x^{\frac{2}{3}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{(1+x)^2}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{(1+x)^2}\left(\frac{3x^{\frac{2}{3}}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(1+x)^2} + \frac{3c_2 x}{2(1+x)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(1+x)^2} + \frac{3c_2 x}{2(1+x)^2}
$$

Verified OK.

2.119.1 Maple step by step solution

Let's solve

$3x^2(1+x)^2 y'' + (11x^3 + 10x^2 - x)\,y' + (5x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+1)y}{3x^2(1+x)^2} - \frac{y'(11x-1)}{3x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'(11x-1)}{3x(1+x)}+\frac{(5x^{2}+1)y}{3x^{2}(1+x)^{2}}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_{2}(x)=\frac{11x-1}{3x(1+x)}, P_{3}(x)=\frac{5x^{2}+1}{3x^{2}(1+x)^{2}}\right]$

  - $(1+x)\cdot P_{2}(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_{2}(x)\right)\Big|_{x=-1}=4$

  - $(1+x)^{2}\cdot P_{3}(x)$ is analytic at $x=-1$

    $\left((1+x)^{2}\cdot P_{3}(x)\right)\Big|_{x=-1}=2$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $3x^{2}(1+x)^{2}y''+x(11x-1)(1+x)y'+(5x^{2}+1)y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(3u^{4}-6u^{3}+3u^{2})\left(\frac{d^{2}}{du^{2}}y(u)\right)+(11u^{3}-23u^{2}+12u)\left(\frac{d}{du}y(u)\right)+(5u^{2}-10u+6)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..2$

    $u^{m}\cdot y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^{m}\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=2..4$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$3a_{0}(2+r)\,(1+r)\,u^{r}+\left(3a_{1}(3+r)\,(2+r)-a_{0}(2+r)\,(5+6r)\right)u^{1+r}+\left(\sum_{k=2}^{\infty}\left(3a_{k}(k+r+2)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$3(2+r)\,(1+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{-2,-1\}$$

- Each term must be 0

  $$3a_{1}(3+r)\,(2+r)-a_{0}(2+r)\,(5+6r)=0$$

- Solve for the dependent coefficient(s)

  $$a_{1}=\frac{a_{0}(5+6r)}{3(3+r)}$$

- Each term in the series must be 0, giving the recursion relation

  $$3(a_{k}+a_{k-2}-2a_{k-1})\,k^{2}+\left(6(a_{k}+a_{k-2}-2a_{k-1})\,r+9a_{k}-4a_{k-2}-5a_{k-1}\right)k+3(a_{k}+a_{k-2}-2a$$

- Shift index using $k->k+2$

  $$3(a_{k+2}+a_{k}-2a_{k+1})\,(k+2)^{2}+\left(6(a_{k+2}+a_{k}-2a_{k+1})\,r+9a_{k+2}-4a_{k}-5a_{k+1}\right)(k+2)+3(a_{k+}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}+6kra_{k}-12kra_{k+1}+3r^{2}a_{k}-6r^{2}a_{k+1}+8ka_{k}-29ka_{k+1}+8ra_{k}-29ra_{k+1}+5a_{k}-33a_{k+1}}{3(k^{2}+2kr+r^{2}+7k+7r+12)}$$

- Recursion relation for $r=-2$

  $$a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}-4ka_{k}-5ka_{k+1}+a_{k}+a_{k+1}}{3(k^{2}+3k+2)}$$

- Solution for $r=-2$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k-2},a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}-4ka_{k}-5ka_{k+1}+a_{k}+a_{k+1}}{3(k^{2}+3k+2)},a_{1}=-\frac{7a_{0}}{3}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-2}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} - 4k a_k - 5k a_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-1}\right), a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} - 4k a_k - 5k a_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 =$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(3*x^2*(1+x)^2*diff(y(x),x$2)-x*(1-10*x-11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x), si
```

$$y(x) = \frac{c_1 x}{(x+1)^2} + \frac{c_2 x^{\frac{1}{3}}}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 29

```
DSolve[3*x^2*(1+x)^2*y''[x]-x*(1-10*x-11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to \frac{2c_1 \sqrt[3]{x} + 3c_2 x}{2(x+1)^2}$$

2.120 problem 122

2.120.1 Maple step by step solution 1160

Internal problem ID [7610]
Internal file name [`OUTPUT/6543_Sunday_June_05_2022_04_58_05_PM_69855881/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 122.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+2x+3)\,y''-x(-15x^2-14x+3)\,y'+(7x^2+3)\,y=0}$$

Writing the ode as

$$(4x^4+8x^3+12x^2)\,y''+(15x^3+14x^2-3x)\,y'+(7x^2+3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+8x^3+12x^2 \\ B &= 15x^3+14x^2-3x \\ C &= 7x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{8} - \left(\frac{1}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{8x} + (-)\,(0) \\ &= \frac{1}{8x} \\ &= \frac{1}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{8x} dx} \\ &= x^{\frac{1}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \ln(x^2+2x+3)} \\ &= z_1 \left(\frac{x^{\frac{1}{8}}}{x^2+2x+3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{x^2+2x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4x^{\frac{3}{4}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{x^2+2x+3}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{x^2+2x+3}\left(\frac{4x^{\frac{3}{4}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+2x+3} + \frac{4c_2 x}{3x^2+6x+9} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+2x+3} + \frac{4c_2 x}{3x^2+6x+9}
$$

Verified OK.

2.120.1 Maple step by step solution

Let's solve

$(4x^4+8x^3+12x^2)\,y''+(15x^3+14x^2-3x)\,y'+(7x^2+3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+3)y}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)} + \frac{(7x^2+3)y}{4x^2(x^2+2x+3)} = 0$$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{4}$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2(x^2+2x+3)\, y'' + x(15x^2+14x-3)\, y' + (7x^2+3)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$$

Rewrite ODE with series expansions

$$3a_0(-1+4r)(-1+r)x^r + (3a_1(3+4r)r + 2a_0r(3+4r))x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3(-1+4r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{4}\right\}$$

- Each term must be 0

$$3a_1(3+4r)r + 2a_0r(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{2a_0}{3}$$

- Each term in the series must be 0, giving the recursion relation

$$(4k+4r-1)(k+r-1)(3a_k + 2a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(4k+4r+7)(k+r+1)(3a_{k+2} + 2a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Solution for $r = \frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}, a_1=-\frac{2a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}, a_1=-\frac{2a_0}{3}, b_{k+2}=-\frac{2b_{k+1}}{3}-\frac{b_k}{3}, b_1=-$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 33

```
dsolve(4*x^2*(3+2*x+x^2)*diff(y(x),x$2)-x*(3-14*x-15*x^2)*diff(y(x),x)+(3+7*x^2)*y(x)=0,y(x)
```

$$y(x)=\frac{c_1 x}{x^2+2x+3}+\frac{c_2 x^{\frac{1}{4}}}{x^2+2x+3}$$

✓ Solution by Mathematica
Time used: 0.058 (sec). Leaf size: 33

```
DSolve[4*x^2*(3+2*x+x^2)*y''[x]-x*(3-14*x-15*x^2)*y'[x]+(3+7*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x)\to\frac{3c_1\sqrt[4]{x}+4c_2 x}{3x^2+6x+9}$$

2.121 problem 123

2.121.1 Maple step by step solution 1170

Internal problem ID [7611]
Internal file name [OUTPUT/6544_Sunday_June_05_2022_04_58_08_PM_8921848/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 123.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(4+x)\,y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 10x - 1 \\ t &= 4x^2(x - 1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{-2}{2} + 2\right) &= \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{-2}{2} + 2\right) &= \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\
&= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{\frac{3}{2}}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}} \mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x\,(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}} \mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x\,(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}}$$

Verified OK.

2.121.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (-x^2-3x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4+x)y}{x^2(x-1)^2} + \frac{(x+3)y'}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+3)y'}{x(x-1)^2}+\frac{(4+x)y}{x^2(x-1)^2}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x+3}{x(x-1)^2}, P_3(x)=\frac{4+x}{x^2(x-1)^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=-3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=4$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x^2(x-1)^2-x(x+3)\,y'+(4+x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term must be 0

  $$a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(1+2r)}{-1+r}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1})\, k + (a_k + a_{k-2} - 2a_{k-1})\, r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k->k+2$

  $$(k+r)\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})\, r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(x^2*(1-2*x+x^2)*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 \mathrm{e}^{-\frac{4}{x-1}}}{x-1} + \frac{c_2 x^2 \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right) \mathrm{e}^{-\frac{4x}{x-1}}}{x-1}$$

✓ Solution by Mathematica

Time used: 0.236 (sec). Leaf size: 54

```
DSolve[x^2*(1-2*x+x^2)*y''[x]-x*(3+x)*y'[x]+(4+x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}} \sqrt{1-x} x^2 \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right) + e^4 c_1\right)}{(x-1)^{3/2}}$$

2.122 problem 124

2.122.1 Maple step by step solution 1179

Internal problem ID [7612]
Internal file name [OUTPUT/6545_Sunday_June_05_2022_04_58_10_PM_18956701/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 124.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+2)\,y''+5x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} + \frac{1}{8x+16} + \frac{5}{16\,(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\,(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16 \left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4 (x + 2)} + \frac{1}{2x} + (-) (0) \\ &= -\frac{1}{4 (x + 2)} + \frac{1}{2x} \\ &= \frac{4 + x}{4x (x + 2)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x+2\right)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\left(x+2\right)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\left(x+2\right)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16\left(x^2+2x\right)^2}\right)\right.$$
$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x+2\right)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{\left(x+2\right)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(x+2\right)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{x+2} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\left(2\sqrt{x+2} - 2\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{2c_2 \sqrt{x}\left(\sqrt{x+2} - \sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)}{(x+2)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{2c_2 \sqrt{x}\left(\sqrt{x+2} - \sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)}{(x+2)^{\frac{3}{2}}}
$$

Verified OK.

2.122.1 Maple step by step solution

Let's solve

$(2x^3 + 4x^2)\, y'' + 5x^2 y' + (1+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(1+x)y}{2x^2(x+2)} - \frac{5y'}{2(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2(x+2)} + \frac{(1+x)y}{2x^2(x+2)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{5}{2(x+2)}, P_3(x) = \frac{1+x}{2x^2(x+2)}\right]$
  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{5}{2}$
  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x = -2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$

- Multiply by denominators

  $2x^2(x+2)\,y'' + 5x^2y' + (1+x)\,y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\, u^{-1+r} + \left(4a_1(1+r)(5+2r) - a_0(8r^2+12r+1)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$$

- Shift index using $k->k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+5*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} - \frac{c_2\sqrt{2}\left(-2\sqrt{2}\,\sqrt{x+2} + 4\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\sqrt{x}}{2\,(x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 55

```
DSolve[2*x^2*(2+x)*y''[x]+5*x^2*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\text{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

2.123 problem 125

2.123.1 Maple step by step solution 1190

Internal problem ID [7613]
Internal file name [`OUTPUT/6546_Sunday_June_05_2022_04_58_13_PM_56533208/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 125.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-x^2+2)\,y''-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0}$$

Writing the ode as

$$(-x^4+2x^2)\,y''+(-4x^3-2x)\,y'+(-2x^2+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-2x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 1}{(x^3 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 1 \\ t &= (x^3 - 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 1}{(x^3 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16\left(x - \sqrt{2}\right)^2} + \frac{5}{16\left(x + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{32\left(x - \sqrt{2}\right)} + \frac{3\sqrt{2}}{32\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 1}{(x^3 - 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \\
&= -\frac{1}{x^3-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x-\sqrt{2}\right)^2} + \frac{1}{4\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{}{4\left(\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{\left(x-\sqrt{2}\right)^{\frac{1}{4}}\left(x+\sqrt{2}\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{5\ln\left(x^2-2\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x}{(x^2-2)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{5\ln\left(x^2-2\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x^2-2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{x}{(x^2-2)^{\frac{3}{2}}}\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)}{(x^2-2)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)}{(x^2-2)^{\frac{3}{2}}}$$

Verified OK.

2.123.1 Maple step by step solution

Let's solve

$(-x^4 + 2x^2) y'' + (-4x^3 - 2x) y' + (-2x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2-1)y}{x^2(x^2-2)} - \frac{2(2x^2+1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(2x^2+1)y'}{x(x^2-2)} + \frac{2(x^2-1)y}{x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(2x^2+1)}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(x^2 - 2) + 2x(2x^2 + 1) y' + (2x^2 - 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}(-1+r)^{2} x^{r}-2a_{1}r^{2}x^{1+r}+\left(\sum_{k=2}^{\infty}\left(-2a_{k}(k+r-1)^{2}+a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^{2}=0$
- Values of r that satisfy the indicial equation

  $r=1$
- Each term must be 0

  $-2a_{1}r^{2}=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $-2a_{k}(k+r-1)^{2}+a_{k-2}(k+r)\,(k+r-1)=0$
- Shift index using $k->k+2$

  $-2a_{k+2}(k+r+1)^{2}+a_{k}(k+r+2)\,(k+r+1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(k+r+2)}{2(k+r+1)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-2*x*(1+2*x^2)*diff(y(x),x)+(2-2*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2\sqrt{2}\,x\left(2\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) + \sqrt{2}\,\sqrt{x^2-2}\right)}{2\,(x^2-2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 58

```
DSolve[x^2*(2-x^2)*y''[x]-2*x*(1+2*x^2)*y'[x]+(2-2*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{x\left(-\sqrt{2}c_2\text{arctanh}\left(\sqrt{1-\frac{x^2}{2}}\right) + c_2\sqrt{2-x^2} + c_1\right)}{(2-x^2)^{3/2}}$$

2.124 problem 126

2.124.1 Maple step by step solution 1200

Internal problem ID [7614]
Internal file name [`OUTPUT/6547_Sunday_June_05_2022_04_58_16_PM_35339237/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 126.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2 - 5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - 5x \\ C &= 9 - 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 6x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{6x - 1}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right)\right) &= 0 \\
\frac{1 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x) \, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1 + x) \, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1 + x) \sqrt{x} \, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\
&= z_1\left(x^{\frac{5}{2}}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{-x}+(-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(\frac{\mathrm{e}^{-x}+(-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3(1+x) + c_2 x^3\left(-\mathrm{expIntegral}_1(x)\,x + \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3(1+x) + c_2 x^3\left(-\mathrm{expIntegral}_1(x)\,x + \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\right)
$$

Verified OK.

2.124.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2-5x)\,y' + (9-4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x-9)y}{x^2} - \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-5)y'}{x} - \frac{(4x-9)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-5}{x}, P_3(x) = -\frac{4x-9}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 9$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x-5)\,y' + (9-4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 3$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-3)^2 + a_{k-1}(k-5+r) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}(k-2+r)^2 + a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-4)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$; series terminates at $k = 1$

  $$a_{k+1} = -\frac{a_k(k-1)}{(k+1)^2}$$

- Apply recursion relation for $k = 0$

  $$a_1 = a_0$$

- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

  $$y = a_0 \cdot (1+x)$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*diff(y(x),x$2)-x*(5-x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^3(x+1) + c_2 x^3 \left(\text{expIntegral}_1\left(x\right) x + \text{expIntegral}_1\left(x\right) - \mathrm{e}^{-x}\right)$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]-x*(5-x)*y'[x]+(9-4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} x^3 (c_2 e^x (x+1) \text{ExpIntegralEi}(-x) + c_1 e^x (x+1) + c_2)$$

2.125 problem 127

2.125.1 Maple step by step solution 1209

Internal problem ID [7615]
Internal file name [`OUTPUT/6548_Sunday_June_05_2022_04_58_19_PM_91920693/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 127.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+x+1)\,y''+12x^2(1+x)\,y'+(3x^2+3x+1)\,y=0}$$

Writing the ode as

$$\begin{aligned}(4x^4+4x^3+4x^2)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y&=0 &(1)\\ Ay''+By'+Cy&=0 &(2)\end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+4x^3+4x^2\\ B&=12x^3+12x^2 \qquad (3)\\ C&=3x^2+3x+1\end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \qquad (4)$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \qquad (5)\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 4x - 1 \\ t &= 4\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 235: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{1}{4x^2} + \frac{-\frac{3}{8}-\frac{i\sqrt{3}}{8}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8}+\frac{i\sqrt{3}}{8}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4}-\frac{5i\sqrt{3}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4}+\frac{5i\sqrt{3}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8}-\frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-2i\sqrt{3}-2}}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} + \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{2i\sqrt{3}-2}}{4} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{-2i\sqrt{3}-2}}{4}$	$\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{2i\sqrt{3}-2}}{4}$	$\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2+1}{2x\left(x^2+x+1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \right) + \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{2i\sqrt{3}-2}}{4}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= \sqrt{2}\,\sqrt{x}\,\left(x^2+x+1\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{\left(x^2+x+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x\sqrt{x^2+x+1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}}\left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x\sqrt{x^2+x+1}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}} + \frac{c_2\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{x\sqrt{x^2+x+1}}dx\right)}{2\sqrt{x^2+x+1}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}} + \frac{c_2\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{x\sqrt{x^2+x+1}}dx\right)}{2\sqrt{x^2+x+1}}$$

Verified OK.

2.125.1 Maple step by step solution

Let's solve

$(4x^4+4x^3+4x^2)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}-\frac{3(1+x)y'}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(1+x)y'}{x^2+x+1}+\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3(1+x)}{x^2+x+1}, P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+x+1)\, y'' + 12x^2(1+x)\, y' + (3x^2+3x+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\, x^r + \left(a_1(1+2r)^2 + a_0(3+2r)\,(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-1}(2k\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 + a_0(3+2r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(3+2r)a_0}{1+2r}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left((a_k+a_{k-2}+a_{k-1})k + (a_k+a_{k-2}+a_{k-1})r - \frac{a_k}{2} - \frac{3a_{k-2}}{2} + \frac{a_{k-1}}{2}\right) = 0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})(k+2) + (a_{k+2}+a_k+a_{k+1})r - \frac{a_{k+2}}{2} - \frac{3a_k}{2} + \frac{a_{k+1}}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}, a_1 = -2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 143

```
dsolve(4*x^2*(1+x+x^2)*diff(y(x),x$2)+12*x^2*(1+x)*diff(y(x),x)+(1+3*x+3*x^2)*y(x)=0,y(x), s
```

$$y(x) = c_1\sqrt{\frac{x}{x^2+x+1}}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{2}} + c_2\sqrt{\frac{x}{x^2+x+1}}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{2}}\left(\int\frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{2}}}{x\sqrt{x^2+x+1}}dx\right)$$

✓ Solution by Mathematica

Time used: 1.028 (sec). Leaf size: 93

```
DSolve[4*x^2*(1+x+x^2)*y''[x]+12*x^2*(1+x)*y'[x]+(1+3*x+3*x^2)*y[x]==0,y[x],x,IncludeSingula
```

$$y(x) \to \frac{\sqrt{x}e^{-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}\left(c_2\int_1^x\frac{e^{\sqrt{3}\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

2.126 problem 128

2.126.1 Maple step by step solution 1220

Internal problem ID [7616]
Internal file name [OUTPUT/6549_Sunday_June_05_2022_04_58_28_PM_77393323/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 128.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0}$$

Writing the ode as

$$x^2(x^2+x+1)\,y''+(2x^3+4x^2-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x^2+x+1) \\ B &= 2x^3+4x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 10x^2 - 8x - 1 \\ t &= 4\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2x} - \frac{1}{4x^2} + \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} - \frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b=-\frac{29}{24}+\frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right)\\
&= 1-(1)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2 - 2x + 1}{2x\left(x^2 + x + 1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{-138+42i}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right.} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} - \frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \right) dx} \\
&= \sqrt{2}\,\sqrt{x}\,\left(x^2 + x + 1\right)^{\frac{1}{4}} e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)} \, dx} \\
&= z_1 e^{-\frac{3 \ln\left(x^2+x+1\right)}{4} - \frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3 \ln\left(x^2+x+1\right)}{2} - \frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3} + \ln(x)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x\sqrt{x^2+x+1}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} \right) \\
&\quad + c_2 \left(\frac{\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x\sqrt{x^2+x+1}} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1 \sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} \\
& + \frac{c_2 \sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x\sqrt{x^2+x+1}} dx \right)}{2\sqrt{x^2+x+1}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} + \frac{c_2 \sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x\sqrt{x^2+x+1}} dx \right)}{2\sqrt{x^2+x+1}}
$$

Verified OK.

2.126.1 Maple step by step solution

Let's solve

$x^2(x^2+x+1)\, y'' + (2x^3+4x^2-x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)y'}{x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(2x^2+4x-1)y'}{x(x^2+x+1)} + \frac{y}{x^2(x^2+x+1)} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$(x \cdot P_2(x))\bigg|_{x=0} = -1$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $$x^2(x^2 + x + 1)\, y'' + x(2x^2 + 4x - 1)\, y' + y = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$$
  - Shift index using $k- >k + 2 - m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (a_1 r^2 + a_0 r(3+r)) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k+r-1)(k+2+r) \cdot\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 + a_0 r(3+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(3+r)a_0}{r}$
- Each term in the series must be 0, giving the recursion relation

  $((a_k + a_{k-2} + a_{k-1})k + (a_k + a_{k-2} + a_{k-1})r - a_k - 2a_{k-2} + 2a_{k-1})(k+r-1) = 0$
- Shift index using $k- >k+2$

  $((a_{k+2} + a_k + a_{k+1})(k+2) + (a_{k+2} + a_k + a_{k+1})r - a_{k+2} - 2a_k + 2a_{k+1})(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + 4a_{k+1}}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{ka_k + ka_{k+1} + a_k + 5a_{k+1}}{k+2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{ka_k + ka_{k+1} + a_k + 5a_{k+1}}{k+2}, a_1 = -4a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 137

```
dsolve(x^2*(1+x+x^2)*diff(y(x),x$2)-x*(1-4*x-2*x^2)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1x\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{7i\sqrt{3}}{6}}}{\sqrt{x^2+x+1}}+\frac{c_2x\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{7i\sqrt{3}}{6}}\left(\int\frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{7i\sqrt{3}}{6}}}{x\sqrt{x^2+x+1}}dx\right)}{\sqrt{x^2+x+1}}$$

✓ Solution by Mathematica

Time used: 1.035 (sec). Leaf size: 90

```
DSolve[x^2*(1+x+x^2)*y''[x]-x*(1-4*x-2*x^2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{xe^{-\frac{7\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{7\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

2.127 problem 129

2.127.1 Maple step by step solution 1231

Internal problem ID [7617]
Internal file name [OUTPUT/6550_Sunday_June_05_2022_04_58_37_PM_36070381/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 129.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 3x\left(-2x^2 + 3x + 5\right)y' + \left(-14x^2 + 12x + 1\right)y = 0}$$

Writing the ode as

$$9x^2y'' + \left(-6x^3 + 9x^2 + 15x\right)y' + \left(-14x^2 + 12x + 1\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3 + 9x^2 + 15x \\ C &= -14x^2 + 12x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 12x^3 + 33x^2 - 18x - 9$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v=1$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{1} a_i x^i$$
$$= -\frac{1}{2} + \frac{x}{3} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4} - \frac{1}{3}x + \frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2 - \frac{1}{3}x + \frac{11}{12}\right) + \left(\frac{-18x-9}{36x^2}\right) \\ &= \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} + \frac{-18x-9}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{11}{12}\right) - \left(\frac{1}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{1}{2} + \frac{x}{3} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{4x^4-12x^3+33x^2-18x-9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{2}+\frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{1}{2}$ then

$$\begin{aligned}d&=\alpha_\infty^+-(\alpha_{c_1}^+)\\&=\frac{1}{2}-\left(\frac{1}{2}\right)\\&=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=\left((+)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^+}{x-c_1}\right)+(+)[\sqrt{r}]_\infty\\&=\frac{1}{2x}+\left(-\frac{1}{2}+\frac{x}{3}\right)\\&=\frac{1}{2x}-\frac{1}{2}+\frac{x}{3}\\&=\frac{1}{2x}-\frac{1}{2}+\frac{x}{3}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{\frac{x(-3+x)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x^3+9x^2+15x}{9x^2}\, dx} \\ &= z_1 e^{\frac{x^2}{6} - \frac{x}{2} - \frac{5\ln(x)}{6}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x(-3+x)}{6}}}{x^{\frac{5}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2} dx$$
$$= y_1 \left(\int \frac{e^{-\frac{x(-3+x)}{3}}}{x} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} \right) + c_2 \left(\frac{e^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} \left(\int \frac{e^{-\frac{x(-3+x)}{3}}}{x} dx \right) \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{x(-3+x)}{3}} \left(\int \frac{e^{-\frac{x(-3+x)}{3}}}{x} dx \right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{x(-3+x)}{3}} \left(\int \frac{e^{-\frac{x(-3+x)}{3}}}{x} dx \right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.127.1 Maple step by step solution

Let's solve

$9x^2y'' + (-6x^3 + 9x^2 + 15x)\, y' + (-14x^2 + 12x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(14x^2-12x-1)y}{9x^2} + \frac{(2x^2-3x-5)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-3x-5)y'}{3x} - \frac{(14x^2-12x-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{5}{3}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{9}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' - 3x(2x^2 - 3x - 5)\, y' + (-14x^2 + 12x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)^2\, x^r + \left(a_1(4+3r)^2 + 3a_0(4+3r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)^2 + 3a_{k-1}(3k+3r+1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{3}$$

- Each term must be 0

$$a_1(4+3r)^2 + 3a_0(4+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{3a_0}{4+3r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r+1)^2 + (3k+3r+1)\,(-2a_{k-2} + 3a_{k-1}) = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(3k+3r+7)^2 + (3k+3r+7)\,(-2a_k + 3a_{k+1}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(5+3*x-2*x^2)*diff(y(x),x)+(1+12*x-14*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1 e^{\frac{1}{3}x^2-x}}{x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{1}{3}x^2-x} \left(\int \frac{e^{-\frac{1}{3}x^2+x}}{x}dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.552 (sec). Leaf size: 52

```
DSolve[9*x^2*y''[x]+3*x*(5+3*x-2*x^2)*y'[x]+(1+12*x-14*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{e^{\frac{1}{3}(x-3)x}\left(c_2 \int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]} dK[1] + c_1\right)}{\sqrt[3]{x}}$$

2.128 problem 130

2.128.1 Maple step by step solution 1243

Internal problem ID [7618]
Internal file name [`OUTPUT/6551_Sunday_June_05_2022_04_58_41_PM_33225453/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 130.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 2x^3+x^2\\ B &= 3x^3+14x^2+5x\\ C &= 12x^2+18x+4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 12x^3 - 16x^2 - 4x - 1$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{1}{4x^2} - \frac{15}{64\left(x + \frac{1}{2}\right)^2} - \frac{21}{16\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \\ &= Q + \frac{R}{16x^4 + 16x^3 + 4x^2} \\ &= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right) \\ &= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{21}{16}\right) - (0) \\ &= -\frac{21}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &&= -\frac{7}{8} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &&= \frac{7}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\ &= \frac{7}{8} - \left(\frac{7}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\
&= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4} \\
&= \frac{-3x^2+2x+1}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)^2 - \left(\frac{9x^4 - 12x}{4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right) dx} \\
&= \sqrt{x}\,(2x+1)^{\frac{3}{8}}\, e^{-\frac{3x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{3x}{4}-\frac{5\ln(x)}{2}-\frac{5\ln(2x+1)}{8}} \\
&= z_1 \left(\frac{e^{-\frac{3x}{4}}}{x^{\frac{5}{2}}\left(2x+1\right)^{\frac{5}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(2x+1)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{e^{\frac{3x}{2}}}{x\left(2x+1\right)^{\frac{3}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}\right) + c_2 \left(\frac{e^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}\left(\int \frac{e^{\frac{3x}{2}}}{x\left(2x+1\right)^{\frac{3}{4}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{3x}{2}}\left(\int \frac{e^{\frac{3x}{2}}}{x(2x+1)^{\frac{3}{4}}}dx\right)}{x^2\left(2x+1\right)^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 e^{-\frac{3x}{2}}}{x^2 (2x+1)^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{3x}{2}} \left(\int \frac{e^{\frac{3x}{2}}}{x(2x+1)^{\frac{3}{4}}} dx\right)}{x^2 (2x+1)^{\frac{1}{4}}}$$

Verified OK.

2.128.1 Maple step by step solution

Let's solve

$(2x^3 + x^2) y'' + (3x^3 + 14x^2 + 5x) y' + (12x^2 + 18x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(6x^2+9x+2)y}{x^2(2x+1)} - \frac{(3x^2+14x+5)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+14x+5)y'}{x(2x+1)} + \frac{2(6x^2+9x+2)y}{x^2(2x+1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3x^2+14x+5}{x(2x+1)}, P_3(x) = \frac{2(6x^2+9x+2)}{x^2(2x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x+1) y'' + x(3x^2 + 14x + 5) y' + (12x^2 + 18x + 4) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)^2 x^r + \left(a_1(3+r)^2 + 2a_0(3+r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)^2 + 2a_{k-1}(k+r+2)^2 + 3a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term must be 0

$$a_1(3+r)^2 + 2a_0(3+r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -2a_0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(\left(2k+2r+4\right)a_{k-1}+a_k(k+r+2)+3a_{k-2}\right)\left(k+r+2\right)=0$$

- Shift index using $k->k+2$

$$\left(\left(2k+8+2r\right)a_{k+1}+a_{k+2}(k+r+4)+3a_k\right)\left(k+r+4\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2ka_{k+1}+2ra_{k+1}+3a_k+8a_{k+1}}{k+r+4}$$

- Recursion relation for $r=-2$

$$a_{k+2}=-\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k-2},a_{k+2}=-\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2},a_1=-2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+14*x+3*x^2)*diff(y(x),x)+(4+18*x+12*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{c_1 e^{-\frac{3x}{2}}}{(2x+1)^{\frac{1}{4}} x^2} + \frac{c_2 e^{-\frac{3x}{2}} \left(\int \frac{e^{\frac{3x}{2}}}{(2x+1)^{\frac{3}{4}} x} dx\right)}{(2x+1)^{\frac{1}{4}} x^2}$$

✓ Solution by Mathematica

Time used: 14.745 (sec). Leaf size: 61

```
DSolve[x^2*(1+2*x)*y''[x]+x*(5+14*x+3*x^2)*y'[x]+(4+18*x+12*x^2)*y[x]==0,y[x],x,IncludeSingu
```

$$y(x)\to \frac{e^{-3x/2}\left(c_2\int_1^x \frac{e^{\frac{3K[1]}{2}}}{K[1](2K[1]+1)^{3/4}}dK[1]+c_1\right)}{x^2\sqrt[4]{2x+1}}$$

2.129 problem 131

2.129.1 Maple step by step solution 1254

Internal problem ID [7619]
Internal file name [`OUTPUT/6552_Sunday_June_05_2022_04_58_45_PM_28348095/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 131.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{16x^2y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0}$$

Writing the ode as

$$16x^2y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3 + 4x^2 + 24x \\ C &= 18x^2 + 5x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 + 4x^3 - 31x^2 - 8x - 16$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{8x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{1}{8} + \frac{x}{4} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2
$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\
&= Q + \frac{R}{64x^2} \\
&= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x-16}{64x^2}\right) \\
&= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x-16}{64x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} + \frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{1}{8} + \frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x(1+x)}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^3+4x^2+24x}{16x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{8} - \frac{x}{8} - \frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{e^{-\frac{x(1+x)}{8}}}{x^{\frac{3}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2} dx$$

$$= y_1 \left(\int \frac{e^{\frac{x(1+x)}{4}}}{x} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} \right) + c_2 \left(\frac{e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} \left(\int \frac{e^{\frac{x(1+x)}{4}}}{x} dx \right) \right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{x(1+x)}{4}} \left(\int \frac{e^{\frac{x(1+x)}{4}}}{x} dx \right)}{x^{\frac{1}{4}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{x(1+x)}{4}} \left(\int \frac{e^{\frac{x(1+x)}{4}}}{x} dx \right)}{x^{\frac{1}{4}}}$$

Verified OK.

2.129.1 Maple step by step solution

Let's solve

$16x^2y'' + (8x^3 + 4x^2 + 24x) y' + (18x^2 + 5x + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(18x^2+5x+1)y}{16x^2} - \frac{(2x^2+x+6)y'}{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+x+6)y'}{4x} + \frac{(18x^2+5x+1)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{16}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2\, x^r + \left(a_1(5+4r)^2 + a_0(5+4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-1}(4k+4r+1) \cdot \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{4}$$

- Each term must be 0

$$a_1(5+4r)^2 + a_0(5+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0}{5+4r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k+4r+1)^2 + (4k+4r+1)\,(2a_{k-2} + a_{k-1}) = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(4k+4r+9)^2 + (4k+4r+9)\,(2a_k + a_{k+1}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(16*x^2*diff(y(x),x$2)+4*x*(6+x+2*x^2)*diff(y(x),x)+(1+5*x+18*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1 e^{-\frac{1}{4}x^2-\frac{1}{4}x}}{x^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{1}{4}x^2-\frac{1}{4}x} \left(\int \frac{e^{\frac{1}{4}x^2+\frac{1}{4}x}}{x} dx\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.404 (sec). Leaf size: 51

```
DSolve[16*x^2*y''[x]+4*x*(6+x+2*x^2)*y'[x]+(1+5*x+18*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-\frac{1}{4}x(x+1)}\left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]}dK[1] + c_1\right)}{\sqrt[4]{x}}$$

2.130 problem 132

2.130.1 Maple step by step solution 1266

Internal problem ID [7620]
Internal file name [`OUTPUT/6553_Sunday_June_05_2022_04_58_48_PM_7956812/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 132.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0}$$

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+9x^2 \\ B &= -3x^3+33x^2+15x \\ C &= -7x^2+16x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 + 6x^3 + 3x^2 - 18x - 9 \\ t &= 36\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36\,(1+x)^2} - \frac{1}{4x^2} + \frac{1}{9x+9}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \\ &= Q + \frac{R}{36x^4 + 72x^3 + 36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2}\right) \\ &= \frac{1}{36} + \frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{9}\right) - (0) \\ &= \frac{1}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &= \frac{1}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &= -\frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \left(\frac{1}{6}\right) \\
&= -\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6} \\
&= -\frac{1}{6+6x} + \frac{1}{2x} + \frac{1}{6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{6\,(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)^2 - \left(\frac{x^4 + 6x^3 +}{36\,(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6(1+x)} + \frac{1}{2x} + \frac{1}{6}\right) dx} \\
&= \frac{\sqrt{x}\,\mathrm{e}^{\frac{x}{6}}}{(1+x)^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{7\ln(1+x)}{6}} \\
&= z_1 \left(\frac{e^{\frac{x}{6}}}{x^{\frac{5}{6}}\,(1+x)^{\frac{7}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}\,(1+x)^{\frac{4}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{(1+x)^{\frac{1}{3}}\,e^{-\frac{x}{3}}}{x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}\,(1+x)^{\frac{4}{3}}}\right) + c_2\left(\frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}\,(1+x)^{\frac{4}{3}}}\left(\int \frac{(1+x)^{\frac{1}{3}}\,e^{-\frac{x}{3}}}{x}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{\frac{x}{3}}}{x^{\frac{1}{3}}\,(1+x)^{\frac{4}{3}}} + \frac{c_2 e^{\frac{x}{3}}\left(\int \frac{(1+x)^{\frac{1}{3}}\,e^{-\frac{x}{3}}}{x}dx\right)}{x^{\frac{1}{3}}\,(1+x)^{\frac{4}{3}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 e^{\frac{x}{3}}}{x^{\frac{1}{3}} (1+x)^{\frac{4}{3}}} + \frac{c_2 e^{\frac{x}{3}} \left(\int \frac{(1+x)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x} dx\right)}{x^{\frac{1}{3}} (1+x)^{\frac{4}{3}}}$$

Verified OK.

2.130.1 Maple step by step solution

Let's solve

$(9x^3 + 9x^2) y'' + (-3x^3 + 33x^2 + 15x) y' + (-7x^2 + 16x + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(7x^2-16x-1)y}{9x^2(1+x)} + \frac{(x^2-11x-5)y'}{3x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-11x-5)y'}{3x(1+x)} - \frac{(7x^2-16x-1)y}{9x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-11x-5}{3x(1+x)}, P_3(x) = -\frac{7x^2-16x-1}{9x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = \frac{7}{3}$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $9x^2(1+x) y'' - 3x(x^2 - 11x - 5) y' + (-7x^2 + 16x + 1) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(9u^3-18u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right)+(-7u^2+30u-22)\,y(u) =$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(4+3r)\,u^{-1+r}+\left(3a_1(1+r)\,(7+3r)-2a_0(9r^2+21r+11)\right)u^r+\left(3a_2(2+r)\,(10+3r)-2\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(4+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{4}{3}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(7+3r)-2a_0(9r^2+21r+11)=0, 3a_2(2+r)\,(10+3r)-2a_1(9r^2+39r+41)+3a$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$9(-2a_k + a_{k-1} + a_{k+1})\,k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1})\,r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})\,k + 9($$

- Shift index using $k- >k+2$

  $$9(-2a_{k+2} + a_{k+1} + a_{k+3})\,(k+2)^2 + 3(6(-2a_{k+2} + a_{k+1} + a_{k+3})\,r - 14a_{k+2} - a_k + 5a_{k+1} + 10a_{k}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^2a_{k+1}-18r^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-3ra_k+51ra_{k+1}-114ra}{3(3k^2+6kr+3r^2+22k+22r+39)}$$

- Recursion relation for $r = 0$

  $$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21}, a_2\right.$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21},\right.$$

- Recursion relation for $r = -\frac{4}{3}$

  $$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}$$

- Solution for $r = -\frac{4}{3}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3}, a_2\right.$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3},\right.$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{4}{3}}\right), a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a}{3(3k^2+22k+39)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 48

```
dsolve(9*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x-x^2)*diff(y(x),x)+(1+16*x-7*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{c_1 e^{\frac{x}{3}}}{(x+1)^{\frac{4}{3}} x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{x}{3}} \left(\int \frac{(x+1)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x} dx\right)}{(x+1)^{\frac{4}{3}} x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 7.827 (sec). Leaf size: 50

```
DSolve[9*x^2*(1+x)*y''[x]+3*x*(5+11*x-x^2)*y'[x]+(1+16*x-7*x^2)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to \frac{e^{x/3}\left(c_1 - \sqrt[3]{3e}c_2\Gamma\left(\frac{1}{3}, \frac{x+1}{3}\right)\right)}{\sqrt[3]{x}(x+1)^{4/3}}$$

2.131 problem 133

2.131.1 Maple step by step solution 1277

Internal problem ID [7621]
Internal file name [`OUTPUT/6554_Sunday_June_05_2022_04_58_52_PM_4578488/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 133.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{36x^2(1-2x)\, y''+24x(1-9x)\, y'+(1-70x)\, y=0}$$

Writing the ode as

$$\left(-72x^3+36x^2\right) y''+\left(-216x^2+24x\right) y'+(1-70x)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -72x^3+36x^2 \\ B &= -216x^2+24x \\ C &= 1-70x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 48x - 9 \\ t &= 36\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{2}{9}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-32x^2+48x-9}{36\,(2x^2-x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} + (-)(0) \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} \\ &= \frac{4x-3}{12x^2-6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{6\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)dx} \\ &= \frac{\sqrt{x}}{(2x-1)^{\frac{1}{6}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-216x^2+24x}{-72x^3+36x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{3} - \frac{7\ln(2x-1)}{6}} \\ &= z_1\left(\frac{1}{x^{\frac{1}{3}}(2x-1)^{\frac{7}{6}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-216x^2+24x}{-72x^3+36x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2\ln(x)}{3}-\frac{7\ln(2x-1)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(3(2x-1)^{\frac{1}{3}} + \frac{\ln\left((2x-1)^{\frac{2}{3}} - (2x-1)^{\frac{1}{3}} + 1\right)}{2} \right. \\
&\quad \left. - \sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - \ln\left((2x-1)^{\frac{1}{3}}+1\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}\right) + c_2\left(\frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}\left(3(2x-1)^{\frac{1}{3}} + \frac{\ln\left((2x-1)^{\frac{2}{3}} - (2x-1)^{\frac{1}{3}} + 1\right)}{2} \right.\right. \\
&\quad \left.\left. - \sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - \ln\left((2x-1)^{\frac{1}{3}}+1\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} - \frac{c_2\left(\sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - 3(2x-1)^{\frac{1}{3}} + \ln\left((2x-1)^{\frac{1}{3}}+1\right) - \frac{\ln\left((2x-1)^{\frac{2}{3}}-(2x-1)^{\frac{1}{3}}+1\right)}{2}\right)x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} - \frac{c_2 \left(\sqrt{3}\, \arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - 3(2x-1)^{\frac{1}{3}} + \ln\left((2x-1)^{\frac{1}{3}}+1\right) - \frac{\ln\left((2x-1)^{\frac{2}{3}}-(2x-1)^{\frac{1}{3}}+1\right)}{2}\right) x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}$$

Verified OK.

2.131.1 Maple step by step solution

Let's solve

$(-72x^3+36x^2)\, y'' + (-216x^2+24x)\, y' + (1-70x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(70x-1)y}{36x^2(2x-1)} - \frac{2(9x-1)y'}{3x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(9x-1)y'}{3x(2x-1)} + \frac{(70x-1)y}{36x^2(2x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(9x-1)}{3x(2x-1)}, P_3(x) = \frac{70x-1}{36x^2(2x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{36}$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$36y''x^2(2x-1)+24x(9x-1)\,y'+y(70x-1)=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k->k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..2$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..3$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+6r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(6k+6r-1)^2+2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+6r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=\frac{1}{6}$$

- Each term in the series must be 0, giving the recursion relation

$$-36\left(k+r-\frac{1}{6}\right)\left(\left(-2k-2r-\frac{1}{3}\right)a_{k-1}+a_k\left(k+r-\frac{1}{6}\right)\right)=0$$

- Shift index using $k- >k+1$

$$-36\left(k+\frac{5}{6}+r\right)\left(\left(-2k-\frac{7}{3}-2r\right)a_k+a_{k+1}\left(k+\frac{5}{6}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{2(6k+6r+7)a_k}{6k+6r+5}$$

- Recursion relation for $r=\frac{1}{6}$

$$a_{k+1}=\frac{2(6k+8)a_k}{6k+6}$$

- Solution for $r=\frac{1}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{6}}, a_{k+1}=\frac{2(6k+8)a_k}{6k+6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(36*x^2*(1-2*x)*diff(y(x),x$2)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^{\frac{1}{6}}}{(-1+2x)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{6}} \left(\int \frac{(-1+2x)^{\frac{1}{3}}}{x} dx\right)}{(-1+2x)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 111

```
DSolve[36*x^2*(1-2*x)*y''[x]+24*x*(1-9*x)*y'[x]+(1-70*x)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{\sqrt[6]{x}\left(-2\sqrt{3}c_2 \arctan\left(\frac{2\sqrt[3]{1-2x}+1}{\sqrt{3}}\right) + 6c_2\sqrt[3]{1-2x} + 2c_2 \log\left(\sqrt[3]{1-2x}-1\right) - c_2 \log\left((1-2x)^{2/3} + \sqrt[3]{1}\right.}{2(1-2x)^{4/3}}$$

2.132 problem 134

2.132.1 Maple step by step solution 1288

Internal problem ID [7622]
Internal file name [OUTPUT/6555_Sunday_June_05_2022_04_58_57_PM_14299732/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 134.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-x(3-x)\,y'+4y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(x^2-3x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= x^2-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 249: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x - 1}{2x(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - 2\ln(1+x)} \\
&= z_1 \left(\frac{x^{\frac{3}{2}}}{(1+x)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\,x^2}{(1+x)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-1)\,x^2}{(1+x)^3}\right) + c_2 \left(\frac{(x-1)\,x^2}{(1+x)^3}\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x-1)\,x^2}{(1+x)^3} + \frac{c_2(\ln(x)\,(x-1)-4)\,x^2}{(1+x)^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x-1)\,x^2}{(1+x)^3} + \frac{c_2(\ln(x)\,(x-1)-4)\,x^2}{(1+x)^3}$$

Verified OK.

2.132.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+(x^2-3x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2(1+x)}-\frac{(-3+x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-3+x)y'}{x(1+x)}+\frac{4y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{-3+x}{x(1+x)}, P_3(x)=\frac{4}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''+x(-3+x)\,y'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-5u+4)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, u^{-1+r} + \left(a_1(1+r)\,(4+r) - a_0(2r^2+3r-4)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+4+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1(1+r)\,(4+r) - a_0(2r^2+3r-4) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+4+r) - a_k(2k^2+4kr+2r^2+3k+3r-4) + a_{k-1}(k+r-1)^2 = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) - a_{k+1}\left(2(k+1)^2+4(k+1)\,r+2r^2+3k-1+3r\right) + a_k(k+r)^2 =$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-7ka_{k+1}-7ra_{k+1}-a_{k+1}}{(k+2+r)(k+5+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$
- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 6k a_k + 5k a_{k+1} + 9a_k + 2a_{k+1}}{(k-1)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3-x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 (x-1)}{(x+1)^3} + \frac{c_2 x^2 (x\ln(x) - \ln(x) - 4)}{(x+1)^3}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 33

```
DSolve[x^2*(1+x)*y''[x]-x*(3-x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2(c_1(x-1) + c_2(x-1)\log(x) - 4c_2)}{(x+1)^3}$$

2.133 problem 135

2.133.1 Maple step by step solution 1297

Internal problem ID [7623]
Internal file name [OUTPUT/6556_Sunday_June_05_2022_04_59_00_PM_6684988/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 135.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-2x)\, y'' - x(5-4x)\, y' + (9-4x)\, y = 0}$$

Writing the ode as

$$\left(-2x^3+x^2\right) y'' + \left(4x^2-5x\right) y' + (9-4x)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 4x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 1}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 1 \\ t &= 4\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 1}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 251: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4x^2} + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} \\ &= -\frac{1}{2x\left(2x-1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{\sqrt{2x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2-5x}{-2x^3+x^2} \, dx} \\ &= z_1 e^{\frac{5\ln(x)}{2} - \frac{3\ln(2x-1)}{2}} \\ &= z_1\left(\frac{x^{\frac{5}{2}}}{(2x-1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3}{(2x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5\ln(x)-3\ln(2x-1)}}{(y_1)^2}\, dx \\
&= y_1(2x - \ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3}{(2x-1)^2}\right) + c_2 \left(\frac{x^3}{(2x-1)^2}(2x - \ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 x^3}{(2x-1)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(2x-1)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^3}{(2x-1)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(2x-1)^2}
$$

Verified OK.

2.133.1 Maple step by step solution

Let's solve
$(-2x^3 + x^2)\, y'' + (4x^2 - 5x)\, y' + (9 - 4x)\, y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = -\frac{(4x-9)y}{x^2(2x-1)} + \frac{(4x-5)y'}{x(2x-1)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' - \frac{(4x-5)y'}{x(2x-1)} + \frac{(4x-9)y}{x^2(2x-1)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = -\frac{4x-5}{x(2x-1)}, P_3(x) = \frac{4x-9}{x^2(2x-1)}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 9$
  - ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- ● Multiply by denominators

  $y''x^2(2x-1) - x(4x-5)\,y' + (4x-9)\,y = 0$
- ● Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - ○ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 (-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k (k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 3$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_k (k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$$

- Shift index using $k->k+1$

  $$-a_{k+1} (k+r-2)^2 + 2a_k (k+r-1)(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{2a_k (k+r-1)}{k+r-2}$$

- Recursion relation for $r = 3$

  $$a_{k+1} = \frac{2a_k (k+2)}{k+1}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k (k+2)}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^3}{(-1+2x)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(-1+2x)^2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 29

```
DSolve[x^2*(1-2*x)*y''[x]-x*(5-4*x)*y'[x]+(9-4*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{x^3(-2c_2 x + c_2 \log(x) + c_1)}{(1-2x)^2}$$

2.134 problem 136

2.134.1 Maple step by step solution 1306

Internal problem ID [7624]
Internal file name [OUTPUT/6557_Sunday_June_05_2022_04_59_02_PM_33603070/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 136.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x+2)\,y''+x^2y'+(1-x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+x^2y'+(1-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= x^2 \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 8x - 16}{16 (x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 8x - 16$$
$$t = 16(x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 8x - 16}{16 (x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 253: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{8x} - \frac{3}{8(x+2)} - \frac{3}{16(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 8x - 16}{16(x^2 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= \frac{5}{4} - \left(\frac{5}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{4\left(x+2\right)} + \frac{1}{2x} + (0) \\ &= \frac{3}{4\left(x+2\right)} + \frac{1}{2x} \\ &= \frac{5x+4}{4x\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x+2)} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{3}{4(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{3}{4(x+2)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2+8x-16}{16(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \sqrt{x}\,(x+2)^{\frac{3}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{(x+2)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\sqrt{x+2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2}-2}{2\sqrt{x+2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\sqrt{x+2}\right) + c_2\left(\sqrt{x}\,\sqrt{x+2}\left(-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2}-2}{2\sqrt{x+2}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\sqrt{x+2} - \frac{c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2}-2\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\sqrt{x+2} - \frac{c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2}-2\right)}{2}
$$

Verified OK.

2.134.1 Maple step by step solution

Let's solve

$(2x^3+4x^2)\,y''+x^2y'+(1-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-1)y}{2x^2(x+2)} - \frac{y'}{2(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2(x+2)} - \frac{(x-1)y}{2x^2(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{2(x+2)}, P_3(x) = -\frac{x-1}{2x^2(x+2)}\right]$

- ○ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = \frac{1}{2}$

- ○ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

- ○ $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $2x^2(x+2)\, y'' + x^2 y' + (1-x)\, y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 4u + 4)\left(\frac{d}{du}y(u)\right) + (3-u)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(-1+2r)\, u^{-1+r} + \left(4a_1(1+r)\,(1+2r) - a_0(8r^2-4r-3)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(1+2r) - a_0(8r^2-4r-3) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + \left(4(-4a_k + a_{k-1} + 4a_{k+1})\, r + 4a_k - 5a_{k-1} + 12a_{k+1}\right) k + 2(-4a_k +$$

- Shift index using $k- >k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + \left(4(-4a_{k+1} + a_k + 4a_{k+2})\, r + 4a_{k+1} - 5a_k + 12a_{k+2}\right)(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k - a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12 k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20 k a_{k+1} - a_k - 9 a_{k+1}}{4(2k^2+9k+10)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20 k a_{k+1} - a_k - 9 a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20 k a_{k+1} - a_k - 9 a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12 k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 +$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+x^2*diff(y(x),x)+(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x^2+2x} + \frac{c_2\sqrt{2}\left(\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x - \sqrt{2}\,\sqrt{x+2} + 2\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\sqrt{x\,(x+2)}}{2x+4}$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 65

```
DSolve[2*x^2*(2+x)*y''[x]+x^2*y'[x]+(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}\left(2(c_1\sqrt{x+2}+c_2) - \sqrt{2}c_2\sqrt{x+2}\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right)\right)}{2\sqrt[4]{2}}$$

2.135 problem 137

2.135.1 Maple step by step solution 1317

Internal problem ID [7625]
Internal file name [`OUTPUT/6558_Sunday_June_05_2022_04_59_05_PM_11033079/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 137.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(1+x)\,y''-x(-x+6)\,y'+(8-x)\,y=0}$$

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(x^2-6x\right)y'+(8-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2\\ B &= x^2-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 20x - 4$$
$$t = 16\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 255: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{21}{16(1+x)^2} - \frac{1}{4x^2} + \frac{3}{4(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16(x^2+x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(1 + x\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{3}{4\left(1 + x\right)} + \frac{1}{2x} \\
&= -\frac{x - 2}{4x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{3}{4(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2 - 20x - 4}{16(x^2+x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(1+x)^{\frac{3}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-6x}{2x^3+2x^2}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} - \frac{7\ln(1+x)}{4}} \\ &= z_1\left(\frac{x^{\frac{3}{2}}}{(1+x)^{\frac{7}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{(1+x)^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-6x}{2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{7\ln(1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln\left(\sqrt{1+x}-1\right) - \ln\left(\sqrt{1+x}+1\right) + \frac{(2x+8)\sqrt{1+x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{(1+x)^{\frac{5}{2}}}\right) \\
&\quad + c_2 \left(\frac{x^2}{(1+x)^{\frac{5}{2}}} \left(\ln\left(\sqrt{1+x}-1\right) - \ln\left(\sqrt{1+x}+1\right) + \frac{(2x+8)\sqrt{1+x}}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{(1+x)^{\frac{5}{2}}} + \frac{c_2 x^2 \left(2\sqrt{1+x}\,x + 8\sqrt{1+x} + 3\ln\left(\sqrt{1+x}-1\right) - 3\ln\left(\sqrt{1+x}+1\right)\right)}{3(1+x)^{\frac{5}{2}}} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 x^2}{(1+x)^{\frac{5}{2}}} + \frac{c_2 x^2 \left(2\sqrt{1+x}\,x + 8\sqrt{1+x} + 3\ln\left(\sqrt{1+x}-1\right) - 3\ln\left(\sqrt{1+x}+1\right)\right)}{3(1+x)^{\frac{5}{2}}}
$$

Verified OK.

2.135.1 Maple step by step solution

Let's solve

$(2x^3 + 2x^2) y'' + (x^2 - 6x) y' + (8 - x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-8)y}{2x^2(1+x)} - \frac{(x-6)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-6)y'}{2x(1+x)} - \frac{(x-8)y}{2x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-6}{2x(1+x)}, P_3(x) = -\frac{x-8}{2x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = \frac{7}{2}$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(1+x) y'' + x(x-6) y' + (8-x) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 8u + 7)\left(\frac{d}{du}y(u)\right) + (9-u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..1$

  $$u^{m}\cdot y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r(5+2r)\, u^{-1+r}+\left(a_{1}(1+r)\,(7+2r)-a_{0}(4r^{2}+4r-9)\right) u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$r(5+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,-\frac{5}{2}\right\}$$

- Each term must be 0

  $$a_{1}(1+r)\,(7+2r)-a_{0}(4r^{2}+4r-9)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-4a_{k}+2a_{k-1}+2a_{k+1})\,k^{2}+\left((-8a_{k}+4a_{k-1}+4a_{k+1})\,r-4a_{k}-5a_{k-1}+9a_{k+1}\right)k+(-4a_{k}+2a$$

- Shift index using $k->k+1$

  $$(-4a_{k+1}+2a_{k}+2a_{k+2})\,(k+1)^{2}+\left((-8a_{k+1}+4a_{k}+4a_{k+2})\,r-4a_{k+1}-5a_{k}+9a_{k+2}\right)(k+1)+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + 4kr a_k - 8kr a_{k+1} + 2r^2 a_k - 4r^2 a_{k+1} - k a_k - 12k a_{k+1} - r a_k - 12r a_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Recursion relation for $r = -\frac{5}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}$$

- Solution for $r = -\frac{5}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11k a_k + 8k a_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{5}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 +\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(6-x)*diff(y(x),x)+(8-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{(x+1)^{\frac{5}{2}}} + \frac{c_2 x^2 \left(\frac{2\sqrt{x+1}\,x}{3} + \frac{8\sqrt{x+1}}{3} + \ln\left(\sqrt{x+1}-1\right) - \ln\left(\sqrt{x+1}+1\right)\right)}{(x+1)^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 50

```
DSolve[2*x^2*(1+x)*y''[x]-x*(6-x)*y'[x]+(8-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{x^2\left(-6c_2\text{arctanh}\left(\sqrt{x+1}\right) + 2c_2\sqrt{x+1}(x+4) + 3c_1\right)}{3(x+1)^{5/2}}$$

2.136 problem 138

2.136.1 Maple step by step solution 1327

Internal problem ID [7626]
Internal file name [`OUTPUT/6559_Sunday_June_05_2022_04_59_07_PM_69939762/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 138.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''+x(5+9x)\,y'+(4+3x)\,y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 9x^2+5x \\ C &= 4+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^2 + 6x - 1$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 257: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{2x} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{7}{4}-\left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)} \\
&= \frac{1 + 7x}{4x^2 + 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x + \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right)\right)
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x + \frac{1}{2}\right)}\right) dx} \\
&= \sqrt{x}\, (2x + 1)^{\frac{5}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{9x^2 + 5x}{2x^3 + x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln(x)}{2} + \frac{\ln(2x+1)}{4}} \\
&= z_1 \left(\frac{(2x + 1)^{\frac{1}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(2x+1)^{\frac{3}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{\ln(2x+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{2x+1}\,\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}-1\right) - 2\sqrt{2x+1}\,\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}+1\right) + 4x + \frac{8}{3}}{(2x+1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(2x+1)^{\frac{3}{2}}}{x^2}\right) \\
&\quad + c_2 \left(\frac{(2x+1)^{\frac{3}{2}}}{x^2} \left(\frac{2\sqrt{2x+1}\,\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}-1\right) - 2\sqrt{2x+1}\,\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}+1\right) + 4x + \frac{8}{3}}{(2x+1)^{\frac{3}{2}}}\right.\right.
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1 (2x+1)^{\frac{3}{2}}}{x^2} \\
&\quad + \frac{c_2\left((6x+3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}-1\right) + (-6x-3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}+1\right) + 12x + 8\right)}{3x^2}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(2x+1)^{\frac{3}{2}}}{x^2} + \frac{c_2\left((6x+3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}-1\right)+(-6x-3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}+1\right)+12x+8\right)}{3x^2}$$

Verified OK.

2.136.1 Maple step by step solution

Let's solve

$(2x^3+x^2)\,y''+(9x^2+5x)\,y'+(4+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4+3x)y}{x^2(2x+1)}-\frac{(5+9x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(5+9x)y'}{x(2x+1)}+\frac{(4+3x)y}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5+9x}{x(2x+1)}, P_3(x)=\frac{4+3x}{x^2(2x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=4$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+2)^2 + a_{k-1} (k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\,(a_k (k+r+2) + a_{k-1} (2k-1+2r)) = 0$$

- Shift index using $k->k+1$

$$(k+r+3)\,(a_{k+1} (k+r+3) + a_k (2k+2r+1)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 130

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+9*x)*diff(y(x),x)+(4+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x+1)^{\frac{3}{2}}}{x^2} + \frac{c_2\left(-12\ln\left(\sqrt{2x+1}+1\right)x^2 + 12\ln\left(\sqrt{2x+1}-1\right)x^2 + 12\sqrt{2x+1}\,x - 12\ln\left(\sqrt{2x+1}+1\right)x + 12\ln}{3x^2\sqrt{2x+1}}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 56

```
DSolve[x^2*(1+2*x)*y''[x]+x*(5+9*x)*y'[x]+(4+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{2c_2\left(-3(2x+1)^{3/2}\text{arctanh}\left(\sqrt{2x+1}\right)+6x+4\right)+3c_1(2x+1)^{3/2}}{3x^2}$$

2.137 problem 139

2.137.1 Maple step by step solution 1337

Internal problem ID [7627]
Internal file name [`OUTPUT/6560_Sunday_June_05_2022_04_59_10_PM_98445039/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 139.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-2x)\,y''-x(4x+5)\,y'+(9+4x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -4x^2-5x \\ C &= 9+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 56x - 1}{4\,(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 32x^2 + 56x - 1$$
$$t = 4\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 56x - 1}{4\,(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 259: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{13}{x} - \frac{1}{4x^2} + \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= -1-(-2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)} + (-)(0) \\ &= \frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)} \\ &= \frac{-8x - 1}{4x^2 - 2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}\right)\right)$$

$$\frac{-1 + 8a_0}{x\left(2x - 1\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{8}\right) e^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x - \frac{1}{2}\right)}\right) dx} \\ &= \left(x + \frac{1}{8}\right) e^{\frac{\ln(x)}{2} - \frac{5\ln(2x-1)}{2}} \\ &= \frac{\left(x + \frac{1}{8}\right)\sqrt{x}}{\left(2x - 1\right)^{\frac{5}{2}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{7\ln(2x-1)}{2}} \\
&= z_1 \left(\frac{x^{\frac{5}{2}}}{(2x-1)^{\frac{7}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x+\frac{1}{8}\right)x^3}{(2x-1)^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-7\ln(2x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - 64\ln(x) - \frac{3125}{128x+16}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\frac{1}{8}\right)x^3}{(2x-1)^6}\right) + c_2 \left(\frac{\left(x+\frac{1}{8}\right)x^3}{(2x-1)^6}\left(\frac{32x^3}{3} - 44x^2 + \frac{203x}{2} - 64\ln(x) - \frac{3125}{128x+16}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x+\frac{1}{8}\right)x^3}{(2x-1)^6} + \frac{32c_2\left(\left(-6x-\frac{3}{4}\right)\ln(x) + x^4 - 4x^3 + 9x^2 + \frac{609x}{512} - \frac{9375}{4096}\right)x^3}{3(2x-1)^6} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \left(x+\frac{1}{8}\right) x^3}{\left(2x-1\right)^6}+\frac{32 c_2 \left(\left(-6x-\frac{3}{4}\right) \ln\left(x\right)+x^4-4x^3+9x^2+\frac{609x}{512}-\frac{9375}{4096}\right) x^3}{3 \left(2x-1\right)^6}$$

Verified OK.

2.137.1 Maple step by step solution

Let's solve

$(-2x^3+x^2) y''+(-4x^2-5x) y'+(9+4x) y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(9+4x)y}{x^2(2x-1)}-\frac{(4x+5)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x+5)y'}{x(2x-1)}-\frac{(9+4x)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{4x+5}{x(2x-1)}, P_3(x)=-\frac{9+4x}{x^2(2x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0}=-5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0}=9$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(4x+5) y'+(-4x-9) y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k- >k + 1$

$$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 63

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5+4*x)*diff(y(x),x)+(9+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^3(8x+1)}{(-1+2x)^6} + \frac{c_2 x^3\left(\frac{4x^4}{3} - \frac{16x^3}{3} - 8x\ln(x) + 12x^2 - \ln(x) + \frac{203x}{128} - \frac{3125}{1024}\right)}{(-1+2x)^6}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 63

```
DSolve[x^2*(1-2*x)*y''[x]-x*(5+4*x)*y'[x]+(9+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{x^3(c_2(4096x^4 - 16384x^3 + 36864x^2 + 4872x - 9375) - 48c_1(8x+1) - 3072c_2(8x+1)\log(x))}{384(1-2x)^6}$$

2.138 problem 140

2.138.1 Maple step by step solution 1347

Internal problem ID [7628]
Internal file name [`OUTPUT/6561_Sunday_June_05_2022_04_59_13_PM_40607812/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 140.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+x^2\\B&=x^2+7x\\C&=9-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 82x - 1$$
$$t = 4\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 261: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x} + \frac{20}{(x-1)^2} - \frac{1}{4x^2} - \frac{20}{x-1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4\,(x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{1}{2} - \left(-\frac{7}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{4}{x - 1} + (-)\,(0) \\ &= \frac{1}{2x} - \frac{4}{x - 1} \\ &= -\frac{1 + 7x}{2x\,(x - 1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{4}{x-1}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(x-1)^2}\right) + \left(\frac{1}{2x} - \frac{}{x}\right.\right.$$

$$\frac{(a_3 - 16)\, x^3 + (4a_2 - 9a_3)\, x^2 +}{x\,(x - }$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{4}{x-1}\right) dx} \\ &= \left(x^4 + 16x^3 + 36x^2 + 16x + 1\right) \mathrm{e}^{\frac{\ln(x)}{2} - 4\ln(x-1)} \\ &= \frac{\left(x^4 + 16x^3 + 36x^2 + 16x + 1\right)\sqrt{x}}{(x-1)^4} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\, dx} \\ &= z_1 e^{-\frac{7\ln(x)}{2} + 4\ln(x-1)} \\ &= z_1 \left(\frac{(x-1)^4}{x^{\frac{7}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-7\ln(x)+8\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{120x^3 + 450x^2 + 280x + 25}{3x^4 + 48x^3 + 108x^2 + 48x + 3} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3} \right) \\
&\quad + c_2 \left(\frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3} \left(\ln(x) + \frac{120x^3 + 450x^2 + 280x + 25}{3x^4 + 48x^3 + 108x^2 + 48x + 3} \right) \right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} \\
&\quad + \frac{c_2(25 + 3\ln(x)(x^4 + 16x^3 + 36x^2 + 16x + 1) + 120x^3 + 450x^2 + 280x)}{3x^3}
\end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$\begin{aligned}
y &= \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} \\
&\quad + \frac{c_2(25 + 3\ln(x)(x^4 + 16x^3 + 36x^2 + 16x + 1) + 120x^3 + 450x^2 + 280x)}{3x^3}
\end{aligned}$$

Verified OK.

2.138.1 Maple step by step solution

Let's solve

$(-x^3+x^2)\, y''+(x^2+7x)\, y'+(9-x)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-9+x)y}{x^2(x-1)}+\frac{(7+x)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(7+x)y'}{x(x-1)}+\frac{(-9+x)y}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{7+x}{x(x-1)}, P_3(x)=\frac{-9+x}{x^2(x-1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=7$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=9$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(x-1)-x(7+x)\, y'+y(-9+x)=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(3+r)^{2} x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+3)^{2}+a_{k-1}(k-2+r)^{2}\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-(3+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-3$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r+3)^{2}+a_{k-1}(k-2+r)^{2}=0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+4+r)^{2}+a_{k}(k+r-1)^{2}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-1)^{2}}{(k+4+r)^{2}}$

- Recursion relation for $r=-3$; series terminates at $k=4$

  $a_{k+1}=\frac{a_{k}(k-4)^{2}}{(k+1)^{2}}$

- Apply recursion relation for $k=0$

$a_1 = 16a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{9a_1}{4}$
- Express in terms of a_0

  $a_2 = 36a_0$
- Apply recursion relation for $k = 2$

  $a_3 = \frac{4a_2}{9}$
- Express in terms of a_0

  $a_3 = 16a_0$
- Apply recursion relation for $k = 3$

  $a_4 = \frac{a_3}{16}$
- Express in terms of a_0

  $a_4 = a_0$
- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 75

```
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(7+x)*diff(y(x),x)+(9-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4+16x^3+36x^2+16x+1)}{x^3} + \frac{c_2\left(x^4\ln(x)+16x^3\ln(x)+36x^2\ln(x)+40x^3+16x\ln(x)+150x^2+\ln(x)+\frac{280x}{3}+\frac{25}{3}\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 78

```
DSolve[x^2*(1-x)*y''[x]+x*(7+x)*y'[x]+(9-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5c_2(24x^3+90x^2+56x+5)+3c_1(x^4+16x^3+36x^2+16x+1)+3c_2(x^4+16x^3+36x^2+16x+1)\log}{3x^3}$$

2.139 problem 141

2.139.1 Maple step by step solution 1357

Internal problem ID [7629]
Internal file name [`OUTPUT/6562_Sunday_June_05_2022_04_59_15_PM_23048389/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 141.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(1-x^2)\,y' + (x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 263: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 4x^2 - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{4} - 1 - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3 - x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{x^2}{4}} \\ &= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\,\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\,\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}
$$

Verified OK.

2.139.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 - x)\,y' + (x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-1)\,y' + (x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_k(k+r-1) + a_{k-2}\right) = 0$$

- Shift index using $k- >k+2$

$$(k+r+1)\left(a_{k+2}(k+r+1) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{k+r+1}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{k+2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \, e^{-\frac{x^2}{2}} + c_2 x \, e^{-\frac{x^2}{2}} \operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)$$

✓ Solution by Mathematica
Time used: 0.031 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{2}} x \left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.140 problem 142

2.140.1 Maple step by step solution 1367

Internal problem ID [7630]
Internal file name [`OUTPUT/6563_Sunday_June_05_2022_04_59_18_PM_43282443/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 142.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-3x(1-x^2)\,y'+4y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(3x^3-3x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 3x^3-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^4 - 10x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 265: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{i}{4x-4i} - \frac{i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)} + (-)\left(0\right) \\
&= \frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)} \\
&= \frac{1}{2x} - \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2\left(x+i\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x-i\right)^2} + \frac{1}{2\left(x+i\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x-i\right)} - \frac{1}{2(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2}\,dx} \\ &= z_1 e^{\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\ &= z_1 \left(\frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{\left(x^2+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{3\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\ &= y_1 \left(\frac{x^2}{2}+\ln\left(x\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2}{\left(x^2+1\right)^2}\right) + c_2 \left(\frac{x^2}{\left(x^2+1\right)^2}\left(\frac{x^2}{2}+\ln\left(x\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2}{\left(x^2+1\right)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2}+\ln\left(x\right)\right)}{\left(x^2+1\right)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{(x^2+1)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2} + \ln(x)\right)}{(x^2+1)^2}$$

Verified OK.

2.140.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (3x^3 - 3x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{3(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x^2-1)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3(x^2-1)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -3$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1) y'' + 3x(x^2-1) y' + 4y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2 x^r + a_1(-1+r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\,(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term must be 0

  $$a_1(-1+r)^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\,(a_k(k+r-2) + a_{k-2}(k+r)) = 0$$

- Shift index using $k->k+2$

  $$(k+r)\,(a_{k+2}(k+r) + a_k(k+r+2)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r+2)}{k+r}$$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-3*x*(1-x^2)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{(x^2+1)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2} + \ln(x)\right)}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 36

```
DSolve[x^2*(1+x^2)*y''[x]-3*x*(1-x^2)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{x^2(c_2 x^2 + 2c_2 \log(x) + 2c_1)}{2(x^2+1)^2}$$

2.141 problem 143

2.141.1 Maple step by step solution 1376

Internal problem ID [7631]
Internal file name [OUTPUT/6564_Sunday_June_05_2022_04_59_21_PM_63286329/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 143.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2y'x^3 + (3x^2 + 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + 2y'x^3 + (3x^2 + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 267: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{4}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^2 - 4}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \,dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\,\mathrm{e}^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{4x^2}\,dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} - \frac{c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\,\text{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} - \frac{c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\,\text{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}$$

Verified OK.

2.141.1 Maple step by step solution

Let's solve

$4x^2y'' + 2y'x^3 + (3x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{xy'}{2} - \frac{(3x^2+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{2} + \frac{(3x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 2y'x^3 + (3x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

  ○ Shift index using $k->k-2$

$$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + a_1(1+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 34

```
dsolve(4*x^2*diff(y(x),x$2)+2*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{expIntegral}_1\left(-\frac{x^2}{4}\right)$$

✓ Solution by Mathematica
Time used: 0.121 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]+2*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{4}}\sqrt{x}\left(c_2\,\mathrm{ExpIntegralEi}\left(\frac{x^2}{4}\right) + 2c_1\right)$$

2.142 problem 144

2.142.1 Maple step by step solution 1386

Internal problem ID [7632]
Internal file name [`OUTPUT/6565_Sunday_June_05_2022_04_59_24_PM_34465619/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 144.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\, y''-x(-2x^2+1)\, y'+y=0}$$

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(2x^3-x\right) y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 269: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} - \frac{5i}{16(x-i)} + \frac{5i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= 0 \\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\ &= 1 - (1) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty}$$
$$= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)\,(0)$$
$$= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}$$
$$= \frac{1}{2x} + \frac{x}{2x^2+2}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4\,(x-i)^2} - \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+\cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = p e^{\int \omega\, dx}$$
$$= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx}$$
$$= \sqrt{x}\,\left(x^2+1\right)^{\frac{1}{4}}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$
$$= z_1 e^{-\int \frac{1}{2}\frac{2x^3-x}{x^4+x^2}\, dx}$$
$$= z_1 e^{\frac{\ln(x)}{2} - \frac{3\ln\left(x^2+1\right)}{4}}$$
$$= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$y_1 = \frac{x}{\sqrt{x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 \left(\frac{x}{\sqrt{x^2+1}} \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{\sqrt{x^2+1}} - \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{\sqrt{x^2+1}} - \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}}$$

Verified OK.

2.142.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (2x^3 - x) y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x^2+1)} - \frac{(2x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2-1)y'}{x(x^2+1)} + \frac{y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x^2-1}{x(x^2+1)}, P_3(x) = \frac{1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1) y'' + x(2x^2 - 1) y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-2+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term must be 0

  $$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k-2+r)) = 0$$

- Shift index using $k->k+2$

  $$(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k(k+1)}{k+2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-2*x^2)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{\sqrt{x^2+1}} + \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 33

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1-2*x^2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x\left(c_1 - c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right)}{\sqrt{x^2+1}}$$

2.143 problem 145

2.143.1 Maple step by step solution 1395

Internal problem ID [7633]
Internal file name [OUTPUT/6566_Sunday_June_05_2022_04_59_27_PM_176636/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 145.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+2)\,y''+7y'x^3+(3x^2+1)\,y=0}$$

Writing the ode as

$$(2x^4+4x^2)\,y''+7y'x^3+(3x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^4 - 16$$
$$t = 16\left(x^3 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 271: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + 2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{7}{64\left(x - i\sqrt{2}\right)^2} - \frac{7}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} + (0) \\
&= \frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}} \\
&= \frac{1}{2x} + \frac{x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x - i\sqrt{2}\right)^2} - \frac{1}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{1}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{8x - 8i\sqrt{2}} + \frac{1}{8x + 8i\sqrt{2}}\right) dx} \\
&= \left(x^2 + 2\right)^{\frac{1}{8}} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2+2\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(x^2+2\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+2\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x\left(x^2+2\right)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}\left(\int \frac{1}{x\left(x^2+2\right)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{x(x^2+2)^{\frac{1}{4}}}dx\right)}{(x^2+2)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{x(x^2+2)^{\frac{1}{4}}}dx\right)}{(x^2+2)^{\frac{3}{4}}}$$

Verified OK.

2.143.1 Maple step by step solution

Let's solve

$(2x^4+4x^2)\,y''+7y'x^3+(3x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+1)y}{2x^2(x^2+2)}-\frac{7y'x}{2(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{7y'x}{2(x^2+2)}+\frac{(3x^2+1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{7x}{2(x^2+2)}, P_3(x)=\frac{3x^2+1}{2x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=\frac{1}{4}$

  ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2x^2(x^2+2)\,y'' + 7y'x^3 + (3x^2+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^3 \cdot y'$ to series expansion

    $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+2}$

  - Shift index using $k- > k - 2$

    $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k-\frac{1}{2}+r\right)\right)\left(k-\frac{1}{2}+r\right) = 0$
- Shift index using $k- > k+2$

  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+7*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{(x^2+2)^{\frac{1}{4}}x}dx\right)}{(x^2+2)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.189 (sec). Leaf size: 77

```
DSolve[2*x^2*(2+x^2)*y''[x]+7*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sqrt{x}\left(2^{3/4}c_2 \arctan\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) - 2^{3/4}c_2 \operatorname{arctanh}\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) + 2c_1\right)}{2\left(x^2+2\right)^{3/4}}$$

2.144 problem 146

2.144.1 Maple step by step solution 1406

Internal problem ID [7634]
Internal file name [`OUTPUT/6567_Sunday_June_05_2022_04_59_30_PM_34417814/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 146.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+(2x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 4x^3-x \\ C &= 2x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -6x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 273: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{3i}{16(x-i)} - \frac{3i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)} \\
&= \frac{1}{2x^3 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\,(x-i)^2} + \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3 - x}{x^4 + x^2}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{5}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x^2+1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)-\frac{5\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{(x^2+1)^{\frac{3}{2}}}\right) + c_2 \left(\frac{x}{(x^2+1)^{\frac{3}{2}}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}}$$

Verified OK.

2.144.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (4x^3 - x) y' + (2x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x^2+1)y}{x^2(x^2+1)} - \frac{(4x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2-1)y'}{x(x^2+1)} + \frac{(2x^2+1)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4x^2-1}{x(x^2+1)}, P_3(x) = \frac{2x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1) y'' + x(4x^2 - 1) y' + (2x^2 + 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k+r)) = 0$

- Shift index using $k->k+2$

  $(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r+2)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-4*x^2)*diff(y(x),x)+(1+2*x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 45

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1-4*x^2)*y'[x]+(1+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{x\left(-c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right) + c_2\sqrt{x^2+1} + c_1\right)}{(x^2+1)^{3/2}}$$

2.145 problem 147

2.145.1 Maple step by step solution 1415

Internal problem ID [7635]
Internal file name [`OUTPUT/6568_Sunday_June_05_2022_04_59_32_PM_37048508/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 147.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+4)\,y''+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+16x^2 \\ B &= 9x^3+24x \\ C &= -9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 153x^4 + 704x^2 - 256$$
$$t = 64\left(x^3 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 275: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{39}{256 (x - 2i)^2} - \frac{39}{256 (x + 2i)^2} - \frac{377i}{512 (x - 2i)} + \frac{377i}{512 (x + 2i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{16} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{16}
\end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{16} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{16}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{17}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{9}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{17}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{17}{8} - \left(\frac{17}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} + (0) \\ &= \frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)} \\ &= \frac{1}{2x} + \frac{13x}{8x^2+32}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x-2i)} + \frac{13}{16\,(x+2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x-2i)^2} - \frac{13}{16\,(x+2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right) dx} \\ &= \sqrt{x}\,(x^2+4)^{\frac{13}{16}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}-\frac{3\ln\left(x^2+4\right)}{16}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}}\left(x^2+4\right)^{\frac{3}{16}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^3+24x}{4x^4+16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+4\right)}{8}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x\left(x^2+4\right)^{\frac{13}{8}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}\right) + c_2 \left(\frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}\left(\int \frac{1}{x\left(x^2+4\right)^{\frac{13}{8}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2(x^2+4)^{\frac{5}{8}} \left(\int \frac{1}{x(x^2+4)^{\frac{13}{8}}}dx\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2(x^2+4)^{\frac{5}{8}} \left(\int \frac{1}{x(x^2+4)^{\frac{13}{8}}}dx\right)}{x^{\frac{1}{4}}}$$

Verified OK.

2.145.1 Maple step by step solution

Let's solve

$(4x^4+16x^2)\,y''+(9x^3+24x)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(9x^2-1)y}{4x^2(x^2+4)}-\frac{3(3x^2+8)y'}{4x(x^2+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(3x^2+8)y'}{4x(x^2+4)}-\frac{(9x^2-1)y}{4x^2(x^2+4)}=0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x)=\frac{3(3x^2+8)}{4x(x^2+4)}, P_3(x)=-\frac{9x^2-1}{4x^2(x^2+4)}\right]$

- ∘ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{3}{2}$

- ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=\frac{1}{16}$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2(x^2+4)\, y'' + 3x(3x^2+8)\, y' + (-9x^2+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+4r)^2 x^r + a_1(5+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-2}(4k+4r+1)\,(k-3+r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+4r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = -\frac{1}{4}$

- Each term must be 0

$a_1(5+4r)^2=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $16\left(\frac{a_{k-2}(k-3+r)}{4}+a_k\left(k+r+\frac{1}{4}\right)\right)\left(k+r+\frac{1}{4}\right)=0$

- Shift index using $k->k+2$

  $16\left(\frac{a_k(k+r-1)}{4}+a_{k+2}\left(k+\frac{9}{4}+r\right)\right)\left(k+\frac{9}{4}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r-1)}{4k+4r+9}$

- Recursion relation for $r=-\frac{1}{4}$

  $a_{k+2}=-\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$

- Solution for $r=-\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{4}}, a_{k+2}=-\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}, a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(4*x^2*(4+x^2)*diff(y(x),x$2)+3*x*(8+3*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1 (x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2 (x^2+4)^{\frac{5}{8}} \left(\int \frac{1}{(x^2+4)^{\frac{13}{8}} x} dx\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.526 (sec). Leaf size: 198

```
DSolve[4*x^2*(4+x^2)*y''[x]+3*x*(8+3*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{c_2\left(5\ 2^{3/4}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt[8]{x^2+4}}{\sqrt[4]{2}}\right)+5\sqrt[4]{2}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt{2}-\sqrt[4]{x^2+4}}{2^{3/4}\sqrt[8]{x^2+4}}\right)-5\ 2^{3/4}(x^2+4)^{5/8}\,a\right.}{80\sqrt[4]{x}}$$

2.146 problem 148

2.146.1 Maple step by step solution 1426

Internal problem ID [7636]
Internal file name [`OUTPUT/6569_Sunday_June_05_2022_04_59_35_PM_17531537/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 148.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2(x^2+3)\,y''+x(11x^2+3)\,y'+(5x^2+1)\,y=0}$$

Writing the ode as

$$(3x^4+9x^2)\,y''+(11x^3+3x)\,y'+(5x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x^4+9x^2\\ B&=11x^3+3x\\ C&=5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 + 18x^2 - 81$$
$$t = 36\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 277: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{5}{36\left(x-i\sqrt{3}\right)^2} - \frac{5}{36\left(x+i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x-i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x+i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} + (0) \\
&= \frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \\
&= \frac{1}{2x} + \frac{x}{3x^2+9}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\,(x-i\sqrt{3})^2} - \frac{1}{6\,(x+i\sqrt{3})^2} \right) + \left(\frac{1}{2x} + \frac{}{6x} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{6x-6i\sqrt{3}} + \frac{1}{6x+6i\sqrt{3}} \right) dx} \\
&= (x^2+3)^{\frac{1}{6}} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{6}-\frac{5\ln(x^2+3)}{6}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{6}}\left(x^2+3\right)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-\frac{5\ln(x^2+3)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x\left(x^2+3\right)^{\frac{1}{3}}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}\left(\int \frac{1}{x\left(x^2+3\right)^{\frac{1}{3}}}\,dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{1}{x(x^2+3)^{\frac{1}{3}}} dx\right)}{(x^2+3)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{1}{x(x^2+3)^{\frac{1}{3}}} dx\right)}{(x^2+3)^{\frac{2}{3}}}$$

Verified OK.

2.146.1 Maple step by step solution

Let's solve

$(3x^4+9x^2)\, y'' + (11x^3+3x)\, y' + (5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+1)y}{3x^2(x^2+3)} - \frac{(11x^2+3)y'}{3x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(11x^2+3)y'}{3x(x^2+3)} + \frac{(5x^2+1)y}{3x^2(x^2+3)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{11x^2+3}{3x(x^2+3)}, P_3(x) = \frac{5x^2+1}{3x^2(x^2+3)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3x^2(x^2+3)\, y'' + x(11x^2+3)\, y' + (5x^2+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+3r)^2 x^r + a_1(2+3r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)\,(k+r-1)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(\frac{a_{k-2}(k+r-1)}{3} + a_k\left(k+r-\frac{1}{3}\right)\right)\left(k+r-\frac{1}{3}\right) = 0$

- Shift index using $k- > k+2$

  $9\left(\frac{a_k(k+r+1)}{3} + a_{k+2}\left(k+\frac{5}{3}+r\right)\right)\left(k+\frac{5}{3}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(3*x^2*(3+x^2)*diff(y(x),x$2)+x*(3+11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x), singsol
```

$$y(x)=\frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}}+\frac{c_2 x^{\frac{1}{3}}\left(\int \frac{1}{(x^2+3)^{\frac{1}{3}} x}dx\right)}{(x^2+3)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 94

```
DSolve[3*x^2*(3+x^2)*y'[x]+x*(3+11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x)\to \frac{c_1 \exp\left(\frac{1}{3}\text{RootSum}\left[3\#1^3+11\#1^2+9\#1+3\&,\frac{3\#1^2\log(x-\#1)-4\#1\log(x-\#1)+9\log(x-\#1)}{9\#1^2+22\#1+9}\&\right]\right)}{\sqrt[3]{x}}$$

$$y(x)\to 0$$

2.147 problem 149

2.147.1 Maple step by step solution 1437

Internal problem ID [7637]
Internal file name [`OUTPUT/6570_Sunday_June_05_2022_04_59_40_PM_67492986/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 149.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' - 3x(-2x^2 + 7)\, y' + (2x^2 + 25)\, y = 0}$$

Writing the ode as

$$9x^2y'' + (6x^3 - 21x)\, y' + (2x^2 + 25)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3 - 21x \\ C &= 2x^2 + 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 24x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 279: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 24x^2 - 9}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{3}\right) - (0) \\ &= -\frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{3} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\ &= \frac{1}{2x} - \frac{x}{3} \\ &= \frac{1}{2x} - \frac{x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6x^3 - 21x}{9x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{6} + \frac{7\ln(x)}{6}} \\ &= z_1 \left(x^{\frac{7}{6}} \mathrm{e}^{-\frac{x^2}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}}\right) + c_2\left(x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}}\left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} - \frac{c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} - \frac{c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}$$

Verified OK.

2.147.1 Maple step by step solution

Let's solve

$9x^2 y'' + (6x^3 - 21x)\,y' + (2x^2 + 25)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(2x^2+25)y}{9x^2} - \frac{(2x^2-7)y'}{3x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(2x^2-7)y'}{3x} + \frac{(2x^2+25)y}{9x^2} = 0$$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{7}{3}$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{25}{9}$$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $$x_0 = 0$$

- Multiply by denominators

$$9x^2y'' + 3x(2x^2-7)\,y' + (2x^2+25)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-5+3r)^2\, x^r + a_1(-2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-5+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{5}{3}$

- Each term must be 0

  $a_1(-2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{2a_k}{3k+6}$

- Solution for $r = \frac{5}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 34

```
dsolve(9*x^2*diff(y(x),x$2)-3*x*(7-2*x^2)*diff(y(x),x)+(25+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} + c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \operatorname{expIntegral}_1\left(-\frac{x^2}{3}\right)$$

✓ Solution by Mathematica
Time used: 0.112 (sec). Leaf size: 39

```
DSolve[9*x^2*y''[x]-3*x*(7-2*x^2)*y'[x]+(25+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{3}} x^{5/3} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1\right)$$

2.148 problem 150

2.148.1 Maple step by step solution 1447

Internal problem ID [7638]
Internal file name [`OUTPUT/6571_Sunday_June_05_2022_04_59_42_PM_65075313/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 150.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(1-x^2)\,y' + (x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 281: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 4x^2 - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{4} - 1 - \frac{1}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &= -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3 - x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{x^2}{4}} \\ &= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx$$
$$= y_1 \left(-\frac{\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}$$

Verified OK.

2.148.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 - x)\,y' + (x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = -1$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2y'' + x(x^2-1)\,y' + (x^2+1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{k+r+1}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{k+2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \,\mathrm{e}^{-\frac{x^2}{2}} + c_2 x \,\mathrm{e}^{-\frac{x^2}{2}} \,\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{2}} x \left(c_1 \,\mathrm{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.149 problem 151

2.149.1 Maple step by step solution 1457

Internal problem ID [7639]
Internal file name [OUTPUT/6572_Sunday_June_05_2022_04_59_45_PM_71970995/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 151.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''+3xy'+(1+4x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+3xy'+(1+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 3x \\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 16x - 1}{4(2x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 32x^2 + 16x - 1 \\ t &= 4(2x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 16x - 1}{4(2x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 283: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{x} - \frac{1}{4x^2} + \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r=\frac{s}{t}=\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right)\\ &= -1-(-1)\\ &= 0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-1-4x}{4x^2-2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right)\right)
$$

$$
0 = (
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{(2x-1)^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{-2x^3+x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} + \frac{3\ln(2x-1)}{2}} \\
&= z_1 \left(\frac{(2x-1)^{\frac{3}{2}}}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)+3\ln(2x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2\left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2\left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)}{x}$$

Verified OK.

2.149.1 Maple step by step solution

Let's solve

$\left(-2x^3+x^2\right)y''+3xy'+\left(1+4x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{\left(1+4x\right)y}{x^2\left(2x-1\right)}+\frac{3y'}{x\left(2x-1\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3y'}{x\left(2x-1\right)}-\frac{\left(1+4x\right)y}{x^2\left(2x-1\right)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{3}{x\left(2x-1\right)},P_3(x)=-\frac{1+4x}{x^2\left(2x-1\right)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2\left(2x-1\right)-3xy'+\left(-1-4x\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r}$
- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(1+r)^{2}\,x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+1)^{2}+2a_{k-1}(k+r)\,(k-3+r)\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)^{2}=0$
- Values of r that satisfy the indicial equation

  $r=-1$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r+1)^{2}+2a_{k-1}(k+r)\,(k-3+r)=0$
- Shift index using $k->k+1$

  $-a_{k+1}(k+2+r)^{2}+2a_{k}(k+r+1)\,(k+r-2)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_{k}(k+r+1)(k+r-2)}{(k+2+r)^{2}}$
- Recursion relation for $r=-1$; series terminates at $k=3$

  $a_{k+1}=\frac{2a_{k}k(k-3)}{(k+1)^{2}}$
- Apply recursion relation for $k=0$

  $a_{1}=0$
- Apply recursion relation for $k=1$

  $a_{2}=-a_{1}$

- Express in terms of a_0

  $a_2 = 0$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{9}$
- Express in terms of a_0

  $a_3 = 0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot 0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} - \frac{c_2(-8x^3 + 18x^2 + 3\ln(x) - 18x)}{3x}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 36

```
DSolve[x^2*(1-2*x)*y''[x]+3*x*y'[x]+(1+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{3}c_2\left(4x^2-9x+9\right)+\frac{c_1}{x}+\frac{c_2\log(x)}{x}$$

2.150 problem 152

2.150.1 Maple step by step solution 1467

Internal problem ID [7640]
Internal file name [`OUTPUT/6573_Sunday_June_05_2022_04_59_48_PM_16584098/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 152.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)\,y''+(1-x)\,y'+y=0}$$

Writing the ode as

$$\left(x^{2}+x\right)y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^{2}+x\\ B&=1-x\\ C&=1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 10x - 1 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 285: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4(x^2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x\,(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{1+a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\ln(1+x)} \\
&= z_1 \left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4)
$$

Verified OK.

2.150.1 Maple step by step solution

Let's solve

$(x^2+x)\, y''+(1-x)\, y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(1+x)}+\frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-1)y'}{x(1+x)}+\frac{y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x-1}{x(1+x)}, P_3(x)=\frac{1}{x(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-2$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)\, y''+(1-x)\, y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m} \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m = 1..2$

  $u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^{2}\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)^{2} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)^{2}}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)^{2}}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x*(1+x)*diff(y(x),x$2)+(1-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x - 1) + c_2(x \ln(x) - \ln(x) - 4)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 23

```
DSolve[x*(1+x)*y''[x]+(1-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

2.151 problem 153

2.151.1 Maple step by step solution 1476

Internal problem ID [7641]
Internal file name [OUTPUT/6574_Sunday_June_05_2022_04_59_51_PM_4044869/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 153.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= 5x^2-3x \\ C &= 4-5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 6x - 1 \\ t &= 4\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 287: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(x-1)^2} - \frac{1}{4x^2} + \frac{2}{x-1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4(x^2-x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{x - 1} + (0) \\
&= \frac{1}{2x} + \frac{2}{x - 1} \\
&= \frac{5x - 1}{2x\,(x - 1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{x-1}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{x-1}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{x-1}\right) dx} \\ &= \sqrt{x}\,(x-1)^2 \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2 - 3x}{-x^3 + x^2}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} + \ln(x-1)} \\ &= z_1\left(x^{\frac{3}{2}}(x-1)\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2(x-1)^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-3x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)+2\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) - \frac{1}{3(x-1)^3} - \frac{1}{x-1} + \frac{1}{2(x-1)^2} - \ln(x-1)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2(x-1)^3\right) + c_2\left(x^2(x-1)^3\left(\ln(x) - \frac{1}{3(x-1)^3} - \frac{1}{x-1} + \frac{1}{2(x-1)^2} - \ln(x-1)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2(x-1)^3 + c_2\left(-\ln(x-1)(x-1)^3 + \ln(x)(x-1)^3 - x^2 + \frac{5x}{2} - \frac{11}{6}\right)x^2(1)
$$

Verification of solutions

$$
y = c_1 x^2(x-1)^3 + c_2\left(-\ln(x-1)(x-1)^3 + \ln(x)(x-1)^3 - x^2 + \frac{5x}{2} - \frac{11}{6}\right)x^2
$$

Verified OK.

2.151.1 Maple step by step solution

Let's solve

$(-x^3+x^2)\,y'' + (5x^2-3x)\,y' + (4-5x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x-4)y}{x^2(x-1)} + \frac{(5x-3)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(5x-3)y'}{x(x-1)} + \frac{(5x-4)y}{x^2(x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{5x-3}{x(x-1)}, P_3(x) = \frac{5x-4}{x^2(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1) - x(5x-3)\, y' + y(5x-4) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$$

- Shift index using $k->k+1$

  $$-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$$

- Recursion relation for $r=2$; series terminates at $k=3$

  $$a_{k+1} = \frac{a_k(k-3)}{k+1}$$

- Apply recursion relation for $k=0$

  $$a_1 = -3a_0$$

- Apply recursion relation for $k=1$

  $$a_2 = -a_1$$

- Express in terms of a_0

  $$a_2 = 3a_0$$

- Apply recursion relation for $k=2$

  $$a_3 = -\frac{a_2}{3}$$

- Express in terms of a_0

  $$a_3 = -a_0$$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second li

$$y = a_0 \cdot (-x^3 + 3x^2 - 3x + 1)$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve(x^2*(1-x)*diff(y(x),x$2)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 (x^3 - 3x^2 + 3x - 1) + c_2 x^2 \Big(-\ln(x-1)\, x^3 + x^3 \ln(x) + 3\ln(x-1)\, x^2 - 3x^2 \ln(x) - 3\ln(x-1)\, x + 3x\ln(x) - x^2 + \ln(x-1) - \ln(x) + \frac{5x}{2} - \frac{11}{6} \Big)$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 76

```
DSolve[x^2*(1-x)*y''[x]-x*(3-5*x)*y'[x]+(4-5*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{1}{6} x^2 (6c_1 x^3 - 18c_1 x^2 - 6c_2 x^2 + 18c_1 x + 15c_2 x - 6c_2 (x-1)^3 \log(x-1) + 6c_2 (x-1)^3 \log(x) - 6c_1 - 11c_2)$$

2.152 problem 154

2.152.1 Maple step by step solution 1486

Internal problem ID [7642]
Internal file name [`OUTPUT/6575_Sunday_June_05_2022_04_59_53_PM_56262628/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 154.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(9x^2+1)\,y'+(25x^2+1)\,y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(-9x^3-x)\,y'+(25x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= -9x^3-x \\ C &= 25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 - 98x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 289: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)\,(0) \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right)\left(4x^3 + 3x^2a_3 + 2a_2x + a_1\right) + \left(\left(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{}{(x-} \right.\right. \cdots$$

$$\cdots \frac{}{(x^2+1)\,(x^4a \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i}\right) dx} \\
&= \left(x^4 - 4x^2 + 1\right) \mathrm{e}^{\frac{\ln(x)}{2} - 2\ln(x^2+1)} \\
&= \frac{\left(x^4 - 4x^2 + 1\right)\sqrt{x}}{\left(x^2+1\right)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^3-x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}+2\ln(x^2+1)} \\
&= z_1\left(\sqrt{x}\,(x^2+1)^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 4x^3 + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3-x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)+4\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-6x^2+3}{x^4-4x^2+1} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^5-4x^3+x) + c_2\left(x^5-4x^3+x\left(\frac{-6x^2+3}{x^4-4x^2+1}+\ln(x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(x^5-4x^3+x) + c_2 x\left(-6x^2+3+\ln(x)\,(x^4-4x^2+1)\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^5-4x^3+x) + c_2 x\left(-6x^2+3+\ln(x)\,(x^4-4x^2+1)\right)$$

Verified OK.

2.152.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (-9x^3 - x) y' + (25x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2+1)y}{x^2(x^2+1)} + \frac{(9x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(9x^2+1)y'}{x(x^2+1)} + \frac{(25x^2+1)y}{x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{9x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1) y'' - x(9x^2 + 1) y' + (25x^2 + 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2}\,x^{r}+a_{1}r^{2}x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r-1)^{2}+a_{k-2}(k-7+r)^{2}\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$
- Values of r that satisfy the indicial equation

  $r=1$
- Each term must be 0

  $a_{1}r^{2}=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r-1)^{2}+a_{k-2}(k-7+r)^{2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)^{2}+a_{k}(k+r-5)^{2}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r-5)^{2}}{(k+1+r)^{2}}$

- Recursion relation for $r = 1$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k (k-4)^2}{(k+2)^2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1+9*x^2)*diff(y(x),x)+(1+25*x^2)*y(x)=0,y(x), singsol=a
```

$$y(x) = c_1 x(x^4 - 4x^2 + 1) + c_2 (x^4 \ln(x) - 4x^2 \ln(x) - 6x^2 + \ln(x) + 3) x$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 43

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1+9*x^2)*y'[x]+(1+25*x^2)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to c_1 (x^5 - 4x^3 + x) + c_2 x(-6x^2 + (x^4 - 4x^2 + 1) \log(x) + 3)$$

2.153 problem 155

2.153.1 Maple step by step solution 1496

Internal problem ID [7643]
Internal file name [OUTPUT/6576_Sunday_June_05_2022_04_59_56_PM_78652149/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 155.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + 3x(1-x^2)\,y' + (7x^2+1)\,y = 0}$$

Writing the ode as

$$9x^2y'' + (-3x^3+3x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 36x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 36x^2 - 9}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{36} - 1 - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &= -\frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 36x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{2x} - \frac{x}{6} \\ &= \frac{1}{2x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)\right) = 0$$

$$\frac{x^2 a_1 + 2(6 + a_0) x + 3a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6\right) e^{\int \left(\frac{1}{2x} - \frac{x}{6}\right) dx} \\
&= \left(x^2 - 6\right) e^{-\frac{x^2}{12} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 6\right) \sqrt{x}\, e^{-\frac{x^2}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x^3 + 3x}{9x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{6} + \frac{x^2}{12}} \\
&= z_1 \left(\frac{e^{\frac{x^2}{12}}}{x^{\frac{1}{6}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}}\left(x^2 - 6\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{3}}\left(x^2-6\right)\right) + c_2\left(x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\left(x^2-6\right) + c_2 x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}}\left(x^2-6\right) + c_2 x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)$$

Verified OK.

2.153.1 Maple step by step solution

Let's solve

$9x^2y'' + (-3x^3 + 3x)\,y' + (7x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{9x^2} + \frac{(x^2-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{3x} + \frac{(7x^2+1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-1}{3x}, P_3(x) = \frac{7x^2+1}{9x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{9}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9x^2y'' - 3x(x^2 - 1)\,y' + (7x^2 + 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+3r)^{2}\,x^{r}+a_{1}(2+3r)^{2}\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(3k+3r-1)^{2}-a_{k-2}(3k-13+3r)\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=\frac{1}{3}$

- Each term must be 0

  $a_{1}(2+3r)^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(3k+3r-1)^{2}+(-3k+13-3r)\,a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(3k+5+3r)^{2}+a_{k}(-3k-3r+7)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(3k+3r-7)}{(3k+5+3r)^{2}}$

- Recursion relation for $r=\frac{1}{3}$; series terminates at $k=2$

  $a_{k+2}=\frac{a_{k}(3k-6)}{(3k+6)^{2}}$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(1-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}} (x^2 - 6) + c_2 x^{\frac{1}{3}} (x^2 - 6) \left(\int \frac{e^{\frac{x^2}{6}}}{(x^2 - 6)^2 x} dx\right)$$

✓ Solution by Mathematica

Time used: 3.367 (sec). Leaf size: 53

```
DSolve[9*x^2*y''[x]+3*x*(1-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{72}\sqrt[3]{x}\left(c_2(x^2-6)\,\text{ExpIntegralEi}\left(\frac{x^2}{6}\right)+72c_1(x^2-6)-6c_2e^{\frac{x^2}{6}}\right)$$

2.154 problem 156

2.154.1 Maple step by step solution 1506

Internal problem ID [7644]
Internal file name [`OUTPUT/6577_Sunday_June_05_2022_05_00_00_PM_511034/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 156.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+1)\,y''+(1-x^2)\,y'-8yx=0}$$

Writing the ode as

$$\left(x^3+x\right)y''+\left(1-x^2\right)y'-8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^3+x\\B&=1-x^2\\C&=-8x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 22x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 293: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (0) \\ &= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\ &= \frac{1}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int\left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)dx} \\ &= \sqrt{x}\,\left(x^2+1\right)^{\frac{3}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x^2}{x^3+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x^2}{x^3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) + \frac{1}{2x^2+2} + \frac{1}{4\left(x^2+1\right)^2} - \frac{\ln\left(x^2+1\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2 \left(\ln(x) + \frac{1}{2x^2+2} + \frac{1}{4\left(x^2+1\right)^2} - \frac{\ln\left(x^2+1\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+1\right)^2 + c_2\left(\ln(x)\left(x^2+1\right)^2 + \frac{x^2}{2} + \frac{3}{4} - \frac{\ln\left(x^2+1\right)\left(x^2+1\right)^2}{2}\right) \tag{1}
$$

Verification of solutions

$$y = c_1 (x^2+1)^2 + c_2 \left(\ln(x)(x^2+1)^2 + \frac{x^2}{2} + \frac{3}{4} - \frac{\ln(x^2+1)(x^2+1)^2}{2}\right)$$

Verified OK.

2.154.1 Maple step by step solution

Let's solve

$(x^3+x)\,y'' + (1-x^2)\,y' - 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{8y}{x^2+1} + \frac{(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{x(x^2+1)} - \frac{8y}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-1}{x(x^2+1)}, P_3(x) = -\frac{8}{x^2+1}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\,y'' + (1-x^2)\,y' - 8yx = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+r+1)^2 + a_{k-1} (k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1 (1+r)^2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1}\right)(k+1) = 0$$

- Shift index using $k->k+1$

$$((a_k + a_{k+2})(k+1) - 5a_k + a_{k+2})(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k-4)}{k+2}$$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$$a_{k+2} = -\frac{a_k(k-4)}{k+2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(x*(1+x^2)*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 2x^2 + 1) + c_2\left(-\frac{\ln(x^2+1)x^4}{2} + x^4\ln(x) - \ln(x^2+1)x^2 + 2x^2\ln(x) + \frac{x^2}{2} - \frac{\ln(x^2+1)}{2} + \ln(x) + \frac{3}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 55

```
DSolve[x*(1+x^2)*y''[x]+(1-x^2)*y'[x]-8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 (x^2+1)^2 + \frac{1}{4} c_2 \left(2x^2 + 4(x^2+1)^2 \log(x) - 2(x^2+1)^2 \log(x^2+1) + 3\right)$$

2.155 problem 157

2.155.1 Maple step by step solution 1517

Internal problem ID [7645]
Internal file name [OUTPUT/6578_Sunday_June_05_2022_05_00_03_PM_34558943/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 157.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x(-x^2+4)\,y' + (7x^2+1)\,y = 0}$$

Writing the ode as

$$4x^2y'' + (-2x^3+8x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 40x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 295: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 40x^2 - 4}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &= -\frac{11}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &= \frac{9}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 40x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{9}{2} - \left(\frac{1}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(4x^3 + 3a_3x^2 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4}{\cdots}\right.\right.$$
$$\frac{x^4a_3 + 2(16 + a_2)\, x^3 + 3(a_1 + 6a_3)\, x^2 + 4(a_0 + 2a\cdots}{2x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 16x^2 + 32$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x^4 - 16x^2 + 32)\, e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\
&= (x^4 - 16x^2 + 32)\, e^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\
&= (x^4 - 16x^2 + 32)\, \sqrt{x}\, e^{-\frac{x^2}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^3+8x}{4x^2}\, dx} \\
&= z_1 e^{\frac{x^2}{8} - \ln(x)} \\
&= z_1 \left(\frac{e^{\frac{x^2}{8}}}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 - 16x^2 + 32}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{4} - 2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x\,(x^4 - 16x^2 + 32)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \right) + c_2 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x\,(x^4 - 16x^2 + 32)^2} dx \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32) \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x(x^4 - 16x^2 + 32)^2} dx \right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32) \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x(x^4 - 16x^2 + 32)^2} dx \right)}{\sqrt{x}}$$

Verified OK.

2.155.1 Maple step by step solution

Let's solve

$4x^2y'' + (-2x^3 + 8x)\, y' + (7x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{4x^2} + \frac{(x^2-4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-4)y'}{2x} + \frac{(7x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2-4}{2x}, P_3(x) = \frac{7x^2+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 2x(x^2 - 4)\, y' + (7x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)^2\, x^r + a_1(3+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)^2 - a_{k-2}(2k-11+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{2}$

- Each term must be 0

  $a_1(3+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(2k+2r-7)}{(2k+5+2r)^2}$

- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$

  $a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(4*x^2*diff(y(x),x$2)+2*x*(4-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32)\left(\int \frac{e^{\frac{x^2}{4}}}{x(x^4-16x^2+32)^2}dx\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.442 (sec). Leaf size: 68

```
DSolve[4*x^2*y''[x]+2*x*(4-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_2(x^4-16x^2+32)\operatorname{ExpIntegralEi}\left(\frac{x^2}{4}\right)-4c_2e^{\frac{x^2}{4}}(x^2-12)+2048c_1(x^4-16x^2+32)}{2048\sqrt{x}}$$

2.156 problem 158

2.156.1 Maple step by step solution 1526

Internal problem ID [7646]
Internal file name [`OUTPUT/6579_Sunday_June_05_2022_05_00_06_PM_35953695/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 158.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 297: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\ln(1+x)} \\ &= z_1\left(\frac{1}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{1+x}\right) + c_2 \left(\frac{\sqrt{x}}{1+x} (\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}}{1+x} + \frac{c_2 \sqrt{x}\, \ln(x)}{1+x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{x}}{1+x} + \frac{c_2 \sqrt{x}\, \ln(x)}{1+x}
$$

Verified OK.

2.156.1 Maple step by step solution

Let's solve

$(4x^3 + 4x^2) y'' + 8x^2 y' + (1+x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{1+x} - \frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{1+x} + \frac{y}{4x^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{1+x}, P_3(x) = \frac{1}{4x^2}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 2$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\, y'' + 8x^2 y' + (1+x)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (8u^2 - 16u + 8)\left(\frac{d}{du}y(u)\right) + uy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u \cdot y(u)$ to series expansion

  $u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$

- Shift index using $k- >k-1$

  $u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$4a_0r(1+r)\,u^{-1+r} + (4a_1(1+r)\,(2+r) - 8a_0r(1+r))\,u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(k+2+r) -\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 0\}$$

- Each term must be 0

  $$4a_1(1+r)\,(2+r) - 8a_0r(1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(2k-1+2r)^2 - 8\left(\left(-\frac{k}{2} - \frac{r}{2} - 1\right) a_{k+1} + a_k(k+r)\right)(k+r+1) = 0$$

- Shift index using $k- >k+1$

  $$a_k(2k+2r+1)^2 - 8\left(\left(-\frac{k}{2} - \frac{3}{2} - \frac{r}{2}\right) a_{k+2} + a_{k+1}(k+r+1)\right)(k+2+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 4ka_k - 24ka_{k+1} + 4ra_k - 24ra_{k+1} + a_k - 16a_{k+1}}{4(k+3+r)(k+2+r)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}$$

- Solution for $r = -1$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-1}, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 8ka_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 4ka_k - 24ka_{k+1} + a_k - 16a_{k+1}}{4(k+3)(k+2)}, 8a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(1+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k(1+x)^{k}\right), a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-8ka_{k+1}+a_k}{4(k+2)(k+1)}, 0=0, b_{k+2}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+8*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1\sqrt{x}}{x+1}+\frac{c_2\sqrt{x}\,\ln(x)}{x+1}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 24

```
DSolve[4*x^2*(1+x)*y''[x]+8*x^2*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{\sqrt{x}(c_2\log(x)+c_1)}{x+1}$$

2.157 problem 159

2.157.1 Maple step by step solution 1535

Internal problem ID [7647]
Internal file name [OUTPUT/6580_Sunday_June_05_2022_05_00_09_PM_99286175/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 159.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2(x+3)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0}$$

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=9x^3+27x^2\\ B&=21x^2+9x\\ C&=3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 299: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{21x^2+9x}{9x^3+27x^2} \, dx} \\ &= z_1 e^{-\ln(x+3) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(x+3)\, x^{\frac{1}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x+3) - \frac{\ln(x)}{3}}}{(y_1)^2} \, dx \\ &= y_1(\ln(x) - 1) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^{\frac{1}{3}}}{x+3}\right) + c_2\left(\frac{x^{\frac{1}{3}}}{x+3}(\ln(x)-1)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}}(\ln(x)-1)}{x+3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}}(\ln(x)-1)}{x+3}
$$

Verified OK.

2.157.1 Maple step by step solution

Let's solve

$(9x^3+27x^2)\,y''+(21x^2+9x)\,y'+(3+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3+4x)y}{9x^2(x+3)}-\frac{(3+7x)y'}{3x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3+7x)y'}{3x(x+3)}+\frac{(3+4x)y}{9x^2(x+3)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3+7x}{3x(x+3)}, P_3(x)=\frac{3+4x}{9x^2(x+3)}\right]$

- ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3}=2$

- ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

$$\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = 0$$

- $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $$9x^2(x+3)\, y'' + 3x(3+7x)\, y' + (3+4x)\, y = 0$$

- Change variables using $x = u-3$ so that the regular singular point is at $u = 0$

  $$(9u^3 - 54u^2 + 81u)\left(\frac{d^2}{du^2}y(u)\right) + (21u^2 - 117u + 162)\left(\frac{d}{du}y(u)\right) + (-9+4u)\, y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r}+\left(81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(81a_{k+1}(k+r+1)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$81r(1+r)=0$$
- Values of r that satisfy the indicial equation
$$r\in\{-1,0\}$$
- Each term must be 0
$$81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)=0$$
- Each term in the series must be 0, giving the recursion relation
$$81a_{k+1}(k+r+1)\,(k+2+r)-54\left(k+r+\frac{1}{6}\right)a_k(k+r+1)+a_{k-1}(3k-1+3r)^2=0$$
- Shift index using $k->k+1$
$$81a_{k+2}(k+2+r)\,(k+3+r)-54\left(k+\frac{7}{6}+r\right)a_{k+1}(k+2+r)+a_k(3k+3r+2)^2=0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+18kra_k-108kra_{k+1}+9r^2a_k-54r^2a_{k+1}+12ka_k-171ka_{k+1}+12ra_k-171ra_{k+1}+4a_k-126a_{k+1}}{81(k+2+r)(k+3+r)}$$
- Recursion relation for $r=-1$
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}$$
- Solution for $r=-1$
$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$$
- Revert the change of variables $u=x+3$
$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$$
- Recursion relation for $r=0$
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}$$
- Solution for $r=0$
$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$$
- Revert the change of variables $u=x+3$
$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k}\right), a_{k+2} = -\frac{9k^2 a_k - 54k^2 a_{k+1} - 6k a_k - 63k a_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}, 0 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(9*x^2*(3+x)*diff(y(x),x$2)+3*x*(3+7*x)*diff(y(x),x)+(3+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}} \ln(x)}{x+3}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 24

```
DSolve[9*x^2*(3+x)*y''[x]+3*x*(3+7*x)*y'[x]+(3+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x+3}$$

2.158 problem 160

2.158.1 Maple step by step solution 1544

Internal problem ID [7648]
Internal file name [`OUTPUT/6581_Sunday_June_05_2022_05_00_11_PM_97677127/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 160.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-x^2+2)\,y''-x(3x^2+2)\,y'+(-x^2+2)\,y=0}$$

Writing the ode as

$$(-x^4+2x^2)\,y''+(-3x^3-2x)\,y'+(-x^2+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -3x^3-2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 301: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3-2x}{-x^4+2x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(x^2-2)} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3-2x}{-x^4+2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x) - 2\ln(x^2-2)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x}{x^2-2}\right) + c_2\left(\frac{x}{x^2-2}(\ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1x}{x^2-2} + \frac{c_2x\ln(x)}{x^2-2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1x}{x^2-2} + \frac{c_2x\ln(x)}{x^2-2}
$$

Verified OK.

2.158.1 Maple step by step solution

Let's solve

$(-x^4+2x^2)\,y''+(-3x^3-2x)\,y'+(-x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2}-\frac{(3x^2+2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x^2+2)y'}{x(x^2-2)}+\frac{y}{x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{3x^2+2}{x(x^2-2)}, P_3(x)=\frac{1}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x^2(x^2-2) + x(3x^2+2)\,y' + y(x^2-2) = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - ○ Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - ○ Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0(-1+r)^2\,x^r - 2a_1r^2x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_1r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(a_k - \frac{a_{k-2}}{2}\right)(k+r-1)^2 = 0$$

- Shift index using $k- >k+2$

$$-2\left(a_{k+2} - \frac{a_k}{2}\right)(k+r+1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k}{2}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{a_k}{2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-x*(2+3*x^2)*diff(y(x),x)+(2-x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x^2 - 2} + \frac{c_2 x \ln(x)}{x^2 - 2}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 23

```
DSolve[x^2*(2-x^2)*y''[x]-x*(2+3*x^2)*y'[x]+(2-x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{x(c_2 \log(x) + c_1)}{x^2 - 2}$$

2.159 problem 161

2.159.1 Maple step by step solution 1553

Internal problem ID [7649]
Internal file name [OUTPUT/6582_Sunday_June_05_2022_05_00_14_PM_67242074/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 161.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0}$$

Writing the ode as

$$(16x^4+16x^2)\,y''+(72x^3+8x)\,y'+(49x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=16x^4+16x^2\\ B&=72x^3+8x\\ C&=49x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 303: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{72x^3+8x}{16x^4+16x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4} - \ln(x^2+1)} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}} (x^2+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{(y_1)^2} dx \\
&= y_1 \left(\ln(x) - \frac{1}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{x^2+1}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{x^2+1}\left(\ln(x) - \frac{1}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}}(2\ln(x)-1)}{2x^2+2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}}(2\ln(x)-1)}{2x^2+2}
$$

Verified OK.

2.159.1 Maple step by step solution

Let's solve

$(16x^4+16x^2)\, y'' + (72x^3+8x)\, y' + (49x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x^2+1)y}{16x^2(x^2+1)} - \frac{(9x^2+1)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(9x^2+1)y'}{2x(x^2+1)} + \frac{(49x^2+1)y}{16x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{9x^2+1}{2x(x^2+1)}, P_3(x) = \frac{49x^2+1}{16x^2(x^2+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

• Multiply by denominators

  $16x^2(x^2+1)\,y'' + 8x(9x^2+1)\,y' + (49x^2+1)\,y = 0$

• Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+4r)^2 x^r + a_1(3+4r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{4}$$

- Each term must be 0

$$a_1(3+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(4k+4r-1)^2 (a_k + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(4k+4r+7)^2 (a_{k+2} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -a_k$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -a_k$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(16*x^2*(1+x^2)*diff(y(x),x$2)+8*x*(1+9*x^2)*diff(y(x),x)+(1+49*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}} \ln(x)}{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 26

```
DSolve[16*x^2*(1+x^2)*y''[x]+8*x*(1+9*x^2)*y'[x]+(1+49*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt[4]{x}(c_2 \log(x) + c_1)}{x^2+1}$$

2.160 problem 162

2.160.1 Maple step by step solution 1562

Internal problem ID [7650]
Internal file name [OUTPUT/6583_Sunday_June_05_2022_05_00_17_PM_39252533/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 162.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0}$$

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+4x^2 \\ B &= 3x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 305: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2-4x}{3x^3+4x^2}\, dx}\\ &= z_1 e^{\frac{\ln(x)}{2}-\ln(4+3x)}\\ &= z_1\left(\frac{\sqrt{x}}{4+3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{4+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2}\, dx}}{(y_1)^2}\, dx\\ &= y_1 \int \frac{e^{\ln(x)-2\ln(4+3x)}}{(y_1)^2}\, dx\\ &= y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{4+3x}\right) + c_2\left(\frac{x}{4+3x}(\ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x}{4+3x} + \frac{c_2 x \ln(x)}{4+3x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{4+3x} + \frac{c_2 x \ln(x)}{4+3x}$$

Verified OK.

2.160.1 Maple step by step solution

Let's solve

$(3x^3+4x^2)\,y''+(3x^2-4x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2(4+3x)}-\frac{(3x-4)y'}{x(4+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x-4)y'}{x(4+3x)}+\frac{4y}{x^2(4+3x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x-4}{x(4+3x)}, P_3(x)=\frac{4}{x^2(4+3x)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4+3x)\, y'' + x(3x-4)\, y' + 4y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + 3a_{k-1}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2\,(4a_k + 3a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r)^2 (4a_{k+1}+3a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{4}$
- Recursion relation for $r=1$

  $a_{k+1} = -\frac{3a_k}{4}$
- Solution for $r=1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*(4+3*x)*diff(y(x),x$2)-x*(4-3*x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{3x+4} + \frac{c_2 x \ln(x)}{3x+4}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 22

```
DSolve[x^2*(4+3*x)*y''[x]-x*(4-3*x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{3x + 4}$$

2.161 problem 163

2.161.1 Maple step by step solution 1571

Internal problem ID [7651]
Internal file name [`OUTPUT/6584_Sunday_June_05_2022_05_00_19_PM_78327293/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 163.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+3x+1)\,y''+8x^2(3+2x)\,y'+(9x^2+3x+1)\,y=0}$$

Writing the ode as

$$(4x^4+12x^3+4x^2)\,y''+(16x^3+24x^2)\,y'+(9x^2+3x+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=16x^3+24x^2\\C&=9x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 307: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{16x^3+24x^2}{4x^4+12x^3+4x^2} \, dx} \\ &= z_1 e^{-\ln(x^2+3x+1)} \\ &= z_1 \left(\frac{1}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{16x^3+24x^2}{4x^4+12x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2 \left(\frac{\sqrt{x}}{x^2+3x+1} (\ln(x))\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}}{x^2+3x+1} + \frac{c_2 \sqrt{x}\, \ln(x)}{x^2+3x+1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}}{x^2+3x+1} + \frac{c_2 \sqrt{x}\, \ln(x)}{x^2+3x+1}$$

Verified OK.

2.161.1 Maple step by step solution

Let's solve

$(4x^4+12x^3+4x^2)\, y'' + (16x^3+24x^2)\, y' + (9x^2+3x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} - \frac{2(3+2x)y'}{x^2+3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(3+2x)y'}{x^2+3x+1} + \frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(3+2x)}{x^2+3x+1}, P_3(x) = \frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\,y''+8x^2(3+2x)\,y'+(9x^2+3x+1)\,y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2\,x^r + \left(a_1(1+2r)^2+3a_0(1+2r)^2\right)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2+3a_{k-1}(2k+2r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -3a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -3a_{k+1} - a_k$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -3a_{k+1} - a_k$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)+8*x^2*(3+2*x)*diff(y(x),x)+(1+3*x+9*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2\sqrt{x}\,\ln(x)}{x^2+3x+1}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 29

```
DSolve[4*x^2*(1+3*x+x^2)*y''[x]+8*x^2*(3+2*x)*y'[x]+(1+3*x+9*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to \frac{\sqrt{x}(c_2\log(x)+c_1)}{x^2+3x+1}$$

2.162 problem 164

2.162.1 Maple step by step solution 1580

Internal problem ID [7652]
Internal file name [OUTPUT/6585_Sunday_June_05_2022_05_00_22_PM_10726645/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 164.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)^2 y''-x(-3x^2+2x+1) y'+(x^2+1) y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+(3x^3-2x^2-x) y'+(x^2+1) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2(x-1)^2\\ B &= 3x^3-2x^2-x\\ C &= x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3-2x^2-x}{x^2(x-1)^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - 2\ln(x-1)} \\ &= z_1 \left(\frac{\sqrt{x}}{(x-1)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(x-1)^2}dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x)-4\ln(x-1)}}{(y_1)^2} dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(x-1)^2}\right) + c_2\left(\frac{x}{(x-1)^2}(\ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln(x)}{(x-1)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln(x)}{(x-1)^2}
$$

Verified OK.

2.162.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (3x^3 - 2x^2 - x)y' + (x^2+1)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{x^2(x-1)^2} - \frac{y'(3x+1)}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'(3x+1)}{x(x-1)} + \frac{(x^2+1)y}{x^2(x-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = \frac{3x+1}{x(x-1)}, P_3(x) = \frac{x^2+1}{x^2(x-1)^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$y''x^2(x-1)^2 + x(3x+1)(x-1)y' + (x^2+1)y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + \left(-2a_0 r^2 + a_1 r^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_0 r^2 + a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 2a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)^2 \left(a_k - 2a_{k-1} + a_{k-2}\right) = 0$$

- Shift index using $k->k+2$

$$(k+r+1)^2 \left(a_{k+2} - 2a_{k+1} + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = 2a_{k+1} - a_k$$

- Recursion relation for $r = 1$

$$a_{k+2} = 2a_{k+1} - a_k$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*(1-x)^2*diff(y(x),x$2)-x*(1+2*x-3*x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln(x)}{(x-1)^2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 20

```
DSolve[x^2*(1-x)^2*y''[x]-x*(1+2*x-3*x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{(x-1)^2}$$

2.163 problem 165

2.163.1 Maple step by step solution 1589

Internal problem ID [7653]
Internal file name [`OUTPUT/6586_Sunday_June_05_2022_05_00_24_PM_2954354/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 165.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(x^2+x+1)\,y''+3x(13x^2+7x+1)\,y'+(25x^2+4x+1)\,y=0}$$

Writing the ode as

$$(9x^4+9x^3+9x^2)\,y''+(39x^3+21x^2+3x)\,y'+(25x^2+4x+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^3+9x^2 \\ B &= 39x^3+21x^2+3x \\ C &= 25x^2+4x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2} \, dx} \\ &= z_1 e^{-\ln(x^2+x+1) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(x^2+x+1)\, x^{\frac{1}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{x^2 + x + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+x+1)-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{13}{24}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{x^2+x+1}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{x^2+x+1}\left(\ln(x) + \frac{13}{24}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x^2+x+1} + \frac{c_2 x^{\frac{1}{3}}(24\ln(x) + 13)}{24x^2+24x+24} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x^2+x+1} + \frac{c_2 x^{\frac{1}{3}}(24\ln(x) + 13)}{24x^2+24x+24}
$$

Verified OK.

2.163.1 Maple step by step solution

Let's solve

$(9x^4+9x^3+9x^2)\, y'' + (39x^3+21x^2+3x)\, y' + (25x^2+4x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} - \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)} + \frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} = 0$$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $$\left[P_2(x) = \frac{13x^2+7x+1}{3x(x^2+x+1)}, P_3(x) = \frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$$

  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $$(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{3}$$

  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{9}$$

  - ○ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- • Multiply by denominators

  $$9x^2(x^2+x+1)\,y'' + 3x(13x^2+7x+1)\,y' + (25x^2+4x+1)\,y = 0$$

- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- > k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - ○ Shift index using $k- > k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{3}$$

- Each term must be 0

$$a_1(2+3r)^2 + a_0(2+3r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -a_{k+1} - a_k$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -a_{k+1} - a_k$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(9*x^2*(1+x+x^2)*diff(y(x),x$2)+3*x*(1+7*x+13*x^2)*diff(y(x),x)+(1+4*x+25*x^2)*y(x)=0,
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{x^2 + x + 1} + \frac{c_2 x^{\frac{1}{3}} \ln(x)}{x^2 + x + 1}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 27

```
DSolve[9*x^2*(1+x+x^2)*y''[x]+3*x*(1+7*x+13*x^2)*y'[x]+(1+4*x+25*x^2)*y[x]==0,y[x],x,Include
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x^2 + x + 1}$$

2.164 problem 166

2.164.1 Maple step by step solution 1599

Internal problem ID [7654]
Internal file name [OUTPUT/6587_Sunday_June_05_2022_05_00_28_PM_12754025/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 166.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+2)\,y''-x(4-7x)\,y'-(5-3x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(-5+3x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 7x^2-4x \\ C &= -5+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 32x + 128$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2x} + \frac{5}{2(x+2)} + \frac{45}{16(x+2)^2} + \frac{2}{x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{5}{4\left(x+2\right)} + \frac{2}{x} + (0) \\ &= -\frac{5}{4\left(x+2\right)} + \frac{2}{x} \\ &= \frac{3x+16}{4x\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{5}{4\left(x+2\right)}+\frac{2}{x}\right)(0)+\left(\left(\frac{5}{4\left(x+2\right)^2}-\frac{2}{x^2}\right)+\left(-\frac{5}{4\left(x+2\right)}+\frac{2}{x}\right)^2-\left(\frac{-3x^2-32x+128}{16\left(x^2+2x\right)^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{5}{4(x+2)}+\frac{2}{x}\right)dx} \\ &= \frac{x^2}{\left(x+2\right)^{\frac{5}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7x^2-4x}{2x^3+4x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}-\frac{9\ln(x+2)}{4}} \\ &= z_1\left(\frac{\sqrt{x}}{\left(x+2\right)^{\frac{9}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1=\frac{x^{\frac{5}{2}}}{\left(x+2\right)^{\frac{7}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{9\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 2\sqrt{x+2}\,(33x^2+52x+32)}{48x^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}}\right) \\
&\quad + c_2\left(\frac{x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}}\left(\frac{-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 2\sqrt{x+2}\,(33x^2+52x+32)}{48x^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} \\
&\quad + \frac{c_2\left(-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 66x^2\sqrt{x+2} - 104\sqrt{x+2}\,x - 64\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} \\
&\quad + \frac{c_2\left(-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 66x^2\sqrt{x+2} - 104\sqrt{x+2}\,x - 64\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}
\end{aligned}
$$

Verified OK.

2.164.1 Maple step by step solution

Let's solve

$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+\left(-5+3x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-5+3x)y}{2x^2(x+2)}-\frac{(7x-4)y'}{2x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x-4)y'}{2x(x+2)}+\frac{(-5+3x)y}{2x^2(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{7x-4}{2x(x+2)},P_3(x)=\frac{-5+3x}{2x^2(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{9}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)y''+x(7x-4)y'+\left(-5+3x\right)y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $\left(2u^3-8u^2+8u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(7u^2-32u+36\right)\left(\frac{d}{du}y(u)\right)+\left(-11+3u\right)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0r(7+2r)\, u^{-1+r} + (4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(7+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{7}{2}\right\}$$

- Each term must be 0

  $$4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 24a_k + a_{k-1} + 44a_{k+1})\, k + 2(-4a_k +$$

- Shift index using $k- > k + 1$

  $$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 24a_{k+1} + a_k + 44a_{k+2})\,(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2 a_k - 8r^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 4kr + 2r^2 + 15k + 15r + 22)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a_1 - 11a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a_1 - 11a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}$$

- Solution for $r = -\frac{7}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}, -20a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}, -20a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 80

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} - \frac{c_2\sqrt{2}\left(33\sqrt{2}\,\sqrt{x+2}\,x^2 + 15\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 + 52\sqrt{2}\,\sqrt{x+2}\,x + 32\sqrt{2}\,\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.331 (sec). Leaf size: 92

```
DSolve[2*x^2*(2+x)*y''[x]-x*(4-7*x)*y'[x]-(5-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{15\sqrt{2}c_2x^3\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) - 48c_1x^3 + 66c_2\sqrt{x+2}x^2 + 104c_2\sqrt{x+2}x + 64c_2\sqrt{x+2}}{48\sqrt{x}(x+2)^{7/2}}$$

2.165 problem 167

2.165.1 Maple step by step solution 1609

Internal problem ID [7655]
Internal file name [OUTPUT/6588_Sunday_June_05_2022_05_00_31_PM_91862941/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 167.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-2x)\,y''+x(8-9x)\,y'+(6-3x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-9x^2+8x\right)y'+(6-3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21x^2 - 20x + 24 \\ t &= 4\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{19}{x} + \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\
&= \frac{7}{4} - \left(\frac{3}{4}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)} + (0) \\
&= -\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)} \\
&= \frac{4 + 3x}{4x^2 - 2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right)(1) + \left(\left(\frac{2}{x^2} - \frac{11}{4\left(x - \frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right)\right)
$$

$$
\frac{4 - 3a_0}{x\,(2x - 1)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{4}{3}\right) e^{\int \left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{4}{3}\right) e^{-2\ln(x) + \frac{11\ln(2x-1)}{4}} \\
&= \frac{\left(x + \frac{4}{3}\right)(2x - 1)^{\frac{11}{4}}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{-4\ln(x)+\frac{7\ln(2x-1)}{4}} \\
&= z_1 \left(\frac{(2x-1)^{\frac{7}{4}}}{x^4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-8\ln(x)+\frac{7\ln(2x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-231x^3+198x^2-66x+8}{(2x-1)^{\frac{9}{2}}\,(1540+1155x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}\right) + c_2 \left(\frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}\left(\frac{-231x^3+198x^2-66x+8}{(2x-1)^{\frac{9}{2}}\,(1540+1155x)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6} + \frac{c_2(-231x^3+198x^2-66x+8)}{1155x^6} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6} + \frac{c_2(-231x^3+198x^2-66x+8)}{1155x^6}$$

Verified OK.

2.165.1 Maple step by step solution

Let's solve

$(-2x^3+x^2)\,y''+(-9x^2+8x)\,y'+(6-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3(x-2)y}{x^2(2x-1)}-\frac{(9x-8)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(9x-8)y'}{x(2x-1)}+\frac{3(x-2)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{9x-8}{x(2x-1)}, P_3(x)=\frac{3(x-2)}{x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=8$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(9x-8)\,y'+(3x-6)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(6+r)\,(1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r+6)\,(k+r+1) + a_{k-1}(k+2+r)\,(2k-1+2r)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(6+r)\,(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, -1\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+2+r)\left(k-\frac{1}{2}+r\right) a_{k-1} - a_k(k+r+6)\,(k+r+1) = 0$$

- Shift index using $k->k+1$

$$2(k+r+3)\left(k+\frac{1}{2}+r\right) a_k - a_{k+1}(k+7+r)\,(k+2+r) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$
- Recursion relation for $r = -6$; series terminates at $k = 3$

  $a_{k+1} = \frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{33a_0}{4}$
- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$
- Express in terms of a_0

  $a_2 = \frac{99a_0}{4}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{7a_2}{6}$
- Express in terms of a_0

  $a_3 = -\frac{231a_0}{8}$
- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right)$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(231x^3 - 198x^2 + 66x - 8)}{x^6} + \frac{c_2(3x+4)\,(-1+2x)^{\frac{9}{2}}}{x^6}$$

✓ Solution by Mathematica

Time used: 0.224 (sec). Leaf size: 49

```
DSolve[x^2*(1-2*x)*y''[x]+x*(8-9*x)*y'[x]+(6-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_2(231x^3 - 198x^2 + 66x - 8) + 385c_1(3x+4)(1-2x)^{9/2}}{1155x^6}$$

2.166 problem 168

2.166.1 Maple step by step solution 1619

Internal problem ID [7656]
Internal file name [OUTPUT/6589_Sunday_June_05_2022_05_00_34_PM_25115084/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 168.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2}\left(x^{2}+1\right) y''+x\left(10x^{2}+3\right) y'-\left(-14x^{2}+15\right) y=0}$$

Writing the ode as

$$\left(x^{4}+x^{2}\right) y''+\left(10x^{3}+3x\right) y'+\left(14x^{2}-15\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{4}+x^{2} \\ B &= 10x^{3}+3x \\ C &= 14x^{2}-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^4 + 66x^2 + 63 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{99i}{16(x-i)} - \frac{99i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{\frac{9}{2}}}{(x^2+1)^{\frac{3}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x^3+3x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}-\frac{7\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}\left(x^2+1\right)^{\frac{7}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)-\frac{7\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-3\,\text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8+\sqrt{x^2+1}\,(3x^6-2x^4-24x^2-16)}{128x^8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}\right) \\
&\quad + c_2\left(\frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}\left(\frac{-3\,\text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8+\sqrt{x^2+1}\,(3x^6-2x^4-24x^2-16)}{128x^8}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{3c_2\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8 - \left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)(x^2+2)\sqrt{x^2+1}\right)}{128\,(x^2+1)^{\frac{5}{2}}\,x^5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{3c_2\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8 - \left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)(x^2+2)\sqrt{x^2+1}\right)}{128\,(x^2+1)^{\frac{5}{2}}\,x^5}$$

Verified OK.

2.166.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y'' + (10x^3+3x)\,y' + (14x^2-15)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(14x^2-15)y}{x^2(x^2+1)} - \frac{(10x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(10x^2+3)y'}{x(x^2+1)} + \frac{(14x^2-15)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{10x^2+3}{x(x^2+1)}, P_3(x) = \frac{14x^2-15}{x^2(x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = 3$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -15$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\,y''+x(10x^2+3)\,y'+(14x^2-15)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..4$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+r)\,(-3+r)\,x^r+a_1(6+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r+5)\,(k+r-3)+a_{k-2}(k+r-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-5,3\}$

- Each term must be 0

$a_1(6+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+5)\,(a_k(k+r-3)+a_{k-2}(k+r)) = 0$

- Shift index using $k->k+2$

  $(k+r+7)\,(a_{k+2}(k+r-1)+a_k(k+r+2)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)}{k+r-1}$

- Recursion relation for $r = -5$

  $a_{k+2} = -\frac{a_k(k-3)}{k-6}$

- Series not valid for $r = -5$, division by 0 in the recursion relation at $k = 6$

  $a_{k+2} = -\frac{a_k(k-3)}{k-6}$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+5)}{k+2}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+5)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 89

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(3+10*x^2)*diff(y(x),x)-(15-14*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{c_2 \left(-3 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^8 + 3\sqrt{x^2+1}\, x^6 - 2x^4\sqrt{x^2+1} - 24\sqrt{x^2+1}\, x^2 - 16\sqrt{x^2+1}\right)}{128 x^5 (x^2+1)^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 75

```
DSolve[x^2*(1+x^2)*y''[x]+x*(3+10*x^2)*y'[x]-(15-14*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{c_2\left(\sqrt{x^2+1}(3x^6-2x^4-24x^2-16) - 3x^8 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right) + 128 c_1 x^8}{128 x^5 (x^2+1)^{5/2}}$$

2.167 problem 169

2.167.1 Maple step by step solution 1629

Internal problem ID [7657]
Internal file name [`OUTPUT/6590_Sunday_June_05_2022_05_00_37_PM_10888468/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 169.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0}$$

Writing the ode as

$$(-2x^4+x^2)\,y''+(-13x^3+7x)\,y'-14x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^4+x^2 \\ B &= -13x^3+7x \\ C &= -14x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^4 - 68x^2 + 35$$
$$t = 4\left(2x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-x^2+5}{4x^3-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{5}{2x} +\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)}\right) dx} \\
&= \frac{\left(2x-\sqrt{2}\right)^{\frac{9}{8}}\left(2x+\sqrt{2}\right)^{\frac{9}{8}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-13x^3+7x}{-2x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-1\right)}{8}-\frac{7\ln(x)}{2}} \\
&= z_1\left(\frac{\left(2x^2-1\right)^{\frac{1}{8}}}{x^{\frac{7}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3+7x}{-2x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(2x^2-1\right)}{4}-7\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\left(5x^4-20x^2+8\right)2^{\frac{3}{4}}}{120\left(2x^2-1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}\right) + c_2\left(\frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}\left(\frac{\left(5x^4-20x^2+8\right)2^{\frac{3}{4}}}{120\left(2x^2-1\right)^{\frac{5}{4}}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2c_1 2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6} + \frac{c_2 2^{\frac{7}{8}}\left(5x^4-20x^2+8\right)}{60x^6} \tag{1}
$$

Verification of solutions

$$y = \frac{2c_1 2^{\frac{1}{8}} (2x^2 - 1)^{\frac{5}{4}}}{x^6} + \frac{c_2 2^{\frac{7}{8}} (5x^4 - 20x^2 + 8)}{60x^6}$$

Verified OK.

2.167.1 Maple step by step solution

Let's solve

$(-2x^4 + x^2) y'' + (-13x^3 + 7x) y' - 14x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{14y}{2x^2-1} - \frac{(13x^2-7)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(13x^2-7)y'}{x(2x^2-1)} + \frac{14y}{2x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $14yx + (13x^2 - 7) y' + xy''(2x^2 - 1) = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(6+r)\, x^{-1+r} - a_1(1+r)\,(7+r)\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k+r+1)\,(k+7+r) + a_{k-1}(2k+5+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-6, 0\}$

- Each term must be 0

  $-a_1(1+r)\,(7+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r+\frac{5}{2}\right) a_{k-1} - \frac{a_{k+1}(k+7+r)}{2}\right)(k+r+1) = 0$

- Shift index using $k->k+1$

  $2\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$

- Recursion relation for $r=-6$

  $a_{k+2}=\frac{(2k-5)a_k}{k+2}$

- Solution for $r=-6$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-6}, a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{(2k+7)a_k}{k+8}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=\frac{(2k+7)a_k}{k+8}, -7a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-6}\right)+\left(\sum_{k=0}^{\infty}b_k x^k\right), a_{k+2}=\frac{(2k-5)a_k}{k+2}, 5a_1=0, b_{k+2}=\frac{(2k+7)b_k}{k+8}, -7b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(x^2*(1-2*x^2)*diff(y(x),x$2)+x*(7-13*x^2)*diff(y(x),x)-14*x^2*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1(5x^4 - 20x^2 + 8)}{x^6} + \frac{c_2(2x^2 - 1)^{\frac{5}{4}}}{x^6}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 43

```
DSolve[x^2*(1-2*x^2)*y''[x]+x*(7-13*x^2)*y'[x]-14*x^2*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{15c_1(1 - 2x^2)^{5/4} + c_2(-5x^4 + 20x^2 - 8)}{15x^6}$$

2.168 problem 170

2.168.1 Maple step by step solution 1639

Internal problem ID [7658]
Internal file name [`OUTPUT/6591_Sunday_June_05_2022_05_00_40_PM_97341410/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 170.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1+x)\,y''+4x(2x+1)\,y'-(3x+1)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2+4x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4 + 3x}{4x\,(1 + x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 + 3x \\
t &= 4x(1 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4 + 3x}{4x\,(1 + x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 321: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{1}{1+x}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{4+3x}{4x(1+x)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{4+3x}{4x\left(1+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= \frac{3}{2}-\left(\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{x} + \frac{1}{2x+2} \\
&= \frac{1}{x} + \frac{1}{2x+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2x+2}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2x+2}\right)^2 - \left(\frac{4+3x}{4x(1+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} + \frac{1}{2x+2}\right) dx} \\
&= \sqrt{1+x}\, x
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^2+4x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}\left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}} + \frac{c_2(-x\ln(x) + \ln(1+x)\,x - 1)\sqrt{1+x}}{\sqrt{x\,(1+x)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}} + \frac{c_2(-x\ln(x) + \ln(1+x)\,x - 1)\sqrt{1+x}}{\sqrt{x\,(1+x)}}$$

Verified OK.

2.168.1 Maple step by step solution

Let's solve

$(4x^3+4x^2)\, y''+(8x^2+4x)\, y'+(-3x-1)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(3x+1)y}{4x^2(1+x)}-\frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x+1)y'}{x(1+x)}-\frac{(3x+1)y}{4x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x+1}{x(1+x)}, P_3(x)=-\frac{3x+1}{4x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4x^2(1+x)\, y''+4x(2x+1)\, y'+(-3x-1)\, y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(4u^3-8u^2+4u)\left(\frac{d^2}{du^2}y(u)\right)+(8u^2-12u+4)\left(\frac{d}{du}y(u)\right)+(-3u+2)\, y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0 r^2 u^{-1+r} + \left(4a_1(1+r)^2 - 2a_0(4r^2+2r-1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+1+r)^2 - 2a_k(4k^2+8kr -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$4a_1(1+r)^2 - 2a_0(4r^2+2r-1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(4k^2 - 4k - 3)\, a_{k-1} + (-8k^2 - 4k + 2)\, a_k + 4a_{k+1}(k+1)^2 = 0$$

- Shift index using $k- > k + 1$

  $$\left(4(k+1)^2 - 4k - 7\right) a_k + \left(-8(k+1)^2 - 4k - 2\right) a_{k+1} + 4a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}, 4a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}, 4a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + \frac{c_2(\ln(x+1)\,x - x\ln(x) - 1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 32

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(1+2*x)*y'[x]-(1+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{c_1 x + c_2(-x\log(x) + x\log(x+1) - 1)}{\sqrt{x}}$$

2.169 problem 171

2.169.1 Maple step by step solution 1649

Internal problem ID [7659]
Internal file name [`OUTPUT/6592_Sunday_June_05_2022_05_00_42_PM_54927312/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 171.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(3x+2)\,y''+x(4+21x)\,y'-(1-9x)\,y=0}$$

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 21x^2+4x \\ C &= 9x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-27x - 48}{16x\,(3x + 2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -27x - 48 \\ t &= 16x(3x + 2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-27x - 48}{16x\,(3x + 2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(3x+2)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{5}{16\left(x+\frac{2}{3}\right)^2} + \frac{3}{4\left(x+\frac{2}{3}\right)}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-27x - 48}{16x\,(3x+2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-27x - 48}{16x\,(3x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} + (0) \\ &= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} \\ &= \frac{9x+8}{12x^2+8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x + \frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x + \frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x + \frac{2}{3}\right)}\right)^2 - \left(\frac{-27x - 48}{16x\left(3x + 2\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x + \frac{2}{3}\right)}\right) dx} \\ &= \frac{x}{\left(3x + 2\right)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{21x^2 + 4x}{6x^3 + 4x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(3x+2)}{4}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\,\left(3x + 2\right)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(3x + 2\right)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(3x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x - 2\sqrt{3x+2}}{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(3x+2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x}}{(3x+2)^{\frac{3}{2}}} \left(\frac{-3\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x - 2\sqrt{3x+2}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}} - \frac{3c_2\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x + \frac{2\sqrt{3x+2}}{3}\right)}{2\,(3x+2)^{\frac{3}{2}}\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}} - \frac{3c_2\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x + \frac{2\sqrt{3x+2}}{3}\right)}{2\,(3x+2)^{\frac{3}{2}}\sqrt{x}}
$$

Verified OK.

2.169.1 Maple step by step solution

Let's solve

$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+\left(9x-1\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x-1)y}{2x^2(3x+2)}-\frac{(4+21x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4+21x)y'}{2x(3x+2)}+\frac{(9x-1)y}{2x^2(3x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4+21x}{2x(3x+2)},P_3(x)=\frac{9x-1}{2x^2(3x+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(3x+2)y''+x(4+21x)y'+(9x-1)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..2$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m=2..3$

  $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r+\left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1)+3a_{k-1}(2k+2r+1)\,(k+r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_k+\frac{3a_{k-1}(k+r)}{2}\right)\left(k+r+\frac{1}{2}\right)=0$

- Shift index using $k->k+1$

  $4\left(\left(k+r+\frac{1}{2}\right) a_{k+1}+\frac{3a_k(k+r+1)}{2}\right)\left(k+\frac{3}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k(k+r+1)}{2k+2r+1}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=-\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1}=-\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x)=0,y(x), singsol=all
```

$$y(x)=\frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}}+\frac{c_2\sqrt{2}\left(\sqrt{2}\,\sqrt{3x+2}+3\ \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x\right)}{2\sqrt{x}\,(3x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 64

```
DSolve[2*x^2*(2+3*x)*y''[x]+x*(4+21*x)*y'[x]-(1-9*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{3\sqrt{2}c_2 x \operatorname{arctanh}\left(\sqrt{\frac{3x}{2}+1}\right) - 2c_1 x + 2c_2\sqrt{3x+2}}{2\sqrt{x}(3x+2)^{3/2}}$$

2.170 problem 172

2.170.1 Maple step by step solution 1660

Internal problem ID [7660]
Internal file name [OUTPUT/6593_Sunday_June_05_2022_05_00_45_PM_15149787/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 172.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(x+2)\, y' - (2-3x)\, y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+2x\right) y' + (3x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+2x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 8x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 8x + 8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-8x + 8}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &= -2\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 8x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= 2 - (2)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{2}{x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{x} - \frac{1}{2} \\
&= -\frac{x-4}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 8x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2}\right) dx} \\
&= x^2 \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 2x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x^3 - \mathrm{e}^x(x^2+x+2)}{6x^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x}\right) + c_2\left(x\,\mathrm{e}^{-x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x^3 - \mathrm{e}^x(x^2+x+2)}{6x^3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x^3\mathrm{e}^{-x} - x^2 - x - 2)}{6x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x^3\mathrm{e}^{-x} - x^2 - x - 2)}{6x^2}$$

Verified OK.

2.170.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + 2x)\, y' + (3x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{x^2} - \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+2)y'}{x} + \frac{(3x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x+2}{x}, P_3(x) = \frac{3x-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x + 2)\, y' + (3x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\,(a_k(k+r-1) + a_{k-1}) = 0$$

- Shift index using $k->k+1$

$$(k+r+3)\,(a_{k+1}(k+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+r}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k}{k-2}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k}{k-2}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve(x^2*diff(y(x),x$2)+x*(2+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} x + \frac{c_2 \mathrm{e}^{-x} (\mathrm{expIntegral}_1 (-x) x^3 + \mathrm{e}^x x^2 + x \mathrm{e}^x + 2 \mathrm{e}^x)}{6x^2}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 46

```
DSolve[x^2*y''[x]+x*(2+x)*y'[x]-(2-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x} (c_2 (x^3 \,\mathrm{ExpIntegralEi}(x) - e^x (x^2 + x + 2)) + 6c_1 x^3)}{6x^2}$$

2.171 problem 173

2.171.1 Maple step by step solution 1668

Internal problem ID [7661]
Internal file name [OUTPUT/6594_Sunday_June_05_2022_05_00_48_PM_93344797/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 173.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 32x^2+12x \\ C &= 49x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 8x + 8$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{6}{x} + \frac{15}{4(1+x)^2} + \frac{2}{x^2} + \frac{6}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4(x^2 + x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2\left(1 + x\right)} + \frac{2}{x} + (-)\left(0\right) \\ &= -\frac{3}{2\left(1 + x\right)} + \frac{2}{x} \\ &= \frac{4 + x}{2x\left(1 + x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\,(1+x)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2\,(1+x)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2\,(1+x)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2 - 8x + 8}{4\,(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\ &= \frac{x^2}{(1+x)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{32x^2+12x}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2} - \frac{5\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}\,(1+x)^{\frac{5}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(1+x)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-3\ln(x)-5\ln(1+x)}}{(y_1)^2} dx$$

$$= y_1 \left(-\frac{3}{x} - \frac{3}{2x^2} + \ln(x) - \frac{1}{3x^3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\sqrt{x}}{(1+x)^4}\right) + c_2 \left(\frac{\sqrt{x}}{(1+x)^4} \left(-\frac{3}{x} - \frac{3}{2x^2} + \ln(x) - \frac{1}{3x^3}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}}{(1+x)^4} + \frac{c_2 (6x^3 \ln(x) - 18x^2 - 9x - 2)}{6x^{\frac{5}{2}} (1+x)^4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}}{(1+x)^4} + \frac{c_2 (6x^3 \ln(x) - 18x^2 - 9x - 2)}{6x^{\frac{5}{2}} (1+x)^4}$$

Verified OK.

2.171.1 Maple step by step solution

Let's solve

$(4x^3 + 4x^2) y'' + (32x^2 + 12x) y' + (49x - 5) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x-5)y}{4x^2(1+x)} - \frac{(3+8x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(3+8x)y'}{x(1+x)} + \frac{(49x-5)y}{4x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3+8x}{x(1+x)}, P_3(x) = \frac{49x-5}{4x^2(1+x)}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\bigg|_{x=-1} = 5$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\,y'' + 4x(3+8x)\,y' + (49x-5)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (32u^2 - 52u + 20)\left(\frac{d}{du}y(u)\right) + (49u - 54)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(4+r)\, u^{-1+r} + (4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-4, 0\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4a_{k+1}(k+1+r)\,(k+5+r) - 2a_k(4k^2+8kr+4r^2+22k+22r+27) + a_{k-1}(2k+5+2r)^2 = 0$$

- Shift index using $k->k+1$

$$4a_{k+2}(k+2+r)\,(k+6+r) - 2a_{k+1}\left(4(k+1)^2+8(k+1)\,r+4r^2+22k+49+22r\right) + a_k(2k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+8kra_k-16kra_{k+1}+4r^2a_k-8r^2a_{k+1}+28ka_k-60ka_{k+1}+28ra_k-60ra_{k+1}+49a_k-106a_{k+1}}{4(k+2+r)(k+6+r)}$$

- Recursion relation for $r = -4$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x)=0,y(x), singsol=all
```

$$y(x)=\frac{c_1\sqrt{x}}{(x+1)^4}+\frac{c_2(6x^3\ln(x)-18x^2-9x-2)}{6(x+1)^4 x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 52

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(3+8*x)*y'[x]-(5-49*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x)\to\frac{6c_1x^3+6c_2x^3\log(x)-18c_2x^2-9c_2x-2c_2}{6x^{5/2}(x+1)^4}$$

2.172 problem 174

2.172.1 Maple step by step solution 1678

Internal problem ID [7662]
Internal file name [`OUTPUT/6595_Sunday_June_05_2022_05_00_51_PM_88523473/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 174.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-x(3+10x)\,y'+30yx=0}$$

The ODE is

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30yx=0$$

Or

$$x\left(x^2y''-10xy'+xy''+30y-3y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(x^2+x\right)y''+(-3-10x)\,y'+30y=0$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= -10x^2-3x \\ C &= 30x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-48x + 15}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -48x + 15 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{39}{2x} + \frac{63}{4(1+x)^2} + \frac{15}{4x^2} + \frac{39}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-48x + 15}{4(x^2 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^- + \alpha_{c_2}^+) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{7}{2(1+x)} + \frac{5}{2x} + (0) \\ &= -\frac{7}{2(1+x)} + \frac{5}{2x} \\ &= -\frac{2x-5}{2x(1+x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)(1) + \left(\left(\frac{7}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)^2 - \left(\frac{-48x+15}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{5+2a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{2}\right) e^{\int \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{2}\right) e^{\frac{5 \ln(x)}{2} - \frac{7 \ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{2}\right) x^{\frac{5}{2}}}{(1+x)^{\frac{7}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x^2 - 3x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{3 \ln(x)}{2} + \frac{7 \ln(1+x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} (1+x)^{\frac{7}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - \frac{5}{2} x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2 - 3x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3 \ln(x) + 7 \ln(1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(x - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12 \ln(x) - \frac{823543}{6250(2x-5)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x) - \frac{823543}{6250(2x-5)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(12\ln(x)\,x^5 + x^6 - 30\ln(x)\,x^4 - \frac{5x^5}{2} - \frac{299x^4}{4} + 20x^3 + 5x^2 + x + \frac{1}{10}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(12\ln(x)\,x^5 + x^6 - 30\ln(x)\,x^4 - \frac{5x^5}{2} - \frac{299x^4}{4} + 20x^3 + 5x^2 + x + \frac{1}{10}\right)
\end{aligned}
$$

Verified OK. `{x <> 0}`

2.172.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y'' + (-10x^2 - 3x)\,y' + 30yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{30y}{x(1+x)} + \frac{(3+10x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3+10x)y'}{x(1+x)} + \frac{30y}{x(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{3+10x}{x(1+x)}, P_3(x) = \frac{30}{x(1+x)}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = -7$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)\,y'' + (-3-10x)\,y' + 30y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right) + (7-10u)\left(\frac{d}{du}y(u)\right) + 30y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(-8+r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-7+r)+a_k(k+r-5)\,(k+r-6)\right) u^{k+r}\right)=$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,8\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-7+r)+a_k(k+r-5)\,(k+r-6)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r=0$; series terminates at $k=5$

  $a_{k+1}=\frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{30a_0}{7}$
- Apply recursion relation for $k=1$

  $a_2=-\frac{5a_1}{3}$
- Express in terms of a_0

  $a_2=\frac{50a_0}{7}$
- Apply recursion relation for $k=2$

  $a_3=-\frac{4a_2}{5}$
- Express in terms of a_0

  $a_3=-\frac{40a_0}{7}$
- Apply recursion relation for $k=3$

  $a_4=-\frac{3a_3}{8}$
- Express in terms of a_0

  $a_4=\frac{15a_0}{7}$
- Apply recursion relation for $k=4$

  $a_5=-\frac{2a_4}{15}$
- Express in terms of a_0

$a_5 = -\frac{2a_0}{7}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{30}{7}u + \frac{50}{7}u^2 - \frac{40}{7}u^3 + \frac{15}{7}u^4 - \frac{2}{7}u^5\right)$

- Revert the change of variables $u = 1 + x$

$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$

- Recursion relation for $r = 8$

$a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$

- Solution for $r = 8$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

$\left[y = \sum_{k=0}^{\infty} a_k(1 + x)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(1 + x)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3+10*x)*diff(y(x),x)+30*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^5-\frac{5}{2}x^4\right) + c_2\left(3x^5\ln(x)+\frac{x^6}{4}-\frac{15x^4\ln(x)}{2}-\frac{5x^5}{8}-\frac{299x^4}{16}+5x^3+\frac{5x^2}{4}+\frac{x}{4}+\frac{1}{40}\right)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 68

```
DSolve[x^2*(1+x)*y''[x]-x*(3+10*x)*y'[x]+30*x*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1\left(x^5-\frac{5x^4}{2}\right) + \frac{1}{20}c_2\left(20x^6-50x^5-1495x^4+120(2x-5)x^4\log(x)+400x^3+100x^2+20x+2\right)$$

2.173 problem 175

2.173.1 Maple step by step solution 1690

Internal problem ID [7663]
Internal file name [`OUTPUT/6596_Sunday_June_05_2022_05_00_54_PM_47602478/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 175.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(1+x)\,y' - 3(x+3)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (-3x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= -3x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 14x + 35$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 14x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &= \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{7 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{7}{2x}\right)(0) + \left(\left(-\frac{7}{2x^2}\right) + \left(\frac{1}{2} + \frac{7}{2x}\right)^2 - \left(\frac{x^2 + 14x + 35}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{7}{2x}\right) dx} \\
&= x^{\frac{7}{2}} \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\, \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6}{720x^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^3) + c_2\left(x^3\left(\frac{(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\, \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6}{720x^6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 + \frac{c_2((x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\, \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6)}{720x^3} \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + \frac{c_2((x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\, \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6)}{720x^3}$$

Verified OK.

2.173.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\, y' + (-3x - 9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x+3)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} - \frac{3(x+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = -\frac{3(x+3)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1 + x)\, y' + (-3x - 9)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(3+r)\,(-3+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r+3)\,(k+r-3)+a_{k-1}(k-4+r)\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-3,3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+3)\,(k+r-3)+a_{k-1}(k-4+r)=0$
- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)\,(k-2+r)+a_{k}(k+r-3)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_{k}(k+r-3)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r=-3$; series terminates at $k=6$

  $a_{k+1}=-\frac{a_{k}(k-6)}{(k+1)(k-5)}$
- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=5$

  $a_{k+1}=-\frac{a_{k}(k-6)}{(k+1)(k-5)}$
- Recursion relation for $r=3$

  $a_{k+1}=-\frac{a_{k}k}{(k+7)(k+1)}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
dsolve(x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-3*(3+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^3 - \frac{c_2(-\operatorname{expIntegral}_1(x)\, x^6 + \mathrm{e}^{-x} x^5 - \mathrm{e}^{-x} x^4 + 2\,\mathrm{e}^{-x} x^3 - 6x^2 \mathrm{e}^{-x} + 24\,\mathrm{e}^{-x} x - 120\,\mathrm{e}^{-x})}{720x^3}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 60

```
DSolve[x^2*y''[x]+x*(1+x)*y'[x]-3*(3+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 e^{-x}(e^x x^6 \operatorname{ExpIntegralEi}(-x) + x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)}{720x^3} + c_1 x^3$$

2.174 problem 176

2.174.1 Maple step by step solution 1699

Internal problem ID [7664]
Internal file name [OUTPUT/6597_Sunday_June_05_2022_05_00_57_PM_6634393/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 176.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2x+1)\,y''+x(9+13x)\,y'+(5x+7)\,y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(5x+7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 13x^2+9x \\ C &= 5x+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 77x^2 + 86x + 35 \\ t &= 4\left(2x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{2x} + \frac{35}{4x^2} + \frac{45}{16\left(x+\frac{1}{2}\right)^2} + \frac{27}{2\left(x+\frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{11}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{7}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{7}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{7}{4}-\left(-\frac{15}{4}\right)\\
&= 2
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{-15x-5}{4x^2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x+a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2+86x}{4\,(2x^2+}\right.\right.
$$

$$
\frac{(11a_1-8)\,x + 26a_0}{2x^2+x}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) \mathrm{e}^{-\frac{5\ln(x)}{2} - \frac{5\ln(2x+1)}{4}} \\
&= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^{\frac{5}{2}}\,(2x+1)^{\frac{5}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{9\ln(x)}{2}+\frac{5\ln(2x+1)}{4}} \\
&= z_1 \left(\frac{(2x+1)^{\frac{5}{4}}}{x^{\frac{9}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-9\ln(x)+\frac{5\ln(2x+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(5005x^3 - 6435x^2 + 5148x - 2860)(2x+1)^{\frac{7}{2}}}{45045x^2 + 32760x + 6300}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^7}\right) \\
&\quad + c_2 \left(\frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^7}\left(\frac{(5005x^3 - 6435x^2 + 5148x - 2860)(2x+1)^{\frac{7}{2}}}{45045x^2 + 32760x + 6300}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(x^2 + \frac{8}{11}x + \frac{20}{143}\right)}{x^7} + \frac{c_2(2x+1)^{\frac{7}{2}}\left(35x^3 - 45x^2 + 36x - 20\right)}{315x^7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(x^2 + \frac{8}{11}x + \frac{20}{143}\right)}{x^7} + \frac{c_2(2x+1)^{\frac{7}{2}}\left(35x^3 - 45x^2 + 36x - 20\right)}{315x^7}$$

Verified OK.

2.174.1 Maple step by step solution

Let's solve

$(2x^3 + x^2)\, y'' + (13x^2 + 9x)\, y' + (5x + 7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x+7)y}{x^2(2x+1)} - \frac{(9+13x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(9+13x)y'}{x(2x+1)} + \frac{(5x+7)y}{x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{9+13x}{x(2x+1)}, P_3(x) = \frac{5x+7}{x^2(2x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 9$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 7$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(2x+1)\,y''+x(9+13x)\,y'+(5x+7)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=1..2$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=2..3$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+7)\,(k+r+1)+a_{k-1}(k+4+r)\,(2k-1+2r)\right)x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-7,-1\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+4+r)\,\left(k+r-\frac{1}{2}\right)a_{k-1}+a_k(k+r+7)\,(k+r+1)=0$

- Shift index using $k->k+1$

  $2(k+r+5)\left(k+\frac{1}{2}+r\right)a_k+a_{k+1}(k+8+r)\left(k+2+r\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(k+r+5)(2k+2r+1)a_k}{(k+8+r)(k+2+r)}$
- Recursion relation for $r=-7$; series terminates at $k=2$

  $a_{k+1}=-\frac{(k-2)(2k-13)a_k}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_1=\frac{26a_0}{5}$
- Apply recursion relation for $k=1$

  $a_2=\frac{11a_1}{8}$
- Express in terms of a_0

  $a_2=\frac{143a_0}{20}$
- Terminating series solution of the ODE for $r=-7$. Use reduction of order to find the second

  $y=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-1},a_{k+1}=-\frac{(k+4)(2k-1)a_k}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(\frac{143}{20}x^2+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty}b_kx^{k-1}\right),b_{k+1}=-\frac{(k+4)(2k-1)b_k}{(k+7)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(143x^2 + 104x + 20)}{x^7} + \frac{c_2(35x^3 - 45x^2 + 36x - 20)(2x + 1)^{\frac{7}{2}}}{x^7}$$

✓ Solution by Mathematica

Time used: 1.731 (sec). Leaf size: 58

```
DSolve[x^2*(1+2*x)*y''[x]+x*(9+13*x)*y'[x]+(7+5*x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{c_1(13x(11x + 8) + 20)}{143x^7} + \frac{c_2(35x^3 - 45x^2 + 36x - 20)(2x + 1)^{7/2}}{315x^7}$$

2.175 problem 177

2.175.1 Maple step by step solution 1709

Internal problem ID [7665]
Internal file name [`OUTPUT/6598_Sunday_June_05_2022_05_01_00_PM_62592853/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 177.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(2x+1)\,y''-2x(-x+4)\,y'-(5x+7)\,y=0}$$

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=8x^3+4x^2\\ B&=2x^2-8x\\ C&=-5x-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 33x^2 + 132x + 60$$
$$t = 16\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{4x} + \frac{15}{4x^2} + \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)\,(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x\,(2x+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{9}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right.\right.
$$

$$
0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \frac{(2x+1)^{\frac{9}{8}}}{x^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2-8x}{8x^3+4x^2} \, dx} \\
&= z_1 e^{\ln(x) - \frac{9\ln(2x+1)}{8}} \\
&= z_1 \left(\frac{x}{(2x+1)^{\frac{9}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)-\frac{9\ln(2x+1)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\frac{2}{7}x^3 - \frac{4}{7}x^2 - \frac{16}{7}x - \frac{32}{35}}{(2x+1)^{\frac{5}{4}}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}} \right) + c_2 \left(\frac{1}{\sqrt{x}} \left(\frac{\frac{2}{7}x^3 - \frac{4}{7}x^2 - \frac{16}{7}x - \frac{32}{35}}{(2x+1)^{\frac{5}{4}}} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2\left(x^3 - 2x^2 - 8x - \frac{16}{5}\right)}{7\sqrt{x}\,(2x+1)^{\frac{5}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2\left(x^3 - 2x^2 - 8x - \frac{16}{5}\right)}{7\sqrt{x}\,(2x+1)^{\frac{5}{4}}}$$

Verified OK.

2.175.1 Maple step by step solution

Let's solve

$(8x^3 + 4x^2) y'' + (2x^2 - 8x) y' + (-5x - 7) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(5x+7)y}{4x^2(2x+1)} - \frac{(x-4)y'}{2x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-4)y'}{2x(2x+1)} - \frac{(5x+7)y}{4x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x-4}{2x(2x+1)}, P_3(x) = -\frac{5x+7}{4x^2(2x+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{7}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(2x + 1) y'' + 2x(x - 4) y' + (-5x - 7) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-7+2r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(2k+2r+1)\,(2k+2r-7) + a_{k-1}(2k-1+2r)\,(4k-9+4r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-7+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $8\left(k-\frac{9}{4}+r\right)\left(k+r-\frac{1}{2}\right) a_{k-1} + 4\left(k+r+\frac{1}{2}\right) a_k \left(k+r-\frac{7}{2}\right) = 0$

- Shift index using $k->k+1$

  $8\left(k-\frac{5}{4}+r\right)\left(k+r+\frac{1}{2}\right) a_k + 4\left(k+\frac{3}{2}+r\right) a_{k+1} \left(k-\frac{5}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{(4k+4r-5)(2k+2r+1)a_k}{(2k+3+2r)(2k-5+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 3$

$a_{k+1} = -\frac{2(4k-7)ka_k}{(2k+2)(2k-6)}$

- Recursion relation for $r = \frac{7}{2}$

  $a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}$

- Solution for $r = \frac{7}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(4k+9)(2k+8)a_k}{(2k+10)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(4*x^2*(1+2*x)*diff(y(x),x$2)-2*x*(4-x)*diff(y(x),x)-(7+5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2(5x^3 - 10x^2 - 40x - 16)}{(2x+1)^{\frac{5}{4}}\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 47

```
DSolve[4*x^2*(1+2*x)*y''[x]-2*x*(4-x)*y'[x]-(7+5*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\frac{2c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}} + 35c_1}{35\sqrt{x}}$$

2.176 problem 178

2.176.1 Maple step by step solution 1719

Internal problem ID [7666]
Internal file name [`OUTPUT/6599_Sunday_June_05_2022_05_01_02_PM_31558855/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 178.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2(x+3)\,y''-x(15+x)\,y'-20y=0}$$

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=3x^3+9x^2\\B&=-x^2-15x\\C&=-20\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 450x + 1215}{36 \left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 450x + 1215 \\ t &= 36\left(x^2 + 3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 450x + 1215}{36 \left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{10}{9x} + \frac{10}{9(x+3)} + \frac{15}{4x^2} - \frac{2}{9(x+3)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36(x^2+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -\frac{1}{6} - \left(-\frac{7}{6}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3x + 9} - \frac{3}{2x} + (-)(0) \\ &= \frac{1}{3x + 9} - \frac{3}{2x} \\ &= -\frac{7x + 27}{6x(x + 3)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x+9} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3\left(x+3\right)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{3x+9} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2 + 450x + 1215}{36\left(x^2+3x\right)^2}\right)\right) = \frac{-27 + 7a_0}{3x\left(x+3\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{7}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{27}{7}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{27}{7}\right) \mathrm{e}^{\int \left(\frac{1}{3x+9} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) \mathrm{e}^{-\frac{3\ln(x)}{2} + \frac{\ln(x+3)}{3}} \\
&= \frac{\left(x + \frac{27}{7}\right)\left(x+3\right)^{\frac{1}{3}}}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2} \, dx} \\
&= z_1 e^{-\frac{2\ln(x+3)}{3} + \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{x^{\frac{5}{6}}}{\left(x+3\right)^{\frac{2}{3}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(x+3)}{3}+\frac{5\ln(x)}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{21(x+3)^{\frac{1}{3}}\left(x^2-36x-243\right)}{28x+108}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}\right) + c_2 \left(\frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}\left(\frac{21(x+3)^{\frac{1}{3}}\left(x^2-36x-243\right)}{28x+108}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(7x+27)}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}} + \frac{3c_2\left(x^2-36x-243\right)}{4x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(7x+27)}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}} + \frac{3c_2\left(x^2-36x-243\right)}{4x^{\frac{2}{3}}}$$

Verified OK.

2.176.1 Maple step by step solution

Let's solve

$(3x^3+9x^2)\,y''+(-x^2-15x)\,y'-20y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(15+x)y'}{3x(x+3)}+\frac{20y}{3x^2(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(15+x)y'}{3x(x+3)}-\frac{20y}{3x^2(x+3)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{15+x}{3x(x+3)}, P_3(x)=-\frac{20}{3x^2(x+3)}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=\frac{4}{3}$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

  ◦ $x=-3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $3x^2(x+3)\,y''-x(15+x)\,y'-20y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $(3u^3-18u^2+27u)\left(\frac{d^2}{du^2}y(u)\right)+(-u^2-9u+36)\left(\frac{d}{du}y(u)\right)-20y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $9a_0 r(1+3r)\, u^{-1+r} + \left(9a_1(1+r)\,(4+3r) - a_0(18r^2 - 9r + 20)\right) u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)\,(3k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9r(1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{3}\right\}$

- Each term must be 0

  $9a_1(1+r)\,(4+3r) - a_0(18r^2 - 9r + 20) = 0$

- Each term in the series must be 0, giving the recursion relation

  $3(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (6(-6a_k + a_{k-1} + 9a_{k+1})\, r + 9a_k - 10a_{k-1} + 63a_{k+1})\, k + 3(-6a_k +$

- Shift index using $k- >k+1$

  $3(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (6(-6a_{k+1} + a_k + 9a_{k+2})\, r + 9a_{k+1} - 10a_k + 63a_{k+2})\,(k+1)$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3k^2 a_k - 18k^2 a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2 a_k - 18r^2 a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2 + 6kr + 3r^2 + 13k + 13r + 14)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{3k^2 a_k - 18k^2 a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2 + 13k + 14)}$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-4ka_k-27ka_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1-20a_0=0\right]$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+3)^k, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-4ka_k-27ka_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1-20a_0=0\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-6ka_k-15ka_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-6ka_k-15ka_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k(x+3)^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-6ka_k-15ka_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(x+3)^k\right)+\left(\sum_{k=0}^{\infty} b_k(x+3)^{k-\frac{1}{3}}\right), a_{k+2}=-\frac{3k^2a_k-18k^2a_{k+1}-4ka_k-27ka_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 -\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(3*x^2*(3+x)*diff(y(x),x$2)-x*(15+x)*diff(y(x),x)-20*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 36x - 243)}{x^{\frac{2}{3}}} + \frac{c_2(7x + 27)}{x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.263 (sec). Leaf size: 43

```
DSolve[3*x^2*(3+x)*y''[x]-x*(15+x)*y'[x]-20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{21c_2(x^2 - 36x - 243) + \frac{4c_1(7x+27)}{\sqrt[3]{x+3}}}{28x^{2/3}}$$

2.177 problem 179

2.177.1 Maple step by step solution 1729

Internal problem ID [7667]
Internal file name [OUTPUT/6600_Sunday_June_05_2022_05_01_05_PM_53677634/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 179.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -10x^2+x \\ C &= 10x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 80x^2 - 28x + 35$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{49}{2x} + \frac{143}{4(1+x)^2} + \frac{35}{4x^2} + \frac{49}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4(x^2+x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 5$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= 5 - (4) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{13}{2(1+x)} - \frac{5}{2x} + (0) \\ &= \frac{13}{2(1+x)} - \frac{5}{2x} \\ &= \frac{8x - 5}{2x(1+x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2 - 28x + 35}{4(x^2+x)^2}\right)\right) = \frac{-5 - 8a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{-\frac{5\ln(x)}{2} + \frac{13\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{\frac{13}{2}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x^2+x}{x^2(1+x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \frac{11\ln(1+x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{\frac{11}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)+11\ln(1+x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{-\frac{8}{9}x^4-\frac{32}{45}x^3-\frac{16}{55}x^2-\frac{32}{495}x-\frac{8}{1287}}{\left(8x-5\right)\left(1+x\right)^{12}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3}\right) + c_2 \left(\frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3} \left(\frac{-\frac{8}{9}x^4-\frac{32}{45}x^3-\frac{16}{55}x^2-\frac{32}{495}x-\frac{8}{1287}}{\left(8x-5\right)\left(1+x\right)^{12}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3} + \frac{c_2\left(-715x^4-572x^3-234x^2-52x-5\right)}{6435x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3} + \frac{c_2\left(-715x^4-572x^3-234x^2-52x-5\right)}{6435x^3}$$

Verified OK.

2.177.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+(-10x^2+x)\,y'+(10x-9)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(10x-9)y}{x^2(1+x)}+\frac{(10x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(10x-1)y'}{x(1+x)}+\frac{(10x-9)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{10x-1}{x(1+x)},P_3(x)=\frac{10x-9}{x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=-11$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''-x(10x-1)\,y'+(10x-9)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-10u^2+21u-11)\left(\frac{d}{du}y(u)\right)+(10u-19)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-12+r)\, u^{-1+r} + \left(a_1(1+r)\,(-11+r) - a_0(2r^2 - 23r + 19)\right) u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-12+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 12\}$
- Each term must be 0

  $a_1(1+r)\,(-11+r) - a_0(2r^2 - 23r + 19) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a$
- Shift index using $k- >k + 1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})\,(k+1)$ -
- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 11ka_k + 19ka_{k+1} - 11ra_k + 19ra_{k+1} + 10a_k + 2a_{k+1}}{k^2 + 2kr + r^2 - 8k - 8r - 20}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 11ka_k + 19ka_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Recursion relation for $r = 12$

$$a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}$$

- Solution for $r = 12$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+12}, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 13ka_k - 29ka_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{x^3} + \frac{c_2\left(x^{13} + \frac{91}{8}x^{12} + \frac{117}{2}x^{11} + \frac{715}{4}x^{10} + \frac{715}{2}x^9 + \frac{3861}{8}x^8 + 429x^7 + \frac{429}{2}x^6\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 51

```
DSolve[x^2*(1+x)*y''[x]+x*(1-10*x)*y'[x]-(9-10*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{6435c_1(x+1)^{12}(8x-5) - 8c_2(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{51480x^3}$$

2.178 problem 180

2.178.1 Maple step by step solution 1739

Internal problem ID [7668]
Internal file name [OUTPUT/6601_Sunday_June_05_2022_05_01_08_PM_40793223/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 180.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1+x)\,y''+3x^2y'-(-x+6)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 3x^2 \\ C &= x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 20x + 24 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{x} + \frac{3}{4(1+x)^2} + \frac{6}{x^2} + \frac{7}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 20x + 24}{4(x^2+x)^2}$$

Since the gcd$(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2\left(1 + x\right)} - \frac{2}{x} + (-)\left(0\right) \\
&= \frac{3}{2\left(1 + x\right)} - \frac{2}{x} \\
&= -\frac{4 + x}{2x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{3}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{3}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{-x^2 + 20x + 24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4 + a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (4 + x)\, e^{\int \left(\frac{3}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= (4 + x)\, e^{-2\ln(x) + \frac{3\ln(1+x)}{2}} \\
&= \frac{(4 + x)(1 + x)^{\frac{3}{2}}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{3\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{(1+x)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{4+x}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27+27x} + \frac{256}{108+27x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{4+x}{x^2} \right) + c_2 \left(\frac{4+x}{x^2} \left(\ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27+27x} + \frac{256}{108+27x} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(4+x)}{x^2} + \frac{c_2\left(6(1+x)^2(4+x)\ln(1+x) + 60x^2 + 129x + 68\right)}{6x^2(1+x)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(4+x)}{x^2} + \frac{c_2\left(6(1+x)^2(4+x)\ln(1+x) + 60x^2 + 129x + 68\right)}{6x^2(1+x)^2}$$

Verified OK.

2.178.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{1+x}-\frac{(x-6)y}{x^2(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{1+x}+\frac{(x-6)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{1+x},P_3(x)=\frac{x-6}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=3$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-6u+3)\left(\frac{d}{du}y(u)\right)+(u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term must be 0

  $a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2+(2r+2)\,k+r^2+2r+\frac{7}{2}\right) a_k = 0$

- Shift index using $k- >k + 1$

  $a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2+(2r+2)\,(k+1)+r^2+2r+\frac{7}{2}\right) a_{k+1}$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} + 2ka_k - 8ka_{k+1} + 2ra_k - 8ra_{k+1} + a_k - 13a_{k+1}}{(k+r+2)(k+4+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2ka_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 8ka_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 78

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+3*x^2*diff(y(x),x)-(6-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^3+6x^2+9x+4)}{x^2\left(x+1\right)^2} + \frac{c_2\left(\ln\left(x+1\right)x^3+6\ln\left(x+1\right)x^2+9\ln\left(x+1\right)x+10x^2+4\ln\left(x+1\right)+\frac{43x}{2}+\frac{34}{3}\right)}{x^2\left(x+1\right)^2}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 49

```
DSolve[x^2*(1+x)*y''[x]+3*x^2*y'[x]-(6-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\frac{c_2(60x^2+129x+68)}{(x+1)^2}+6c_1(x+4)+6c_2(x+4)\log(x+1)}{6x^2}$$

2.179 problem 181

2.179.1 Maple step by step solution 1749

Internal problem ID [7669]
Internal file name [OUTPUT/6602_Sunday_June_05_2022_05_01_10_PM_33775521/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 181.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\, y'' - 2x(3+14x)\, y' + (6+100x)\, y = 0}$$

Writing the ode as

$$\left(2x^3+x^2\right) y'' + \left(-28x^2-6x\right) y' + (6+100x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= -28x^2-6x \\ C &= 6+100x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^2 - 16x + 6 \\ t &= (2x^2 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 - 16x + 6}{(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{40}{x} + \frac{6}{x^2} + \frac{20}{\left(x + \frac{1}{2}\right)^2} + \frac{40}{x + \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 5 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -4
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} + (0) \\
&= -\frac{2}{x} + \frac{5}{x+\frac{1}{2}} \\
&= \frac{6x-2}{2x^2+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right)^2 - \left(\frac{24x^2-16x+6}{\left(2x^2+x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x+\frac{1}{2}}\right) dx} \\
&= \frac{(2x+1)^5}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-28x^2-6x}{2x^3+x^2}\, dx} \\
&= z_1 e^{3\ln(x)+4\ln(2x+1)} \\
&= z_1 \left(x^3 (2x+1)^4\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x(2x+1)^9$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2-6x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{6\ln(x)+8\ln(2x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-2016x^4-672x^3-144x^2-18x-1}{20160\,(2x+1)^9}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x(2x+1)^9\right) + c_2 \left(x(2x+1)^9 \left(\frac{-2016x^4-672x^3-144x^2-18x-1}{20160\,(2x+1)^9}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x(2x+1)^9 + c_2 \left(-\frac{1}{10}x^5 - \frac{1}{30}x^4 - \frac{1}{140}x^3 - \frac{1}{1120}x^2 - \frac{1}{20160}x\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(2x+1)^9 + c_2 \left(-\frac{1}{10}x^5 - \frac{1}{30}x^4 - \frac{1}{140}x^3 - \frac{1}{1120}x^2 - \frac{1}{20160}x\right)$$

Verified OK.

2.179.1 Maple step by step solution

Let's solve

$(2x^3 + x^2)\, y'' + (-28x^2 - 6x)\, y' + (6 + 100x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(3+50x)y}{x^2(2x+1)} + \frac{2(3+14x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(3+14x)y'}{x(2x+1)} + \frac{2(3+50x)y}{x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(3+14x)}{x(2x+1)}, P_3(x) = \frac{2(3+50x)}{x^2(2x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 6$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2(2x+1)\, y'' - 2x(3+14x)\, y' + (6+100x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)\,(-6+r)\, x^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k}(k+r-1)\,(k+r-6)+2a_{k-1}(k+r-6)\,(k-11+r)\right) x^{k+r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-6+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,6\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-6)\,((2k+2r-22)\, a_{k-1}+a_{k}(k+r-1))=0$

- Shift index using $k->k+1$

  $(k+r-5)\,((2k+2r-20)\, a_{k}+a_{k+1}(k+r))=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2(k+r-10)a_{k}}{k+r}$

- Recursion relation for $r=1$; series terminates at $k=9$

  $a_{k+1}=-\frac{2(k-9)a_{k}}{k+1}$

- Recursion relation that defines the terminating series solution of the ODE for $r=1$

$$\left[y=\sum_{k=0}^{8} a_k x^{k+1}, a_{k+1}=-\frac{2(k-9)a_k}{k+1}\right]$$

- Recursion relation for $r=6$; series terminates at $k=4$

  $a_{k+1}=-\frac{2(k-4)a_k}{k+6}$
- Apply recursion relation for $k=0$

  $a_1=\frac{4a_0}{3}$
- Apply recursion relation for $k=1$

  $a_2=\frac{6a_1}{7}$
- Express in terms of a_0

  $a_2=\frac{8a_0}{7}$
- Apply recursion relation for $k=2$

  $a_3=\frac{a_2}{2}$
- Express in terms of a_0

  $a_3=\frac{4a_0}{7}$
- Apply recursion relation for $k=3$

  $a_4=\frac{2a_3}{9}$
- Express in terms of a_0

  $a_4=\frac{8a_0}{63}$
- Terminating series solution of the ODE for $r=6$. Use reduction of order to find the second li

  $y=a_0\cdot\left(1+\frac{4}{3}x+\frac{8}{7}x^2+\frac{4}{7}x^3+\frac{8}{63}x^4\right)$
- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{8} a_k x^{k+1}\right)+b_0\cdot\left(1+\frac{4}{3}x+\frac{8}{7}x^2+\frac{4}{7}x^3+\frac{8}{63}x^4\right), a_{k+1}=-\frac{2(k-9)a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 55

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x)=0,y(x), singsol=a
```

$$y(x) = c_1 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1) + c_2 x\left(x^9 + \frac{9}{2}x^8 + 9x^7 + \frac{21}{2}x^6 + \frac{63}{8}x^5\right)$$

✓ Solution by Mathematica
Time used: 0.07 (sec). Leaf size: 44

```
DSolve[x^2*(1+2*x)*y''[x]-2*x*(3+14*x)*y'[x]+(6+100*x)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to c_1 x(2x+1)^9 - \frac{c_2 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1)}{20160}$$

2.180 problem 182

2.180.1 Maple step by step solution 1759

Internal problem ID [7670]
Internal file name [OUTPUT/6603_Sunday_June_05_2022_05_01_13_PM_49872496/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 182.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-11x^2-6x\right)y'+(6+32x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-11x^2-6x\\C&=6+32x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 4x + 24}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 4x + 24$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 4x + 24}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{11}{x} + \frac{35}{4(1+x)^2} + \frac{6}{x^2} + \frac{11}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4(x^2 + x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} + (0) \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} \\
&= \frac{3x - 4}{2x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{7}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{7}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{15x^2 + 4x + 24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4 - 3a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{7}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{-2\ln(x) + \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{4}{3}\right)(1+x)^{\frac{7}{2}}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^2 - 6x}{x^2(1+x)} \, dx} \\
&= z_1 e^{3\ln(x) + \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(x^3 (1+x)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x - \frac{4}{3}\right) x(1+x)^6$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^2-6x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{6\ln(x)+5\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-\frac{3}{4}x^3 - \frac{9}{10}x^2 - \frac{9}{20}x - \frac{3}{35}}{(1+x)^6 (3x-4)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x - \frac{4}{3}\right) x(1+x)^6\right) + c_2 \left(\left(x - \frac{4}{3}\right) x(1+x)^6 \left(\frac{-\frac{3}{4}x^3 - \frac{9}{10}x^2 - \frac{9}{20}x - \frac{3}{35}}{(1+x)^6 (3x-4)}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x - \frac{4}{3}\right) x(1+x)^6 + c_2 \left(-\frac{1}{4}x^4 - \frac{3}{10}x^3 - \frac{3}{20}x^2 - \frac{1}{35}x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(x - \frac{4}{3}\right) x(1+x)^6 + c_2 \left(-\frac{1}{4}x^4 - \frac{3}{10}x^3 - \frac{3}{20}x^2 - \frac{1}{35}x\right)$$

Verified OK.

2.180.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+(-11x^2-6x)\,y'+(6+32x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3+16x)y}{x^2(1+x)}+\frac{(6+11x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(6+11x)y'}{x(1+x)}+\frac{2(3+16x)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{6+11x}{x(1+x)}, P_3(x)=\frac{2(3+16x)}{x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=-5$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-6+r)\, u^{-1+r} + (a_1(1+r)\,(-5+r) - 2a_0(r^2 - 9r + 13))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-6+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 6\}$$

- Each term must be 0

  $$a_1(1+r)\,(-5+r) - 2a_0(r^2 - 9r + 13) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1} -$$

- Shift index using $k- >k + 1$

  $$(-2a_{k+1} + a_k + a_{k+2})\, (k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\, (k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 12k a_k + 14k a_{k+1} - 12r a_k + 14r a_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12k a_k + 14k a_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Recursion relation for $r = 6$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10k a_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x(35x^3 + 42x^2 + 21x + 4) + c_2 x\left(x^7 + \frac{14}{3}x^6 + 7x^5\right)$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 45

```
DSolve[x^2*(1+x)*y''[x]-x*(6+11*x)*y'[x]+(6+32*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}c_1 x(x+1)^6(3x-4) - \frac{1}{140}c_2 x\left(35x^3+42x^2+21x+4\right)$$

2.181 problem 183

2.181.1 Maple step by step solution 1768

Internal problem ID [7671]
Internal file name [`OUTPUT/6604_Sunday_June_05_2022_05_01_16_PM_68753975/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 183.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(1+x)\,y''+4x(1+4x)\,y'-(49+27x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^3+4x^2\\B&=16x^2+4x\\C&=-27x-49\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^2 + 80x + 48$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{4}{x} + \frac{3}{4(1+x)^2} + \frac{12}{x^2} + \frac{4}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(1 + x\right)} + \frac{4}{x} + (0) \\ &= -\frac{1}{2\left(1 + x\right)} + \frac{4}{x} \\ &= \frac{7x + 8}{2x\left(1 + x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{4}{x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right)^2 - \left(\frac{35x^2 + 80x + 48}{4(x^2+x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right) dx} \\ &= \frac{x^4}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{16x^2+4x}{4x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{3\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\,(1+x)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{7}{2}}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\ln(x)-3\ln(1+x)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{-7x-6}{42x^7}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{x^{\frac{7}{2}}}{(1+x)^2}\right) + c_2\left(\frac{x^{\frac{7}{2}}}{(1+x)^2}\left(\frac{-7x-6}{42x^7}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{7}{2}}}{(1+x)^2} + \frac{c_2(-7x-6)}{42x^{\frac{7}{2}}(1+x)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{7}{2}}}{(1+x)^2} + \frac{c_2(-7x-6)}{42x^{\frac{7}{2}}(1+x)^2}$$

Verified OK.

2.181.1 Maple step by step solution

Let's solve

$(4x^3+4x^2)\,y'' + (16x^2+4x)\,y' + (-27x-49)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(49+27x)y}{4x^2(1+x)} - \frac{(1+4x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(1+4x)y'}{x(1+x)} - \frac{(49+27x)y}{4x^2(1+x)} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{1+4x}{x(1+x)}, P_3(x) = -\frac{49+27x}{4x^2(1+x)}\right]$$
  - $(1+x) \cdot P_2(x)$ is analytic at $x=-1$

    $$\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 3$$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

    $$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$
  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- Multiply by denominators

  $$4x^2(1+x)\,y'' + 4x(1+4x)\,y' + (-27x-49)\,y = 0$$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $$\left(4u^3-8u^2+4u\right)\left(\frac{d^2}{du^2}y(u)\right) + \left(16u^2-28u+12\right)\left(\frac{d}{du}y(u)\right) + (-27u-22)\,y(u) = 0$$
- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$
  - Shift index using $k->k-m$

    $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0 r(2+r)\, u^{-1+r} + (4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- Each term must be 0

$$4a_1(1+r)(3+r) - 2a_0(4r^2+10r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 4(2(-2a_k + a_{k-1} + a_{k+1})\, r - 5a_k + a_{k-1} + 4a_{k+1})\, k + 4(-2a_k + a_{k-}$$

- Shift index using $k- > k+1$

$$4(-2a_{k+1} + a_k + a_{k+2})\, (k+1)^2 + 4(2(-2a_{k+1} + a_k + a_{k+2})\, r - 5a_{k+1} + a_k + 4a_{k+2})\, (k+1) + 4($$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 8kra_k - 16kra_{k+1} + 4r^2a_k - 8r^2a_{k+1} + 12ka_k - 36ka_{k+1} + 12ra_k - 36ra_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+2kr+r^2+6k+6r+8)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} - 4ka_k - 4ka_{k+1} - 35a_k - 10a_{k+1}}{4(k^2+2k)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2a_k - 8k^2a_{k+1} + 12ka_k - 36ka_{k+1} - 27a_k - 50a_{k+1}}{4(k^2+6k+8)}$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1-22a_0=0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1-22a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+4*x)*diff(y(x),x)-(49+27*x)*y(x)=0,y(x), singsol=al
```

$$y(x)=\frac{c_1(7x+6)}{(x+1)^2 x^{\frac{7}{2}}}+\frac{c_2 x^{\frac{7}{2}}}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 36

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(1+4*x)*y'[x]-(49+27*x)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x)\to\frac{42c_1x^7-7c_2x-6c_2}{42x^{7/2}(x+1)^2}$$

2.182 problem 184

2.182.1 Maple step by step solution 1778

Internal problem ID [7672]
Internal file name [`OUTPUT/6605_Sunday_June_05_2022_05_01_20_PM_83905268/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 184.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^{2}\left(x^{2}+1\right)y''-x\left(-2x^{2}+7\right)y'+12y=0}$$

Writing the ode as

$$\left(x^{4}+x^{2}\right)y''+\left(2x^{3}-7x\right)y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{4}+x^{2} \\ B &= 2x^{3}-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -30x^2 + 15 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{45}{16(x-i)^2} + \frac{45}{16(x+i)^2} + \frac{75i}{16(x-i)} - \frac{75i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right)\\ &= 0 - (0)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} - \frac{5}{4\,(x-i)} - \frac{5}{4\,(x+i)} + (0) \\
&= \frac{5}{2x} - \frac{5}{4\,(x-i)} - \frac{5}{4\,(x+i)} \\
&= \frac{5}{2x\,(x^2+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{5}{2x} - \frac{5}{4\,(x-i)} - \frac{5}{4\,(x+i)}\right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4\,(x-i)^2} + \frac{5}{4\,(x+i)^2}\right) + \left(\frac{5}{2x} - \frac{5}{4\,(x-i)} - \frac{5}{4(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right) dx} \\
&= \frac{x^{\frac{5}{2}}}{(x^2+1)^{\frac{5}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2} \, dx} \\
&= z_1 e^{\frac{7\ln(x)}{2} - \frac{9\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{x^{\frac{7}{2}}}{(x^2+1)^{\frac{9}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^6}{(x^2+1)^{\frac{7}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{7\ln(x)-\frac{9\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-15\ \text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 + (8x^4-9x^2-2)\sqrt{x^2+1}}{8x^4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^6}{(x^2+1)^{\frac{7}{2}}}\right) \\
&\quad + c_2 \left(\frac{x^6}{(x^2+1)^{\frac{7}{2}}} \left(\frac{-15\ \text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 + (8x^4-9x^2-2)\sqrt{x^2+1}}{8x^4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} \\
&\quad + \frac{c_2 x^2 \left(8\sqrt{x^2+1}\, x^4 - 15\ \text{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 - 9x^2\sqrt{x^2+1} - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} + \frac{c_2 x^2 \left(8\sqrt{x^2+1}\, x^4 - 15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 - 9x^2\sqrt{x^2+1} - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}$$

Verified OK.

2.182.1 Maple step by step solution

Let's solve

$(x^4+x^2)\, y'' + (2x^3-7x)\, y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y}{x^2(x^2+1)} - \frac{(2x^2-7)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2-7)y'}{x(x^2+1)} + \frac{12y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(2x^2-7)\, y' + 12y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)\,(-6+r)\, x^r + a_1(-1+r)\,(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)\,(-6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{2, 6\}$$

- Each term must be 0

$$a_1(-1+r)\,(-5+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$$

- Shift index using $k- >k+2$

$$(k+r)\,(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$

- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(7-2*x^2)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} + \frac{c_2 x^2 \left(8x^4\sqrt{x^2+1} - 15x^4 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 9\sqrt{x^2+1}\, x^2 - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 88

```
DSolve[x^2*(1+x^2)*y''[x]-x*(7-2*x^2)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{-15c_2x^6\operatorname{arctanh}\left(\sqrt{x^2+1}\right) - 2c_2\sqrt{x^2+1}x^2 + 8x^6\left(c_2\sqrt{x^2+1}+c_1\right) - 9c_2\sqrt{x^2+1}x^4}{8\left(x^2+1\right)^{7/2}}$$

2.183 problem 185

2.183.1 Maple step by step solution 1788

Internal problem ID [7673]
Internal file name [`OUTPUT/6606_Sunday_June_05_2022_05_01_23_PM_53836214/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 185.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(-x^2+7)\,y' + 12y = 0}$$

Writing the ode as

$$x^2y'' + (x^3 - 7x)\,y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - 7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 12x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 12x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4-12x^2+15}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{x^2}{4}-3\right) + \left(\frac{15}{4x^2}\right) \\
&= \frac{x^2}{4} - 3 + \frac{15}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned}
b &= (-3)-(0) \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}}-1\right) &&= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}}-1\right) &&= \frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 12x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{2x} - \frac{x}{2} \\ &= \frac{5}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\ &= x^{\frac{5}{2}} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3 - 7x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} + \frac{7 \ln(x)}{2}} \\ &= z_1 \left(x^{\frac{7}{2}} e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^6 e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 - 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 - 4\,\mathrm{e}^{\frac{x^2}{2}}}{16x^4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \right) + c_2 \left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 - 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 - 4\,\mathrm{e}^{\frac{x^2}{2}}}{16x^4} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x^2 \left(\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 \mathrm{e}^{-\frac{x^2}{2}} + 2x^2 + 4 \right)}{16} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x^2 \left(\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 \mathrm{e}^{-\frac{x^2}{2}} + 2x^2 + 4 \right)}{16}
$$

Verified OK.

2.183.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 - 7x) y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{12y}{x^2} - \frac{(x^2-7)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-7)y'}{x} + \frac{12y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -7$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 12$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-7)\, y' + 12y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(-2+r)(-6+r)x^r + a_1(-1+r)(-5+r)x^{1+r} + \left(\sum_{k=2}^{\infty}(a_k(k+r-2)(k+r-6) + a_{k-2}(k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0

  $a_1(-1+r)(-5+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_k(k+r-6) + a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $(k+r)(a_{k+2}(k-4+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k-4+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(x^2*diff(y(x),x$2)-x*(7-x^2)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2 x^2 \mathrm{e}^{-\frac{x^2}{2}} \left(\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 + 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 + 4\,\mathrm{e}^{\frac{x^2}{2}}\right)}{16}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 61

```
DSolve[x^2*y''[x]-x*(7-x^2)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16} c_2 e^{-\frac{x^2}{2}} x^6 \,\mathrm{ExpIntegralEi}\left(\frac{x^2}{2}\right) - \frac{1}{8} c_2 \left(x^2 + 2\right) x^2 + c_1 e^{-\frac{x^2}{2}} x^6$$

2.184 problem 186

2.184.1 Maple step by step solution 1799

Internal problem ID [7674]
Internal file name [OUTPUT/6607_Sunday_June_05_2022_05_01_26_PM_77010802/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 186.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(2x^2+1)\,y' - (-10x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (2x^3+x)\,y' + (10x^2-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3 + x \\ C &= 10x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 32x^2 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2 - 8) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-8}{1} - 1\right) &&= -\frac{9}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-8}{1} - 1\right) &&= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} + (-)\,(x) \\
&= \frac{3}{2x} - x \\
&= \frac{3}{2x} - x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right)\right) = 0$$

$$\frac{2x^2 a_1 + (4a_0 + 8)\, x + 3a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^2 - 2)\, e^{\int \left(\frac{3}{2x} - x\right) dx} \\ &= (x^2 - 2)\, e^{-\frac{x^2}{2} + \frac{3\ln(x)}{2}} \\ &= (x^2 - 2)\, x^{\frac{3}{2}} e^{-\frac{x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3 + x}{x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x^2 - 2)\, x\, \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x^2-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{x^2}}{x^3 (x^2-2)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left((x^2-2)\, x\, \mathrm{e}^{-x^2}\right) + c_2 \left((x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3 (x^2-2)^2} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 (x^2-2)\, x\, \mathrm{e}^{-x^2} + c_2 (x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3 (x^2-2)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 (x^2-2)\, x\, \mathrm{e}^{-x^2} + c_2 (x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3 (x^2-2)^2} dx\right)$$

Verified OK.

2.184.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^3 + x)\,y' + (10x^2 - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x^2-1)y}{x^2} - \frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+1)y'}{x} + \frac{(10x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+1}{x}, P_3(x) = \frac{10x^2-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x^2 + 1)\,y' + (10x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r)\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term must be 0

  $$a_1(2+r)\, r = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-1) + 2a_{k-2}(k+3+r) = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+3+r)\,(k+r+1) + 2a_k(k+r+5) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{2a_k(k+r+5)}{(k+3+r)(k+r+1)}$$

- Recursion relation for $r = -1$

  $$a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(k+4)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(x^2*diff(y(x),x$2)+x*(1+2*x^2)*diff(y(x),x)-(1-10*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \, \mathrm{e}^{-x^2} \left(x^2 - 2\right) + c_2 x \, \mathrm{e}^{-x^2} \left(x^2 - 2\right) \left(\int \frac{\mathrm{e}^{x^2}}{\left(x^2 - 2\right)^2 x^3} dx\right)$$

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 68

```
DSolve[x^2*y''[x]+x*(1+2*x^2)*y'[x]-(1-10*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{e^{-x^2}\left(c_2(x^2-2)\,x^2\,\text{ExpIntegralEi}\left(x^2\right)+4c_1x^4-x^2\left(c_2e^{x^2}+8c_1\right)+c_2e^{x^2}\right)}{4x}$$

2.185 problem 187

2.185.1 Maple step by step solution 1810

Internal problem ID [7675]
Internal file name [OUTPUT/6608_Sunday_June_05_2022_05_01_30_PM_44770073/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 187.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-2x^2 + 1)\, y' - 4(2x^2 + 1)\, y = 0}$$

Writing the ode as

$$x^2y'' + (-2x^3 + x)\, y' + (-8x^2 - 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3 + x \\ C &= -8x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 24x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + 6 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(x^2 + 6\right) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{6}{1} - 1\right) &= \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (x) \\ &= \frac{5}{2x} + x \\ &= \frac{5}{2x} + x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} + x\right) dx} \\ &= x^{\frac{5}{2}} e^{\frac{x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3 + x}{x^2} \, dx} \\ &= z_1 e^{\frac{x^2}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{\frac{x^2}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2} dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1\left(x^2\right) x^4 + x^2 \mathrm{e}^{-x^2} - \mathrm{e}^{-x^2}}{4x^4} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^{x^2} \right) + c_2 \left(x^2 \mathrm{e}^{x^2} \left(\frac{-\operatorname{expIntegral}_1\left(x^2\right) x^4 + x^2 \mathrm{e}^{-x^2} - \mathrm{e}^{-x^2}}{4x^4} \right) \right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \mathrm{e}^{x^2} + \frac{c_2 \left(-\operatorname{expIntegral}_1\left(x^2\right) \mathrm{e}^{x^2} x^4 + x^2 - 1 \right)}{4x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \mathrm{e}^{x^2} + \frac{c_2 \left(-\operatorname{expIntegral}_1\left(x^2\right) \mathrm{e}^{x^2} x^4 + x^2 - 1 \right)}{4x^2}$$

Verified OK.

2.185.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-2x^3 + x\right) y' + \left(-8x^2 - 4\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4(2x^2+1)y}{x^2} + \frac{(2x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(2x^2-1)y'}{x}-\frac{4(2x^2+1)y}{x^2}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=-\frac{2x^2-1}{x}, P_3(x)=-\frac{4(2x^2+1)}{x^2}\right]$
  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=1$
  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-4$
  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^2y''-x(2x^2-1)y'+(-8x^2-4)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)(-2+r)x^r + a_1(3+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k+r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term must be 0

  $$a_1(3+r)(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+2)(a_k(k+r-2) - 2a_{k-2}) = 0$$

- Shift index using $k->k+2$

  $$(k+r+4)(a_{k+2}(k+r) - 2a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2a_k}{k+r}$$

- Recursion relation for $r = -2$

  $$a_{k+2} = \frac{2a_k}{k-2}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $$a_{k+2} = \frac{2a_k}{k-2}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = \frac{2a_k}{k+2}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 53

```
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x^2)*diff(y(x),x)-4*(1+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 \mathrm{e}^{x^2} - \frac{c_2 \mathrm{e}^{x^2} \left(-\operatorname{expIntegral}_1\left(x^2\right) x^4 + \mathrm{e}^{-x^2} x^2 - \mathrm{e}^{-x^2}\right)}{4x^2}$$

✓ Solution by Mathematica
Time used: 0.1 (sec). Leaf size: 46

```
DSolve[x^2*y''[x]+x*(1-2*x^2)*y'[x]-4*(1+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_2 \left(e^{x^2} x^4 \operatorname{ExpIntegralEi}\left(-x^2\right) + x^2 - 1\right)}{4x^2} + c_1 e^{x^2} x^2$$

2.186 problem 188

2.186.1 Maple step by step solution 1821

Internal problem ID [7676]
Internal file name [OUTPUT/6609_Sunday_June_05_2022_05_01_33_PM_123018/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 188.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-3x^2 + 1) y' - 4(-3x^2 + 1) y = 0}$$

Writing the ode as

$$x^2y'' + (-3x^3 + x) y' + (12x^2 - 4) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3 + x \\ C &= 12x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 60x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 357: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{9x^4 - 60x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}\end{aligned}$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$\begin{aligned}b &= (-15) - (0) \\ &= -15\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{3x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) &&= -\frac{11}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) &&= \frac{9}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\ &= \frac{5}{2x} - \frac{3x}{2} \\ &= \frac{5}{2x} - \frac{3x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)\right) = 0$$

$$\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - 2\right) e^{\int \left(\frac{5}{2x} - \frac{3x}{2}\right) dx} \\ &= \left(x^2 - 2\right) e^{-\frac{3x^2}{4} + \frac{5\ln(x)}{2}} \\ &= \left(x^2 - 2\right) x^{\frac{5}{2}} e^{-\frac{3x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3 + x}{x^2}\, dx} \\ &= z_1 e^{\frac{3x^2}{4} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{\frac{3x^2}{4}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2(x^2 - 2)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{3x^2}{2}-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 (x^2-2)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x^2(x^2-2)) + c_2\left(x^2(x^2-2)\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 (x^2-2)^2} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2(x^2-2) + c_2 x^2(x^2-2)\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 (x^2-2)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2(x^2-2) + c_2 x^2(x^2-2)\left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 (x^2-2)^2} dx\right)$$

Verified OK.

2.186.1 Maple step by step solution

Let's solve

$x^2y'' + (-3x^3 + x)\, y' + (12x^2 - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4(3x^2-1)y}{x^2} + \frac{(3x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3x^2-1)y'}{x} + \frac{4(3x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{3x^2-1}{x}, P_3(x) = \frac{4(3x^2-1)}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(3x^2 - 1)\, y' + (12x^2 - 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2+r)\,(-2+r)\, x^{r}+a_{1}(3+r)\,(-1+r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(k+r+2)\,(k+r-2)-3a_{k-2}(k-6\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,2\}$

- Each term must be 0

  $a_{1}(3+r)\,(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+2)\,(k+r-2)-3a_{k-2}(k-6+r)=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\,(k+r)-3a_{k}(k+r-4)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{3a_{k}(k+r-4)}{(k+4+r)(k+r)}$

- Recursion relation for $r=-2$; series terminates at $k=6$

  $a_{k+2}=\frac{3a_{k}(k-6)}{(k+2)(k-2)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$

  $a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{3a_k(k-2)}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*(1-3*x^2)*diff(y(x),x)-4*(1-3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 (x^2 - 2) + c_2 x^2 (x^2 - 2) \left(\int \frac{e^{\frac{3x^2}{2}}}{(x^2 - 2)^2 x^5} dx \right)$$

✓ Solution by Mathematica

Time used: 0.26 (sec). Leaf size: 89

```
DSolve[x^2*y''[x]+x*(1-3*x^2)*y'[x]-4*(1-3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{64} \left(27c_2 (x^2 - 2) x^2 \operatorname{ExpIntegralEi}\left(\frac{3x^2}{2}\right) + 64c_1 x^4 - 2x^2 \left(9c_2 e^{\frac{3x^2}{2}} + 64c_1\right) + 24c_2 e^{\frac{3x^2}{2}} + \frac{8c_2 e^{\frac{3x^2}{2}}}{x^2} \right)$$

2.187 problem 189

2.187.1 Maple step by step solution 1831

Internal problem ID [7677]
Internal file name [`OUTPUT/6610_Sunday_June_05_2022_05_01_36_PM_73461191/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 189.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0}$$

The ODE is

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0$$

Or

$$x(x^3y''+11x^2y'+24yx+xy''+5y')=0$$

For $x\neq 0$ the above simplifies to

$$(x^3+x)\,y''+11x^2y'+24yx+5y'=0$$

Writing the ode as

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2\\ B &= 11x^3+5x \\ C &= 24x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 + 6x^2 + 15 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 359: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{15}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{9i}{4(x-i)} - \frac{9i}{4(x+i)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

For the pole at $x=i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x=-i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{3x^4+6x^2+15}{4\left(x^3+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^4+6x^2+15}{4\left(x^3+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (0) \\ &= -\frac{3}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\ &= -\frac{3}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{11x^3+5x}{x^4+x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2} - \frac{3\ln\left(x^2+1\right)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{2}} (x^2+1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5\ln(x)-3\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-2x^2-1}{4\,(x^2+1)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(\frac{-2x^2-1}{4\,(x^2+1)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^4} + \frac{c_2(-2x^2-1)}{4x^4\,(x^2+1)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^4} + \frac{c_2(-2x^2-1)}{4x^4\,(x^2+1)^2}
$$

Verified OK. `{x <> 0}`

2.187.1 Maple step by step solution

Let's solve

$(x^4 + x^2)\, y'' + (11x^3 + 5x)\, y' + 24x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{24y}{x^2+1} - \frac{(11x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(11x^2+5)y'}{x(x^2+1)} + \frac{24y}{x^2+1} = 0$$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{11x^2+5}{x(x^2+1)}, P_3(x) = \frac{24}{x^2+1}\right]$$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 5$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $$24yx + (11x^2 + 5)\, y' + x(x^2 + 1)\, y'' = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$
  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$$
  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$
  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(4+r)\, x^{-1+r} + a_1(1+r)(5+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k+5+r) + a_{k-1}(k+5+r)(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-4, 0\}$$

- Each term must be 0

$$a_1(1+r)(5+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+5+r)(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$$

- Shift index using $k- > k+1$

$$(k+r+6)(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$$

- Recursion relation for $r = -4$

$$a_{k+2} = -\frac{a_k k}{k-2}$$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = -\frac{a_k k}{k-2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k(k+4)}{k+2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+11*x^2)*diff(y(x),x)+24*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x^2+1)}{(x^2+1)^2 x^4} + \frac{c_2}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 36

```
DSolve[x^2*(1+x^2)*y''[x]+x*(5+11*x^2)*y'[x]+24*x^2*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{-4c_1x^4 + 2c_2x^2 + c_2}{4x^4(x^2+1)^2}$$

2.188 problem 190

2.188.1 Maple step by step solution 1841

Internal problem ID [7678]
Internal file name [`OUTPUT/6611_Sunday_June_05_2022_05_01_39_PM_26555285/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 190.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+8xy'-(-x^2+35)\,y=0}$$

Writing the ode as

$$(4x^4+4x^2)\,y''+8xy'+(x^2-35)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 8x \\ C &= x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 22x^2 + 35$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 361: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{21i}{4(x-i)} - \frac{21i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 22x^2 + 35}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{3}{2\left(x - i\right)} + \frac{3}{2\left(x + i\right)} + (-)\left(0\right) \\
&= -\frac{5}{2x} + \frac{3}{2\left(x - i\right)} + \frac{3}{2\left(x + i\right)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2 + 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{3}{2\left(x - i\right)} + \frac{3}{2\left(x + i\right)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2\left(x - i\right)^2} - \frac{3}{2\left(x + i\right)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2\left(x - i\right)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{\left(x^2 + 1\right)^{\frac{3}{2}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{1}{4\left(x^2+1\right)^2}+\frac{1}{x^2+1}+\frac{\ln\left(x^2+1\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}\right) + c_2\left(\frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}\left(-\frac{1}{4\left(x^2+1\right)^2}+\frac{1}{x^2+1}+\frac{\ln\left(x^2+1\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+1\right)^2}{x^{\frac{7}{2}}} + \frac{c_2\left(\ln\left(x^2+1\right)\left(x^2+1\right)^2+2x^2+\frac{3}{2}\right)}{2x^{\frac{7}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^2}{x^{\frac{7}{2}}} + \frac{c_2\left(\ln\left(x^2+1\right)\left(x^2+1\right)^2+2x^2+\frac{3}{2}\right)}{2x^{\frac{7}{2}}}$$

Verified OK.

2.188.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\,y''+8xy'+(x^2-35)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-35)y}{4x^2(x^2+1)} - \frac{2y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x(x^2+1)} + \frac{(x^2-35)y}{4x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x(x^2+1)}, P_3(x) = \frac{x^2-35}{4x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{35}{4}$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\,y''+8xy'+(x^2-35)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(7+2r)\,(-5+2r)\, x^r + a_1(9+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)\,(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(7+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(9+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right)\left(k+r-\frac{5}{2}\right) = 0$$

- Shift index using $k- >k + 2$

$$4\left(\left(k-\frac{1}{2}+r\right)a_k+a_{k+2}\left(k+\frac{11}{2}+r\right)\right)\left(k-\frac{1}{2}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r-1)a_k}{2k+11+2r}$$

- Recursion relation for $r=-\frac{7}{2}$; series terminates at $k=4$

$$a_{k+2}=-\frac{(2k-8)a_k}{2k+4}$$

- Solution for $r=-\frac{7}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{7}{2}}, a_{k+2}=-\frac{(2k-8)a_k}{2k+4}, a_1=0\right]$$

- Recursion relation for $r=\frac{5}{2}$

$$a_{k+2}=-\frac{(2k+4)a_k}{2k+16}$$

- Solution for $r=\frac{5}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{(2k+4)a_k}{2k+16}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{7}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{(2k-8)a_k}{2k+4}, a_1=0, b_{k+2}=-\frac{(2k+4)b_k}{2k+16}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+8*x*diff(y(x),x)-(35-x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4 + 2x^2 + 1)}{x^{\frac{7}{2}}} + \frac{c_2\left(\frac{\ln(x^2+1)x^4}{2} + \ln(x^2 + 1)x^2 + x^2 + \frac{\ln(x^2+1)}{2} + \frac{3}{4}\right)}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 53

```
DSolve[4*x^2*(1+x^2)*y''[x]+8*x*y'[x]-(35-x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{4c_1(x^2 + 1)^2 + c_2(4x^2 + 3) + 2c_2(x^2 + 1)^2 \log(x^2 + 1)}{4x^{7/2}}$$

2.189 problem 191

2.189.1 Maple step by step solution 1851

Internal problem ID [7679]
Internal file name [`OUTPUT/6612_Sunday_June_05_2022_05_01_42_PM_97155788/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 191.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-x^2+5)\,y'-(25x^2+7)\,y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(x^3-5x)\,y'+(-25x^2-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=x^3-5x\\C&=-25x^2-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 99x^4 + 150x^2 + 63$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 363: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{9}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= -\frac{7}{2x} - \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{1}{x^{\frac{7}{2}} \sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{x^{\frac{5}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{1}{10}x^{10}+\frac{1}{8}x^8\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x\left(x^2+1\right)^2}\right) + c_2\left(\frac{1}{x\left(x^2+1\right)^2}\left(\frac{1}{10}x^{10}+\frac{1}{8}x^8\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x\left(x^2+1\right)^2} + \frac{c_2 x^7\left(4x^2+5\right)}{40\left(x^2+1\right)^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{x\left(x^2+1\right)^2} + \frac{c_2 x^7 (4x^2+5)}{40\left(x^2+1\right)^2}$$

Verified OK.

2.189.1 Maple step by step solution

Let's solve

$(x^4+x^2)\, y'' + (x^3-5x)\, y' + (-25x^2-7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(25x^2+7)y}{x^2(x^2+1)} - \frac{(x^2-5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-5)y'}{x(x^2+1)} - \frac{(25x^2+7)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2-5}{x(x^2+1)}, P_3(x) = -\frac{25x^2+7}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -7$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + x(x^2-5)\, y' + (-25x^2-7)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\, x^r + a_1(2+r)\,(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 7\}$$

- Each term must be 0

$$a_1(2+r)\,(-6+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-7)\left(a_k(k+r+1)+a_{k-2}(k+3+r)\right)=0$$

- Shift index using $k- >k+2$

$$(k+r-5)\left(a_{k+2}(k+3+r)+a_k(k+r+5)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+5)}{k+3+r}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{a_k(k+4)}{k+2}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0\right]$$

- Recursion relation for $r=7$

$$a_{k+2}=-\frac{a_k(k+12)}{k+10}$$

- Solution for $r=7$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+2}=-\frac{a_k(k+12)}{k+10}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0, b_{k+2}=-\frac{b_k(k+12)}{k+10}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(5-x^2)*diff(y(x),x)-(7+25*x^2)*y(x)=0,y(x), singsol=all
```

$$y(x) = \frac{c_1}{(x^2+1)^2 x} + \frac{c_2\left(x^{10}+\frac{5}{4}x^8\right)}{(x^2+1)^2 x}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 37

```
DSolve[x^2*(1+x^2)*y''[x]-x*(5-x^2)*y'[x]-(7+25*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{c_2(4x^2+5)\,x^8+40c_1}{40x\,(x^2+1)^2}$$

2.190 problem 192

2.190.1 Maple step by step solution 1861

Internal problem ID [7680]
Internal file name [`OUTPUT/6613_Sunday_June_05_2022_05_01_45_PM_57141050/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 192.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3+5x)\,y'-21y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3+5x \\ C &= -21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 78x^2 + 99$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{78x^2 + 99}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 365: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{99}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{219i}{16(x-i)} - \frac{219i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} + (-)(0) \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \\
&= -\frac{9}{2x} + \frac{7x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right)(2x + a_1) + \left(\left(\frac{9}{2x^2} - \frac{7}{4(x-i)^2} - \frac{7}{4(x+i)^2}\right) + \left(-\frac{9}{2x} + \frac{7}{4(x-
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 8, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 8
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 8\right) e^{\int \left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right) dx} \\
&= \left(x^2 + 8\right) e^{-\frac{9\ln(x)}{2} + \frac{7\ln\left(x^2+1\right)}{4}} \\
&= \frac{\left(x^2 + 8\right)\left(x^2 + 1\right)^{\frac{7}{4}}}{x^{\frac{9}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}+\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\left(x^2+1\right)^{\frac{3}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{-35x^6-140x^4-168x^2-64}{\left(x^2+1\right)^{\frac{5}{2}}\left(35x^2+280\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}\right) + c_2 \left(\frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}\left(\frac{-35x^6-140x^4-168x^2-64}{\left(x^2+1\right)^{\frac{5}{2}}\left(35x^2+280\right)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7} + \frac{c_2\left(-35x^6-140x^4-168x^2-64\right)}{35x^7} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2+8)(x^2+1)^{\frac{5}{2}}}{x^7} + \frac{c_2(-35x^6-140x^4-168x^2-64)}{35x^7}$$

Verified OK.

2.190.1 Maple step by step solution

Let's solve

$(x^4+x^2)y''+(2x^3+5x)y'-21y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{21y}{x^2(x^2+1)}-\frac{(2x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x^2+5)y'}{x(x^2+1)}-\frac{21y}{x^2(x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x^2+5}{x(x^2+1)}, P_3(x)=-\frac{21}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-21$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)y''+x(2x^2+5)y'-21y=0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(7+r)\,(-3+r)\, x^r + a_1(8+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-7, 3\}$

- Each term must be 0

  $a_1(8+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2+r)\,(k+r-1) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+9+r)\,(k+r-1) + a_k(k+r)\,(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$

- Recursion relation for $r = -7$; series terminates at $k = 6$

  $a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$

- Solution for $r = -7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-7}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0, b_{k+2} = -\frac{b_k(k+3)(k+4)}{(k+12)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+2*x^2)*diff(y(x),x)-21*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(35x^6 + 140x^4 + 168x^2 + 64)}{x^7} + \frac{c_2(x^2 + 1)^{\frac{5}{2}}(x^2 + 8)}{x^7}$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 52

```
DSolve[x^2*(1+x^2)*y''[x]+x*(5+2*x^2)*y'[x]-21*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{35c_1(x^2+1)^{5/2}(x^2+8) - c_2(35x^6+140x^4+168x^2+64)}{35x^7}$$

2.191 problem 193

2.191.1 Maple step by step solution 1871

Internal problem ID [7681]
Internal file name [`OUTPUT/6614_Sunday_June_05_2022_05_01_48_PM_40280/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 193.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+4x(x^2+2)\,y'-(x^2+15)\,y=0}$$

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(4x^3+8x\right)y'+\left(-x^2-15\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 + 15}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 10x^2 + 15 \\ t &= 4(x^3 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 + 15}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 367: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{35i}{16(x-i)} - \frac{35i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{5}{4(x-i)^2} - \frac{5}{4(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{}{4}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{5}{4}}}{x^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x^3+8x}{4x^4+4x^2} \, dx} \\
&= z_1 e^{-\ln(x) + \frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{(x^2+1)^{\frac{1}{4}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\frac{\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-3x^2-2}{3(x^2+1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}\right) + c_2 \left(\frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}} \left(\frac{-3x^2-2}{3(x^2+1)^{\frac{3}{2}}}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}} + \frac{c_2(-3x^2-2)}{3x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}} + \frac{c_2(-3x^2-2)}{3x^{\frac{5}{2}}}$$

Verified OK.

2.191.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\, y''+(4x^3+8x)\, y'+(-x^2-15)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2+2)y'}{x(x^2+1)}+\frac{(x^2+15)y}{4x^2(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2+2)y'}{x(x^2+1)}-\frac{(x^2+15)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{x^2+2}{x(x^2+1)}, P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=2$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{15}{4}$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\, y''+4x(x^2+2)\, y'+(-x^2-15)\, y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(5+2r)\,(-3+2r)\, x^{r}+a_{1}(7+2r)\,(-1+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(2k+2r+5)\,(2k+2r-3)+a_{k-}\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{5}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_{1}(7+2r)\,(-1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{5}{2}\right) a_{k-2}+a_{k}\left(k+r+\frac{5}{2}\right)\right)=0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right) a_{k}+a_{k+2}\left(k+\frac{9}{2}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{(2k+2r-1)a_{k}}{2k+9+2r}$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{(2k-6)a_k}{2k+4}$

- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{(2k+2)a_k}{2k+12}$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{2k+12}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{2k+12}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(2+x^2)*diff(y(x),x)-(15+x^2)*y(x)=0,y(x), singsol=a
```

$$y(x) = \frac{c_1(3x^2+2)}{x^{\frac{5}{2}}} + \frac{c_2(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 39

```
DSolve[4*x^2*(1+x^2)*y''[x]+4*x*(2+x^2)*y'[x]-(15+x^2)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{3c_1(x^2+1)^{3/2} - c_2(3x^2+2)}{3x^{5/2}}$$

2.192 problem 194

2.192.1 Maple step by step solution 1881

Internal problem ID [7682]
Internal file name [`OUTPUT/6615_Sunday_June_05_2022_05_01_50_PM_77920895/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 194.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{2(t+1)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0}$$

Writing the ode as

$$y'' + \frac{(-2t-2)\,y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \qquad (5)$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{(t^2 + 2t - 1)^2} \qquad (6)$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = (t^2 + 2t - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{(t^2 + 2t - 1)^2}\right) z(t) \qquad (7)$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 369: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 + 2t - 1)^2$. There is a pole at $t = \sqrt{2} - 1$ of order 2. There is a pole at $t = -1 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t - \sqrt{2} + 1\right)^2} + \frac{3}{4\left(t + 1 + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t - \sqrt{2} + 1\right)} + \frac{3\sqrt{2}}{8\left(t + 1 + \sqrt{2}\right)}$$

For the pole at $t = \sqrt{2} - 1$ let b be the coefficient of $\frac{1}{\left(t - \sqrt{2} + 1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $t = -1 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t + 1 + \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{6}{\left(t^2+2t-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\sqrt{2}-1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^2+2t-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\left(t-\right.}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\
&= \frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}}}{\sqrt{t-\sqrt{2}+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\
&= z_1\left(\sqrt{t^2+2t-1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\ln(t^2+2t-1)}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{-t-1}{\left(t+1+\sqrt{2}\right)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}}\right) + c_2 \left(\frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} \left(\frac{-t-1}{\left(t+1+\sqrt{2}\right)^2}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 \left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} + \frac{c_2 \sqrt{t^2+2t-1}\,(-t-1)}{\sqrt{t+1+\sqrt{2}}\,\sqrt{t-\sqrt{2}+1}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 \left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} + \frac{c_2 \sqrt{t^2+2t-1}\,(-t-1)}{\sqrt{t+1+\sqrt{2}}\,\sqrt{t-\sqrt{2}+1}}$$

Verified OK.

2.192.1 Maple step by step solution

Let's solve

$y'' + \frac{(-2t-2)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(t+1)y'}{t^2+2t-1} - \frac{2y}{t^2+2t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(t+1)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$

  ◦ $(t+1+\sqrt{2}) \cdot P_2(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2}) \cdot P_2(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

  ◦ $(t+1+\sqrt{2})^2 \cdot P_3(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2})^2 \cdot P_3(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

  ◦ $t = -1-\sqrt{2}$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1-\sqrt{2}$

- Multiply by denominators

  $y''(t^2+2t-1) + (-2t-2)\,y' + 2y = 0$

- Change variables using $t = u-1-\sqrt{2}$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2\sqrt{2})\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\, r(r-2)\, a_0 u^{r-1} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)\,(k+r-1)\, a_{k+1} + a_k(k+r-1)\,(k+r-2)\right) u\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\, r(r-2) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-2)\right)(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)\sqrt{2}}{4(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)\sqrt{2}}{4(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0\sqrt{2}}{2}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{8}$
- Express in terms of a_0

$a_2 = \frac{a_0}{8}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u\sqrt{2}}{2} + \frac{u^2}{8}\right)$

- Revert the change of variables $u = t + 1 + \sqrt{2}$

  $\left[y = a_0 \left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right)\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$

- Revert the change of variables $u = t + 1 + \sqrt{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(t + 1 + \sqrt{2}\right)^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_k \left(t + 1 + \sqrt{2}\right)^{k+2}\right), b_{k+1} = \frac{b_k k\sqrt{2}}{4(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t$2)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t)=0,y(t), singsol=al
```

$$y(t) = c_1(t+1) + c_2(t^2+1)$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 64

```
DSolve[y''[t]-2*(t+1)/(t^2+2*t-1)*y'[t]+2/(t^2+2*t-1)*y[t]==0,y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{\sqrt{t^2+2t-1}\left(c_1\left(t^2-2\left(\sqrt{2}-1\right)t-2\sqrt{2}+3\right)+c_2(t+1)\right)}{\sqrt{-t^2-2t+1}}$$

2.193 problem 195

2.193.1 Maple step by step solution 1888

Internal problem ID [7683]
Internal file name [`OUTPUT/6616_Sunday_June_05_2022_05_01_54_PM_44015855/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 195.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4ty' + (4t^2 - 2) y = 0}$$

Writing the ode as

$$y'' - 4ty' + (4t^2 - 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 371: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\, dt} \\
&= z_1 e^{t^2} \\
&= z_1 \left(\mathrm{e}^{t^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{2t^2}}{(y_1)^2}\, dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{t^2}\right) + c_2\left(e^{t^2}(t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{t^2} + c_2 e^{t^2} t \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{t^2} + c_2 e^{t^2} t
$$

Verified OK.

2.193.1 Maple step by step solution

Let's solve

$y'' - 4ty' + (4t^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

- ○ Convert $t^m \cdot y$ to series expansion for $m = 0..2$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

- ○ Shift index using $k- > k - m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- ○ Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k k\, t^k$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^{k}$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, t + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) t^k\right) = 0$$

- The coefficients of each power of t must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(t),t$2)-4*t*diff(y(t),t)+(4*t^2-2)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{t^2} + c_2 e^{t^2} t$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[y''[t]-4*t*y'[t]+(4*t^2-2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t^2} (c_2 t + c_1)$$

2.194 problem 196

2.194.1 Maple step by step solution 1897

Internal problem ID [7684]
Internal file name [`OUTPUT/6617_Sunday_June_05_2022_05_01_56_PM_57359124/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 196.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-t^2+1)\,y''-2ty'+2y=0}$$

Writing the ode as

$$\begin{aligned} (-t^2+1)\,y''-2ty'+2y&=0 && (1)\\ Ay''+By'+Cy&=0 && (2)\end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-t^2+1\\ B&=-2t && (3)\\ C&=2\end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2t^2 - 3}{(t^2 - 1)^2}
\tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2t^2 - 3 \\
t &= (t^2 - 1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{2t^2 - 3}{(t^2 - 1)^2}\right) z(t)
\tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 373: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(t^2 - 1\right)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4\,(t+1)} - \frac{1}{4\,(t+1)^2} + \frac{5}{4\,(t-1)} - \frac{1}{4\,(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{\left(t^2 - 1\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} \\
&= \frac{t}{t^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = -\frac{2a_0}{t^2-1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= (t)\, e^{\frac{\ln(t-1)}{2} + \frac{\ln(t+1)}{2}} \\
&= t\sqrt{t-1}\,\sqrt{t+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\,dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{1}{t} - \frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} \left(\frac{1}{t} - \frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} + \frac{c_2\sqrt{t^2-1}\,(\ln(t-1)\,t - \ln(t+1)\,t + 2)}{2\sqrt{t-1}\,\sqrt{t+1}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} + \frac{c_2\sqrt{t^2-1}\,(\ln(t-1)\,t - \ln(t+1)\,t + 2)}{2\sqrt{t-1}\,\sqrt{t+1}}$$

Verified OK.

2.194.1 Maple step by step solution

Let's solve

$(-t^2+1)\,y'' - 2ty' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2ty'}{t^2-1} + \frac{2y}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2ty'}{t^2-1} - \frac{2y}{t^2-1} = 0$

☐ Check to see if t_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{2}{t^2-1}\right]$

- ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1} = 1$

- ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1} = 0$

- ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $y''(t^2-1) + 2ty' - 2y = 0$

- Change variables using $t = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = t + 1$

  $[y = -a_0 t]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((1-t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 \left(-\frac{\ln(t+1)\, t}{2} + \frac{\ln(t-1)\, t}{2} + 1 \right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 33

```
DSolve[(1-t^2)*y''[t]-2*t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

2.195 problem 197

Internal problem ID [7685]
Internal file name [`OUTPUT/6618_Sunday_June_05_2022_05_01_58_PM_79991775/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 197.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__y__method__2", "second__order__ode__non__constant__coeff__transformation__on__B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(t^2+1)\,y''-2ty'+2y=0}$$

Writing the ode as

$$(t^2+1)\,y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{\left(t^2+1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= \left(t^2+1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(-\frac{3}{\left(t^2+1\right)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 375: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}
$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(t^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)} + (-)\,(0) \\ &= -\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)} \\ &= \frac{t-2i}{t^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{3}{(t^2 -}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\
&= \frac{(t^2+1)^{\frac{3}{2}}}{(it+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1}\, dt} \\
&= z_1 e^{\frac{\ln(t^2+1)}{2}} \\
&= z_1 \left(\sqrt{t^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{t}{(t+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2\left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(t^2+1)^2}{(it+1)^2} + \frac{c_2(t^2+1)^2\,t}{(t-i)^2\,(t+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(t^2+1)^2}{(it+1)^2} + \frac{c_2(t^2+1)^2\,t}{(t-i)^2\,(t+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 (t^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 21

```
DSolve[(1+t^2)*y''[t]-2*t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

2.196 problem 198

2.196.1 Maple step by step solution 1913

Internal problem ID [7686]
Internal file name [`OUTPUT/6619_Sunday_June_05_2022_05_02_01_PM_19497511/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 198.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-t^2+1\right) y''-2ty'+6y=0}$$

Writing the ode as

$$\left(-t^2+1\right) y''-2ty'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6t^2 - 7$$
$$t = (t^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6t^2 - 7}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2-1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{13}{4(t+1)} - \frac{1}{4(t+1)^2} + \frac{13}{4(t-1)} - \frac{1}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2-1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 3 - (1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\
&= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\
&= \frac{t}{t^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{(t^2-1)^2}\right.\right.$$

$$\frac{-4a_1 t - 6a_0 - 2}{t^2 - 1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\frac{\ln(t-1)}{2} + \frac{\ln(t+1)}{2}} \\
&= \left(t^2 - \frac{1}{3}\right) \sqrt{t-1}\,\sqrt{t+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2t}{-t^2+1} \, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^2 - \frac{1}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\frac{9\ln(t+1)}{8} + \frac{27t}{12t^2-4} + \frac{9\ln(t-1)}{8} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2 - \frac{1}{3} \right) + c_2 \left(t^2 - \frac{1}{3} \left(-\frac{9\ln(t+1)}{8} + \frac{27t}{12t^2-4} + \frac{9\ln(t-1)}{8} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(t^2 - \frac{1}{3} \right) + c_2 \left(\frac{9\ln(t-1)\, t^2}{8} - \frac{9\ln(t+1)\, t^2}{8} - \frac{3\ln(t-1)}{8} + \frac{3\ln(t+1)}{8} + \frac{9t}{4} \right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(t^2 - \frac{1}{3} \right) + c_2 \left(\frac{9\ln(t-1)\, t^2}{8} - \frac{9\ln(t+1)\, t^2}{8} - \frac{3\ln(t-1)}{8} + \frac{3\ln(t+1)}{8} + \frac{9t}{4} \right)$$

Verified OK.

2.196.1 Maple step by step solution

Let's solve

$(-t^2+1)\,y''-2ty'+6y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2ty'}{t^2-1}+\frac{6y}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2ty'}{t^2-1}-\frac{6y}{t^2-1}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{2t}{t^2-1}, P_3(t)=-\frac{6}{t^2-1}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\bigg|_{t=-1}=1$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\bigg|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $y''(t^2-1)+2ty'-6y=0$

- Change variables using $t=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-6y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+3)(k-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -3a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1}{2}$$

- Express in terms of a_0

$$a_2 = \frac{3a_0}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$$

- Revert the change of variables $u = t + 1$

$$\left[y = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 50

```
dsolve((1-t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+6*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(-3t^2 + 1) + c_2\left(-\frac{3\ln(t+1)\,t^2}{8} + \frac{3\ln(t-1)\,t^2}{8} + \frac{\ln(t+1)}{8} - \frac{\ln(t-1)}{8} + \frac{3t}{4}\right)$$

✓ Solution by Mathematica
Time used: 0.024 (sec). Leaf size: 55

```
DSolve[(1-t^2)*y''[t]-2*t*y'[t]+6*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}c_1(3t^2 - 1) - \frac{1}{4}c_2\left((3t^2 - 1)\log(1-t) + (1 - 3t^2)\log(t+1) + 6t\right)$$

2.197 problem 199

2.197.1 Maple step by step solution 1922

Internal problem ID [7687]
Internal file name [`OUTPUT/6620_Sunday_June_05_2022_05_02_04_PM_38710744/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 199.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2t+1)\,y''-4(t+1)\,y'+4y=0}$$

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1 \\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 2}{(2t + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4t^2 + 2$$
$$t = (2t + 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 2}{(2t + 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t + \frac{1}{2}\right)^2} - \frac{1}{t + \frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2 + 2}{4t^2 + 4t + 1} \\ &= Q + \frac{R}{4t^2 + 4t + 1} \\ &= (1) + \left(\frac{-4t + 1}{4t^2 + 4t + 1}\right) \\ &= 1 + \frac{-4t + 1}{4t^2 + 4t + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4t^2 + 2}{(2t + 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + 1 \\
&= \frac{2t}{2t+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2+2}{(2t+1)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right) dt} \\
&= \frac{\mathrm{e}^t}{\sqrt{2t+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4t-4}{2t+1} \, dt} \\
&= z_1 e^{t + \frac{\ln(2t+1)}{2}} \\
&= z_1 \left(\sqrt{2t+1}\, \mathrm{e}^t\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2}\, dt \\ &= y_1\left(-(t+1)\,\mathrm{e}^{-2t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2t}\right) + c_2\left(\mathrm{e}^{2t}\left(-(t+1)\,\mathrm{e}^{-2t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2t} + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2t} + c_2(-t-1)$$

Verified OK.

2.197.1 Maple step by step solution

Let's solve

$(2t+1)\, y'' + (-4t-4)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{4y}{2t+1} + \frac{4(t+1)y'}{2t+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4(t+1)y'}{2t+1} + \frac{4y}{2t+1} = 0$

☐ Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{4(t+1)}{2t+1}, P_3(t) = \frac{4}{2t+1}\right]$

- ○ $\left(t+\frac{1}{2}\right) \cdot P_2(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right) \cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}} = -1$

- ○ $\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}} = 0$

- ○ $t = -\frac{1}{2}$is a regular singular point

  Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

  $t_0 = -\frac{1}{2}$

- Multiply by denominators

  $(2t+1)\,y'' + (-4t-4)\,y' + 4y = 0$

- Change variables using $t = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + (-4u-2)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) - 4a_k(k+r-1)\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(a_{k+1}(k+1+r) - 2a_k)(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{2a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k}{k+1}\right]$$

- Revert the change of variables $u = t + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{2a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$$

- Revert the change of variables $u = t + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty} b_k \left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1}=\frac{2a_k}{k+1}, b_{k+1}=\frac{2b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((2*t+1)*diff(y(t),t$2)-4*(t+1)*diff(y(t),t)+4*y(t)=0,y(t), singsol=all)
```

$$y(t)=c_1(t+1)+c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 23

```
DSolve[(2*t+1)*y''[t]-4*(t+1)*y'[t]+4*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t)\to c_1 e^{2t+1}-c_2(t+1)$$

2.198 problem 200

2.198.1 Maple step by step solution 1929

Internal problem ID [7688]
Internal file name [`OUTPUT/6621_Sunday_June_05_2022_05_02_08_PM_56558202/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 200.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 380: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\ &= z_1 e^{-\frac{\ln(t)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(t)}{\sqrt{t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(t)}{\sqrt{t}}\right) + c_2 \left(\frac{\cos(t)}{\sqrt{t}} (\tan(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(t)}{\sqrt{t}} + \frac{c_2 \sin(t)}{\sqrt{t}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(t)}{\sqrt{t}} + \frac{c_2 \sin(t)}{\sqrt{t}}
$$

Verified OK.

2.198.1 Maple step by step solution

Let's solve

$y''t^2 + ty' + \left(t^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4t^2-1)y}{4t^2} - \frac{y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{t} + \frac{(4t^2-1)y}{4t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\Big|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = -\frac{1}{4}$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $4y''t^2 + 4ty' + (4t^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..2$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r}$

- Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, t^r + a_1(3 + 2r)\,(1 + 2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(t^2*diff(y(t),t$2)+t*diff(y(t),t)+(t^2-1/4)*y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 \sin (t)}{\sqrt{t}} + \frac{c_2 \cos (t)}{\sqrt{t}}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 39

```
DSolve[t^2*y''[t]+t*y'[t]+(t^2-1/4)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-it}(2c_1 - ic_2e^{2it})}{2\sqrt{t}}$$

2.199 problem 201

Internal problem ID [7689]
Internal file name [`OUTPUT/6622_Sunday_June_05_2022_05_02_11_PM_73596890/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 201.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0}$$

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}
$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(t^{2}+1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} \\
&= \frac{t-2i}{t^{2}+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)(0) + \left(\left(\frac{1}{2(t-i)^2} - \frac{3}{2(t+i)^2}\right) + \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right)^2 - \left(-\frac{3}{(t^2 -\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{\frac{3}{2}}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2t}{t^2+1}}{1}\, dt} \\ &= z_1 e^{\frac{\ln(t^2+1)}{2}} \\ &= z_1 \left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1} dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2} \, dt \\
&= y_1 \left(-\frac{t}{(t+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2 t}{(t-i)^2 (t+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2 t}{(t-i)^2 (t+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t$2)-2*t/(1+t^2)*diff(y(t),t)+2/(1+t^2)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 (t^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 21

```
DSolve[y''[t]-2*t/(1+t^2)*y'[t]+2/(1+t^2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

2.200 problem 202

2.200.1 Maple step by step solution 1946

Internal problem ID [7690]
Internal file name [`OUTPUT/6623_Sunday_June_05_2022_05_02_13_PM_58045217/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 202.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (t^2 + 2t + 1)\, y' - (4t + 4)\, y = 0}$$

Writing the ode as

$$y'' + (t+1)^2 y' + (-4t - 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (t+1)^2 \\ C &= -4t - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 + 4t^3 + 6t^2 + 24t + 21 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{2} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\ &= \frac{1}{2}t^2 + t + \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\
&= Q + \frac{R}{4} \\
&= \left(6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\
&= 6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned}
b &= (6) - (1) \\
&= 5
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) &= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) &= -6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = 6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2 + t + \frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{+} = 4$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_{\infty} \\
&= 0 + \left(\frac{1}{2} t^2 + t + \frac{1}{2} \right) \\
&= \frac{1}{2} t^2 + t + \frac{1}{2} \\
&= \frac{(t+1)^2}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)(4t^3 + 3t^2 a_3 + 2ta_2 + a_1) + \left((t+1) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(6t + \right.\right.
$$

$$
(-a_3 + 4) t^4 + 2(2 - a_2 + a_3) t^3 + 3(4 - a_1 + a_3) t^2 + 2(-2a_0 - a_1 + a_2 + 3a_3)
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega \, dt} \\
&= \left(t^4+4t^3+6t^2+8t+5\right) \mathrm{e}^{\int \left(\frac{1}{2}t^2+t+\frac{1}{2}\right)dt} \\
&= \left(t^4+4t^3+6t^2+8t+5\right) \mathrm{e}^{\frac{(t+1)^3}{6}} \\
&= (t+1)\left(t^3+3t^2+3t+5\right) \mathrm{e}^{\frac{(t+1)^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(t+1)^2}{1}\, dt} \\
&= z_1 e^{-\frac{(t+1)^3}{6}} \\
&= z_1 \left(\mathrm{e}^{-\frac{(t+1)^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (t+1)\left(t^3+3t^2+3t+5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(t+1)^2}{1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{(t+1)^3}{3}}}{(y_1)^2}\, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((t+1)\left(t^3+3t^2+3t+5\right)\big) \\
&\quad + c_2\left((t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{e^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1(t+1)\left(t^3+3t^2+3t+5\right) \\
&\quad + c_2(t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{e^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1(t+1)\left(t^3+3t^2+3t+5\right) \\
&\quad + c_2(t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{e^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)
\end{aligned}
$$

Verified OK.

2.200.1 Maple step by step solution

Let's solve

$y'' + (t+1)^2 y' + (-4t-4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  - Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

  - Shift index using $k->k-m$

    $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $$t^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$$

- Shift index using $k- >k+1-m$

  $$t^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$$

  Rewrite ODE with series expansions

  $$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

  $$2a_2 + a_1 - 4a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$k^2 a_{k+2} + (2a_k + a_{k-1} + a_{k+1} + 3a_{k+2})\, k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+1$

  $$(k+1)^2 a_{k+3} + (2a_{k+1} + a_k + a_{k+2} + 3a_{k+3})\,(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1} k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2 + 5k + 6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve(diff(y(t),t$2)+(t^2+2*t+1)*diff(y(t),t)-(4+4*t)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^4+4t^3+6t^2+8t+5) + c_2(t^4+4t^3+6t^2+8t+5)\left(\int \frac{e^{-\frac{1}{3}t^3-t^2-t}}{(t+1)^2(t^3+3t^2+3t+5)^2}dt\right)$$

✓ Solution by Mathematica

Time used: 3.024 (sec). Leaf size: 132

```
DSolve[y''[t]+(t^2+2*t+1)*y'[t]-(4+4*t)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{36}e^{-\frac{1}{3}t(t^2+3t+3)}\left(-3c_2(t^3+3t^2+3t+4) + 3^{2/3}c_2e^{\frac{1}{3}(t+1)^3}\sqrt[3]{(t+1)^3}(t^3+3t^2+3t+5)\,\Gamma\left(\frac{2}{3},\frac{1}{3}(t+1)^3\right) + 36c_1e^{\frac{t^3}{3}+t^2+t}(t^4+4t^3+6t^2+8t+5)\right)$$

2.201 problem 204

2.201.1 Maple step by step solution 1957

Internal problem ID [7691]
Internal file name [`OUTPUT/6624_Sunday_June_05_2022_05_02_17_PM_44288758/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 204.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{2ty'' + (1-2t)\,y' - y = 0}$$

Writing the ode as

$$2ty'' + (1-2t)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1-2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2 + 4t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4t^2 + 4t - 3}{16t^2} \\
&= Q + \frac{R}{16t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{4t - 3}{16t^2}\right) \\
&= \frac{1}{4} + \frac{4t - 3}{16t^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{4}\right) - (0) \\
&= \frac{1}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4t^2 + 4t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{4t} \\
&= \frac{1}{2} + \frac{1}{4t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2 + 4t - 3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\
&= t^{\frac{1}{4}} \mathrm{e}^{\frac{t}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{1-2t}{2t} \, dt} \\
&= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{t}{2}}}{t^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^t \sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^t \sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)$$

Verified OK.

2.201.1 Maple step by step solution

Let's solve

$2ty'' + (1-2t)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2t} + \frac{(-1+2t)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(-1+2t)y'}{2t} - \frac{y}{2t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{-1+2t}{2t}, P_3(t) = -\frac{1}{2t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = \frac{1}{2}$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2ty'' + (1-2t)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot y''$ to series expansion

$$t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k- >k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(a_{k+1}(k+1+r) - a_k)\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{k+\frac{3}{2}}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k+\frac{3}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+\frac{3}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*t*diff(y(t),t$2)+(1-2*t)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t)=c_1 e^t+c_2 e^t\left(\int \frac{e^{-t}}{\sqrt{t}} dt\right)$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 21

```
DSolve[2*t*y''[t]+(1-2*t)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t \left(c_1 - c_2 \Gamma\left(\frac{1}{2}, t\right)\right)$$

2.202 problem 205

2.202.1 Maple step by step solution 1968

Internal problem ID [7692]
Internal file name [OUTPUT/6625_Sunday_June_05_2022_05_02_20_PM_30528628/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 205.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2ty'' + (t+1)\,y' - 2y = 0}$$

Writing the ode as

$$\begin{align*} 2ty'' + (t+1)\,y' - 2y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 2t \\ B &= t+1 \tag{3} \\ C &= -2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 18t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 387: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{9}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= \frac{9}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{9}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 18t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{9}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{9}{4} - \left(\frac{1}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4t} + \frac{1}{4} \\
&= \frac{t+1}{4t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right)\right) &= 0 \\
\frac{(-a_1 + 6)\, t - 2a_0 + a_1}{2t} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 3\right) e^{\int \left(\frac{1}{4t} + \frac{1}{4}\right) dt} \\
&= \left(t^2 + 6t + 3\right) e^{\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= \left(t^2 + 6t + 3\right) t^{\frac{1}{4}} e^{\frac{t}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t+1}{2t}\,dt} \\
&= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{t}{4}}}{t^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t+1}{2t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (t^2+6t+3) + c_2 \left(t^2+6t+3\left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 (t^2+6t+3) + c_2 (t^2+6t+3)\left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right) \tag{1}
$$

Verification of solutions

$$y = c_1(t^2 + 6t + 3) + c_2(t^2 + 6t + 3)\left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2 + 6t + 3)^2}dt\right)$$

Verified OK.

2.202.1 Maple step by step solution

Let's solve

$2ty'' + (t + 1)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t+1)y'}{2t} + \frac{y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t+1)y'}{2t} - \frac{y}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{t+1}{2t}, P_3(t) = -\frac{1}{t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\Big|_{t=0} = \frac{1}{2}$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2ty'' + (t + 1)\,y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k- > k+1$

  $t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+r-2)\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+1+r)\, a_{k+1} + a_k(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-2)}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(2k+1)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{6}$

- Express in terms of a_0

$a_2 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k \left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k \left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), b_{k+1} = -\frac{b_k \left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(2*t*diff(y(t),t$2)+(1+t)*diff(y(t),t)-2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^2 + 6t + 3) + c_2(t^2 + 6t + 3)\left(\int \frac{e^{-\frac{t}{2}}}{(t^2 + 6t + 3)^2 \sqrt{t}} dt\right)$$

✓ Solution by Mathematica

Time used: 10.591 (sec). Leaf size: 71

```
DSolve[2*t*y''[t]+(1+t)*y'[t]-2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{24}\left(\sqrt{2\pi}c_2(t^2 + 6t + 3)\,\mathrm{erf}\left(\frac{\sqrt{t}}{\sqrt{2}}\right) + 24c_1(t^2 + 6t + 3) + 2c_2e^{-t/2}\sqrt{t}(t + 5)\right)$$

2.203 problem 206

2.203.1 Maple step by step solution 1977

Internal problem ID [7693]
Internal file name [`OUTPUT/6626_Sunday_June_05_2022_05_02_24_PM_51459646/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 206.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2t^2y'' - ty' + (t+1)\,y = 0}$$

Writing the ode as

$$2t^2y'' - ty' + (t+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= t+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 - 8t \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3 - 8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 389: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{t-c} \\
&= \frac{1}{2}\left(\frac{1}{(t-(0))}\right) \\
&= \frac{1}{2t}
\end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2t}
\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\
&= t^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\,\sqrt{-t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2} \, dt} \\
&= z_1 e^{\frac{\ln(t)}{4}} \\
&= z_1 \left(t^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2} \, dt \\
&= y_1 \left(\frac{\sqrt{2}\,\sqrt{-t}\left(-1 + \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\right) + c_2 \left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \left(\frac{\sqrt{2}\,\sqrt{-t}\left(-1 + \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{t}\,e^{\sqrt{2}\,\sqrt{-t}} - \frac{c_2\sqrt{2}\,\sqrt{-t}\left(e^{\sqrt{2}\,\sqrt{-t}} - e^{-\sqrt{2}\,\sqrt{-t}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{t}\,e^{\sqrt{2}\,\sqrt{-t}} - \frac{c_2\sqrt{2}\,\sqrt{-t}\left(e^{\sqrt{2}\,\sqrt{-t}} - e^{-\sqrt{2}\,\sqrt{-t}}\right)}{2}$$

Verified OK.

2.203.1 Maple step by step solution

Let's solve

$2y''t^2 - ty' + (t+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2t} - \frac{(t+1)y}{2t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2t} + \frac{(t+1)y}{2t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{t+1}{2t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -\frac{1}{2}$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = \frac{1}{2}$

  ◦ $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $2y''t^2 - ty' + (t+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

    $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  ◦ Shift index using $k- >k - m$

    $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  ◦ Convert $t \cdot y'$ to series expansion

    $t \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$

  ◦ Convert $t^2 \cdot y''$ to series expansion

    $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k+r-\frac{1}{2}\right) a_k + a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2(k+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+r)(2k+1+2r)}$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+3)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(2*t^2*diff(y(t),t$2)-t*diff(y(t),t)+(1+t)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 \sin\left(\sqrt{2}\,\sqrt{t}\right)\sqrt{t} + c_2\sqrt{t}\,\cos\left(\sqrt{2}\,\sqrt{t}\right)$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 62

```
DSolve[2*t^2*y''[t]-t*y'[t]+(1+t)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-i\sqrt{2}\sqrt{t}}\sqrt{t}\left(2c_1e^{2i\sqrt{2}\sqrt{t}} + i\sqrt{2}c_2\right)$$

2.204 problem 207

2.204.1 Maple step by step solution 1988

Internal problem ID [7694]
Internal file name [OUTPUT/6627_Sunday_June_05_2022_05_02_27_PM_71602461/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 207.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2t^2y'' + (t^2 - t)\,y' + y = 0}$$

Writing the ode as

$$2t^2y'' + (t^2 - t)\,y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 391: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{1}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2-2t-3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t-3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t-3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= \frac{1}{4} - \left(\frac{1}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\
&= \frac{1}{4t} - \frac{1}{4} \\
&= -\frac{t-1}{4t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2-2t-3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\
&= t^{\frac{1}{4}} \mathrm{e}^{-\frac{t}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2 - t}{2t^2} \, dt} \\
&= z_1 e^{-\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= z_1 \left(t^{\frac{1}{4}} \mathrm{e}^{-\frac{t}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\
&= y_1 \left(-i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(-i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} - ic_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} - ic_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)$$

Verified OK.

2.204.1 Maple step by step solution

Let's solve

$2y''t^2 + (t^2 - t)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2t^2} - \frac{(t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t-1)y'}{2t} + \frac{y}{2t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{t-1}{2t}, P_3(t) = \frac{1}{2t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -\frac{1}{2}$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = \frac{1}{2}$

  ◦ $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $2y''t^2 + t(t-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r-1)\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right) = 0$$

- Shift index using $k- >k+1$

$$2(k+r)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{2k+1+2r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{2k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{2k+3}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k+3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(2*t^2*diff(y(t),t$2)+(t^2-t)*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} + c_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(\int \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}dt\right)$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 46

```
DSolve[2*t^2*y''[t]+(t^2-t)*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t/2}\left(c_2\sqrt{t} + \sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2}, -\frac{t}{2}\right)\right)$$

2.205 problem 208

2.205.1 Maple step by step solution 1999

Internal problem ID [7695]
Internal file name [OUTPUT/6628_Sunday_June_05_2022_05_02_30_PM_43243240/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 208.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2y'' + (-t^2+t)\, y' - y = 0}$$

Writing the ode as

$$t^2y'' + (-t^2+t)\, y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 393: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 2t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{-2t + 3}{4t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2}\, dt} \\
&= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^t}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2} dt}}{(y_1)^2} dt \\ &= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2} dt \\ &= y_1 \left(-(t+1) e^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^t}{t}\right) + c_2 \left(\frac{e^t}{t} \left(-(t+1) e^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^t}{t} + \frac{c_2(-t-1)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^t}{t} + \frac{c_2(-t-1)}{t}$$

Verified OK.

2.205.1 Maple step by step solution

Let's solve

$y''t^2 + (-t^2 + t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{t^2} + \frac{(t-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t-1)y'}{t} - \frac{y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t-1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = -1$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 - t(t-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- > k+1-m$

$$t^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

  $$t^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}(1+r)\,(-1+r)\, t^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_{k}(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

  $$(k+r)\,(a_{k+1}(k+2+r) - a_{k}) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_{k}}{k+2+r}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = \frac{a_{k}}{k+1}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_{k} t^{k-1}, a_{k+1} = \frac{a_{k}}{k+1}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{a_{k}}{k+3}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_{k} t^{k+1}, a_{k+1} = \frac{a_{k}}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_{k} t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_{k} t^{k+1}\right), a_{k+1} = \frac{a_{k}}{k+1}, b_{k+1} = \frac{b_{k}}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(t^2*diff(y(t),t$2)+(t-t^2)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1(t+1)}{t} + \frac{c_2 \mathrm{e}^t}{t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[t^2*y''[t]+(t-t^2)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 e^t - c_1(t+1)}{t}$$

2.206 problem 209

2.206.1 Maple step by step solution 2009

Internal problem ID [7696]
Internal file name [`OUTPUT/6629_Sunday_June_05_2022_05_02_33_PM_39514339/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 209.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{ty'' - (t^2 + 2)\, y' + yt = 0}$$

Writing the ode as

$$ty'' + (-t^2 - 2)\, y' + yt = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t^2 - 2 \\ C &= t \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 2t^2 + 8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^4 - 2t^2 + 8$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 395: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 2t^2 + 8}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{t} + \left(\frac{t}{2}\right) \\
&= -\frac{1}{t} + \frac{t}{2} \\
&= -\frac{1}{t} + \frac{t}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt} \\
&= \frac{e^{\frac{t^2}{4}}}{t}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2 - 2}{t} \, dt} \\
&= z_1 e^{\frac{t^2}{4} + \ln(t)} \\
&= z_1 \left(t \, e^{\frac{t^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2}{t}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{t^2}{2}} \left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{t^2}{2}} + c_2 \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)\mathrm{e}^{\frac{t^2}{2}}}{2} - t\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{t^2}{2}} + c_2 \left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)\mathrm{e}^{\frac{t^2}{2}}}{2} - t\right)
$$

Verified OK.

2.206.1 Maple step by step solution

Let's solve

$ty'' + (-t^2 - 2)\, y' + yt = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(t^2+2)y'}{t} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t^2+2)y'}{t} + y = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\bigg|_{t=0} = -2$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t^2 - 2)\, y' + yt = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t \cdot y$ to series expansion

  $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- Shift index using $k->k-1$

  $t\cdot y=\sum_{k=1}^{\infty} a_{k-1} t^{k+r}$
- Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

  $t^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, t^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $t^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
- Convert $t\cdot y''$ to series expansion

  $t\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r-1}$
- Shift index using $k->k+1$

  $t\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-3+r)\, t^{-1+r}+a_{1}(1+r)\,(-2+r)\, t^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r)-a_{k-1}(k-2+r)\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-3+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,3\}$
- Each term must be 0

  $a_{1}(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $(k-2+r)\,(a_{k+1}(k+r+1)-a_{k-1})=0$
- Shift index using $k->k+1$

  $(k+r-1)\,(a_{k+2}(k+2+r)-a_{k})=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}}{k+2+r}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_{k}}{k+2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{a_k}{k+5}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{k+5}, 4b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve(t*diff(y(t),t$2)-(t^2+2)*diff(y(t),t)+t*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{\frac{t^2}{2}} + \frac{c_2 e^{\frac{t^2}{2}} \left(-\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,t}{2}\right) + 2t\,e^{-\frac{t^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 52

```
DSolve[t*y''[t]-(t^2+2)*y'[t]+t*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{\frac{\pi}{2}} c_2 e^{\frac{t^2}{2}} \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + c_1 e^{\frac{t^2}{2}} - c_2 t$$

2.207 problem 210

2.207.1 Maple step by step solution 2020

Internal problem ID [7697]
Internal file name [`OUTPUT/6630_Sunday_June_05_2022_05_02_36_PM_12955923/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 210.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2 y'' + t(t+1)\, y' - y = 0}$$

Writing the ode as

$$t^2 y'' + (t^2 + t)\, y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 397: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 + 2t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{2t + 3}{4t^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2} - \frac{1}{2t} \\
&= -\frac{t+1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2+2t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2} \, dt} \\
&= z_1 e^{-\frac{t}{2} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-t}}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left((t-1)\,\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-t}}{t}\right) + c_2 \left(\frac{\mathrm{e}^{-t}}{t} \left((t-1)\,\mathrm{e}^{t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-t}}{t} + \frac{c_2 (t-1)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-t}}{t} + \frac{c_2 (t-1)}{t}$$

Verified OK.

2.207.1 Maple step by step solution

Let's solve

$y''t^2 + (t^2 + t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{t^2} - \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t+1)y'}{t} - \frac{y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{t+1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = -1$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 + t(t+1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r+1) + a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 22

```
dsolve(t^2*diff(y(t),t$2)+t*(t+1)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1(t-1)}{t} + \frac{c_2 e^{-t}}{t}$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 26

```
DSolve[t^2*y''[t]+t*(t+1)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-t}(c_1 e^t(t-1) + c_2)}{t}$$

2.208 problem 211

2.208.1 Maple step by step solution 2030

Internal problem ID [7698]
Internal file name [OUTPUT/6631_Sunday_June_05_2022_05_02_39_PM_79419308/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 211.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[_Laguerre]
```

$$\boxed{ty'' - (t+4)\,y' + 2y = 0}$$

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 24$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 + 24}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\
&= \frac{1}{4} + \frac{6}{t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 24}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{2}{t} - \frac{1}{2} \\
&= -\frac{t+4}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2 + 24}{4t^2}\right)\right) &= 0 \\
\frac{(a_1 - 6)\, t + 2a_0 - 4a_1}{t} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 12, a_1 = 6\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^2 + 6t + 12
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 12\right) \mathrm{e}^{\int \left(-\frac{2}{t} - \frac{1}{2}\right) dt} \\
&= \left(t^2 + 6t + 12\right) \mathrm{e}^{-\frac{t}{2} - 2\ln(t)} \\
&= \frac{\left(t^2 + 6t + 12\right) \mathrm{e}^{-\frac{t}{2}}}{t^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\,dt} \\
&= z_1 e^{\frac{t}{2}+2\ln(t)} \\
&= z_1 \left(t^2 \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 12
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+4\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{(t^2-6t+12)\,\mathrm{e}^t}{t^2+6t+12}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2+6t+12\right) + c_2 \left(t^2+6t+12 \left(\frac{(t^2-6t+12)\,\mathrm{e}^t}{t^2+6t+12}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(t^2+6t+12\right) + c_2 \left(t^2-6t+12\right)\mathrm{e}^t \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(t^2+6t+12\right) + c_2 \left(t^2-6t+12\right)\mathrm{e}^t
$$

Verified OK.

2.208.1 Maple step by step solution

Let's solve

$ty'' + (-4-t)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{t} + \frac{(t+4)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+4)y'}{t} + \frac{2y}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{t+4}{t}, P_3(t) = \frac{2}{t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -4$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-4-t)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot y''$ to series expansion

$$t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k->k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-5+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-4+r) - a_k(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-5+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-4+r) - a_k(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-2)}{(k+1+r)(k-4+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_k(k-2)}{(k+1)(k-4)}$$

- Apply recursion relation for $k = 0$

$$a_1 = \frac{a_0}{2}$$

- Apply recursion relation for $k = 1$

$$a_2 = \frac{a_1}{6}$$

- Express in terms of a_0

$$a_2 = \frac{a_0}{12}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right)$$

- Recursion relation for $r = 5$

$$a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}$$

- Solution for $r = 5$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+5}\right), b_{k+1} = \frac{b_k(k+3)}{(k+6)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(t*diff(y(t),t$2)-(4+t)*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^2 + 6t + 12) + c_2 e^t (t^2 - 6t + 12)$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 85

```
DSolve[t*y''[t]-(4+t)*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2e^{t/2}\sqrt{t}\left((c_2t^2 - 6ic_1t + 12c_2)\cosh\left(\frac{t}{2}\right) + i(c_1(t^2 + 12) + 6ic_2t)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

2.209 problem 212

2.209.1 Maple step by step solution 2040

Internal problem ID [7699]
Internal file name [OUTPUT/6632_Sunday_June_05_2022_05_02_42_PM_56203535/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 212.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2y'' + (t^2 - 3t)\, y' + 3y = 0}$$

Writing the ode as

$$t^2y'' + (t^2 - 3t)\, y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 6t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 6t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{-6t + 3}{4t^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 6t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= \frac{3}{2} - \left(\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty}\\ &= \frac{3}{2t} + (-)\left(\frac{1}{2}\right)\\ &= \frac{3}{2t} - \frac{1}{2}\\ &= -\frac{t-3}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{3}{2t} - \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2t^2}\right) + \left(\frac{3}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2-6t+3}{4t^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt}\\ &= e^{\int \left(\frac{3}{2t} - \frac{1}{2}\right) dt}\\ &= t^{\frac{3}{2}} e^{-\frac{t}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{t^2-3t}{t^2}\, dt}\\ &= z_1 e^{-\frac{t}{2} + \frac{3\ln(t)}{2}}\\ &= z_1\left(t^{\frac{3}{2}} e^{-\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = t^3 \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-t)\, t^2 - (t+1)\,\mathrm{e}^{t}}{2t^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^3 \mathrm{e}^{-t}\right) + c_2 \left(t^3 \mathrm{e}^{-t} \left(\frac{-\operatorname{expIntegral}_1(-t)\, t^2 - (t+1)\,\mathrm{e}^{t}}{2t^2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 t^3 \mathrm{e}^{-t} - \frac{c_2 t\left(\operatorname{expIntegral}_1(-t)\, t^2 \mathrm{e}^{-t} + t + 1\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 t^3 \mathrm{e}^{-t} - \frac{c_2 t\left(\operatorname{expIntegral}_1(-t)\, t^2 \mathrm{e}^{-t} + t + 1\right)}{2}$$

Verified OK.

2.209.1 Maple step by step solution

Let's solve

$y''t^2 + (t^2 - 3t)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{t^2} - \frac{(t-3)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t-3)y'}{t} + \frac{3y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = \frac{t-3}{t}, P_3(t) = \frac{3}{t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -3$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 3$

  - $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 + t(t-3)\, y' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- ○ Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-3+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-3) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-3) + a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k-2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k-2+r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k-1}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k}{k-1}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(t^2*diff(y(t),t$2)+(t^2-3*t)*diff(y(t),t)+3*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t^3 \mathrm{e}^{-t} + \frac{c_2 t\, \mathrm{e}^{-t} \left(\mathrm{expIntegral}_1\left(-t\right) t^2 + \mathrm{e}^t t + \mathrm{e}^t\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 41

```
DSolve[t^2*y''[t]+(t^2-3*t)*y'[t]+3*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} e^{-t} \left(c_1 t^3 \,\mathrm{ExpIntegralEi}(t) + 2 c_2 t^3 - c_1 e^t (t+1) t\right)$$

2.210 problem 213

2.210.1 Maple step by step solution 2049

Internal problem ID [7700]
Internal file name [OUTPUT/6633_Sunday_June_05_2022_05_02_45_PM_84046727/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 213.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + ty' + 2y = 0}$$

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t-8}{4t} \tag{6}$$

Comparing the above to (5) shows that

$$s = t - 8$$
$$t = 4t$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t-8}{4t}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t-8}{4t} \\
&= Q + \frac{R}{4t} \\
&= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\
&= \frac{1}{4} - \frac{2}{t}
\end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t-8}{4t}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{t} - \frac{1}{2} \\
&= \frac{1}{t} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right)\right) = 0$$
$$\frac{2 + a_0}{t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= (t-2)\, \mathrm{e}^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\ &= (t-2)\, \mathrm{e}^{-\frac{t}{2} + \ln(t)} \\ &= (t-2)\, t\, \mathrm{e}^{-\frac{t}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{t}{t} \, dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (t-2)\, t\, \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-t)\,t^2 - \mathrm{e}^t t + 2\operatorname{expIntegral}_1(-t)\,t + \mathrm{e}^t}{2t\,(t-2)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((t-2)\,t\,\mathrm{e}^{-t}\right) \\
&\quad + c_2\left((t-2)\,t\,\mathrm{e}^{-t}\left(\frac{-\operatorname{expIntegral}_1(-t)\,t^2 - \mathrm{e}^t t + 2\operatorname{expIntegral}_1(-t)\,t + \mathrm{e}^t}{2t\,(t-2)}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(t-2)\,t\,\mathrm{e}^{-t} + c_2\left(-\frac{(t-2)\,t\,\mathrm{e}^{-t}\operatorname{expIntegral}_1(-t)}{2} - \frac{t}{2} + \frac{1}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1(t-2)\,t\,\mathrm{e}^{-t} + c_2\left(-\frac{(t-2)\,t\,\mathrm{e}^{-t}\operatorname{expIntegral}_1(-t)}{2} - \frac{t}{2} + \frac{1}{2}\right)$$

Verified OK.

2.210.1 Maple step by step solution

Let's solve

$ty'' + ty' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{2y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{2y}{t}=0$

☐ Check to see if $t_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(t)=1, P_3(t)=\frac{2}{t}\right]$

  - $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=0$

  - $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

  - $t=0$is a regular singular point

    Check to see if $t_0=0$ is a regular singular point

    $t_0=0$

- Multiply by denominators

  $ty''+ty'+2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $t\cdot y'$ to series expansion

    $t\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$

  - Convert $t\cdot y''$ to series expansion

    $t\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

  - Shift index using $k->k+1$

    $t\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(-1+r)\, t^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r)+a_k(k+r+2)\right) t^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k(k+3)}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k(k+2)}{(k+1)k}, b_{k+1} = -\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(t*diff(y(t),t$2)+t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{-t}(t-2)\,t + \frac{c_2(\text{expIntegral}_1\,(-t)\,t^2 + e^t t - 2\ \text{expIntegral}_1\,(-t)\,t - e^t)\,e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 51

```
DSolve[t*y''[t]+t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(c_2(t-2)t\,\text{ExpIntegralEi}(t) + 2c_1t^2 - t\left(c_2e^t + 4c_1\right) + c_2e^t\right)$$

2.211 problem 214

2.211.1 Maple step by step solution 2060

Internal problem ID [7701]
Internal file name [OUTPUT/6634_Sunday_June_05_2022_05_02_48_PM_11663268/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 214.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + (-t^2 + 1)\, y' + 4yt = 0}$$

Writing the ode as

$$\begin{align*} ty'' + (-t^2 + 1)\, y' + 4yt &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= t\\ B &= -t^2 + 1 \tag{3}\\ C &= 4t \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 20t^2 - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\
&= \frac{t}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1}=t^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4-20t^2-1}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{t^2}{4}-5\right) + \left(-\frac{1}{4t^2}\right) \\
&= \frac{t^2}{4} - 5 - \frac{1}{4t^2}
\end{aligned}$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$\begin{aligned}
b &= (-5) - (0) \\
&= -5
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}}-1\right) &&= -\frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}}-1\right) &&= \frac{9}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 20t^2 - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{9}{2} - \left(\frac{1}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\
&= \frac{1}{2t} - \frac{t}{2} \\
&= \frac{1}{2t} - \frac{t}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12t^2 + 6ta_3 + 2a_2\right) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)\left(4t^3 + 3a_3t^2 + 2a_2t + a_1\right) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.$$

$$\frac{t^4 a_3 + 2(8 + a_2)\, t^3 + 3(a_1 + 3a_3)\, t^2 + 4(a_0 + a_2)\, t}{t}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 - 8t^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\ &= \left(t^4 - 8t^2 + 8\right) \mathrm{e}^{\frac{\ln(t)}{2} - \frac{t^2}{4}} \\ &= \left(t^4 - 8t^2 + 8\right) \sqrt{t}\, \mathrm{e}^{-\frac{t^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-t^2+1}{t} \, dt} \\ &= z_1 e^{-\frac{\ln(t)}{2} + \frac{t^2}{4}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{t^2}{4}}}{\sqrt{t}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t^4 - 8t^2 + 8$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\int \frac{e^{\frac{t^2}{2}}}{t\,(t^4 - 8t^2 + 8)^2} dt\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8 \left(\int \frac{e^{\frac{t^2}{2}}}{t\,(t^4 - 8t^2 + 8)^2} dt\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8\right) \left(\int \frac{e^{\frac{t^2}{2}}}{t\,(t^4 - 8t^2 + 8)^2} dt\right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8\right) \left(\int \frac{e^{\frac{t^2}{2}}}{t\,(t^4 - 8t^2 + 8)^2} dt\right)$$

Verified OK.

2.211.1 Maple step by step solution

Let's solve

$ty'' + (-t^2 + 1)\, y' + 4yt = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(t^2-1)y'}{t} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t^2-1)y'}{t} + 4y = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{t^2-1}{t}, P_3(t) = 4\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $(t \cdot P_2(t))\bigg|_{t=0} = 1$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

- ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t^2 + 1)\, y' + 4yt = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $t \cdot y$ to series expansion

  $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- Shift index using $k->k-1$

  $t\cdot y=\sum_{k=1}^{\infty} a_{k-1} t^{k+r}$

- Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

  $t^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t\cdot y''$ to series expansion

  $t\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

  $t\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r^{2}t^{-1+r}+a_{1}(1+r)^{2}\, t^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)^{2}-a_{k-1}(k-5+r)\right) t^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r^{2}=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term must be 0

  $a_{1}(1+r)^{2}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^{2}-a_{k-1}(k-5)=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2)^{2}-a_{k}(k-4)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(k-4)}{(k+2)^{2}}$

- Recursion relation for $r=0$; series terminates at $k=4$

$a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(t*diff(y(t),t$2)+(1-t^2)*diff(y(t),t)+4*t*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^4 - 8t^2 + 8) + c_2(t^4 - 8t^2 + 8)\left(\int \frac{e^{\frac{t^2}{2}}}{(t^4 - 8t^2 + 8)^2 t} dt\right)$$

✓ Solution by Mathematica

Time used: 0.42 (sec). Leaf size: 61

```
DSolve[t*y''[t]+(1-t^2)*y'[t]+4*t*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{128} c_2\left((t^4 - 8t^2 + 8)\,\text{ExpIntegralEi}\left(\frac{t^2}{2}\right) - 2e^{\frac{t^2}{2}}(t^2 - 6)\right) + c_1(t^4 - 8t^2 + 8)$$

2.212 problem 215

2.212.1 Maple step by step solution 2070

Internal problem ID [7702]
Internal file name [OUTPUT/6635_Sunday_June_05_2022_05_02_52_PM_57968365/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 215.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2 y'' - t(t+1)\, y' + y = 0}$$

Writing the ode as

$$t^2 y'' + \left(-t^2 - t\right) y' + y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 2t - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2} \right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 2t - 1}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-1 + 2t}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-1 + 2t}{4t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 2t - 1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2t} \\
&= \frac{t+1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2+2t-1}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\
&= \sqrt{t}\, \mathrm{e}^{\frac{t}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2-t}{t^2} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^t t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2} dt}}{(y_1)^2} dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2} dt \\ &= y_1(-\operatorname{expIntegral}_1(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t t\right) + c_2 \left(\mathrm{e}^t t(-\operatorname{expIntegral}_1(t))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t t - c_2 \mathrm{e}^t t \operatorname{expIntegral}_1(t) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t t - c_2 \mathrm{e}^t t \operatorname{expIntegral}_1(t)$$

Verified OK.

2.212.1 Maple step by step solution

Let's solve

$y''t^2 + (-t^2 - t) y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t^2} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y}{t^2} - \frac{(t+1)y'}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

$\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t^2}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

$\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 1$

- $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$y''t^2 - t(t+1)\, y' + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

$t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t^2 \cdot y''$ to series expansion

$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation
  $r = 1$
- Each term in the series must be 0, giving the recursion relation
  $(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$
- Shift index using $k- > k+1$
  $(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$
  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$
  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(t^2*diff(y(t),t$2)-t*(1+t)*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = e^t c_1 t + c_2 e^t t \operatorname{expIntegral}_1(t)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[t^2*y''[t]-t*(1+t)*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t t(c_1 \operatorname{ExpIntegralEi}(-t) + c_2)$$

2.213 problem 216

2.213.1 Maple step by step solution 2077

Internal problem ID [7703]
Internal file name [OUTPUT/6636_Sunday_June_05_2022_05_02_55_PM_78335772/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 216.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + \left(4x^2 + 6\right) y = 0}$$

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 6\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1\left(\mathrm{e}^{-x^2}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)\,\mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(2x)\, \mathrm{e}^{-x^2}\right) + c_2 \left(\cos(2x)\, \mathrm{e}^{-x^2} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2x)\, \mathrm{e}^{-x^2} + \frac{c_2 \sin(2x)\, \mathrm{e}^{-x^2}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos(2x)\, \mathrm{e}^{-x^2} + \frac{c_2 \sin(2x)\, \mathrm{e}^{-x^2}}{2}
$$

Verified OK.

2.213.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + 6a_0 + (6a_3 + 10a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+3) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 6a_k + 4a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 6a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 7a_{k+2})}{k^2 + 7k + 12}, a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} \cos(2x) + c_2 \mathrm{e}^{-x^2} \sin(2x)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 37

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-x(x+2i)} \left(4c_1 - ic_2 e^{4ix}\right)$$

2.214 problem 217

2.214.1 Maple step by step solution 2086

Internal problem ID [7704]
Internal file name [OUTPUT/6637_Sunday_June_05_2022_05_02_57_PM_14701786/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 217.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-z^2+1)\,y''-3zy'+\lambda y=0}$$

Writing the ode as

$$(-z^2+1)\,y''-3zy'+\lambda y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-z^2+1\\B&=-3z\\C&=\lambda\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4\lambda\, z^2 + 3z^2 - 4\lambda - 6}{4\left(z^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4\lambda\, z^2 + 3z^2 - 4\lambda - 6$$
$$t = 4\left(z^2 - 1\right)^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4\lambda\, z^2 + 3z^2 - 4\lambda - 6}{4\left(z^2 - 1\right)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2-1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(z-1\right)^2} + \frac{\frac{9}{16}+\frac{\lambda}{2}}{z-1} - \frac{3}{16\left(z+1\right)^2} + \frac{-\frac{9}{16}-\frac{\lambda}{2}}{z+1}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4\lambda\, z^2 + 3z^2 - 4\lambda - 6}{4\left(z^2-1\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{z-c} \\ &= \frac{1}{2}\left(\frac{1}{(z-(1))} + \frac{1}{(z-(-1))}\right) \\ &= \frac{1}{2z-2} + \frac{1}{2z+2} \end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z-2} + \frac{1}{2z+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right)w + \frac{-4\lambda\, z^2 - 3z^2 + 4\lambda + 4}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= e^{\int \omega\, dz} \\ &= \mathrm{e}^{\int \frac{z+2\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}}{2(z-1)(z+1)} dz} \\ &= \left(z^2-1\right)^{\frac{1}{4}} \left(\frac{\sqrt{\left(z^2-1\right)\left(\lambda+1\right)}\,\sqrt{\lambda+1} + \lambda z + z}{\sqrt{\lambda+1}}\right)^{\sqrt{\lambda+1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1}\, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1 \left(\frac{1}{\left(z-1\right)^{\frac{3}{4}}\left(z+1\right)^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(z^2-1\right)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{\left(z-1\right)^{\frac{3}{4}}\left(z+1\right)^{\frac{3}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\, dz}}{(y_1)^2}\, dz \\ &= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2}-\frac{3\ln(z+1)}{2}}}{(y_1)^2}\, dz \\ &= y_1 \left(-\frac{\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{-2\sqrt{\lambda+1}}}{2\sqrt{\lambda+1}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}}\right) \\ &\quad + c_2 \left(\frac{(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}}\left(-\frac{\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{-2\sqrt{\lambda+1}}}{2\sqrt{\lambda+1}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{c_1(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}} \\ & - \frac{c_2(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{-\sqrt{\lambda+1}}}{2(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}\sqrt{\lambda+1}} \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{\sqrt{\lambda+1}}}{(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}} - \frac{c_2(z^2-1)^{\frac{1}{4}}\left(\sqrt{\lambda+1}\left(z+\sqrt{z^2-1}\right)\right)^{-\sqrt{\lambda+1}}}{2(z-1)^{\frac{3}{4}}(z+1)^{\frac{3}{4}}\sqrt{\lambda+1}}$$

Verified OK.

2.214.1 Maple step by step solution

Let's solve

$(-z^2+1)\, y'' - 3zy' + \lambda y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3zy'}{z^2-1} + \frac{\lambda y}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3zy'}{z^2-1} - \frac{\lambda y}{z^2-1} = 0$

□ Check to see if z_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = \frac{3z}{z^2-1}, P_3(z) = -\frac{\lambda}{z^2-1}\right]$

  ◦ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

  $\left((z+1)\cdot P_2(z)\right)\Big|_{z=-1} = \frac{3}{2}$

  ◦ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

  $\left((z+1)^2\cdot P_3(z)\right)\Big|_{z=-1} = 0$

  ◦ $z=-1$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0 = -1$

- Multiply by denominators

  $y''(z^2-1) + 3zy' - \lambda y = 0$

- Change variables using $z = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u-3)\left(\frac{d}{du}y(u)\right) - \lambda y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(2k+3+2r) + a_k(k^2+2kr+r^2+2k-\lambda+2r)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} + a_k\left(k^2+(2r+2)\, k+r^2+2r-\lambda\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k^2+2kr+r^2+2k-\lambda+2r)}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(2k+3)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(2k+3)(k+1)}\right]$$

- Revert the change of variables $u = z+1$

$$\left[y = \sum_{k=0}^{\infty} a_k(z+1)^k, a_{k+1} = \frac{a_k(k^2+2k-\lambda)}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k \left(k^2+k-\lambda-\frac{3}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k^2+k-\lambda-\frac{3}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u = z+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (z+1)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k^2+k-\lambda-\frac{3}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (z+1)^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k \left(k^2+2k-\lambda\right)}{(2k+3)(k+1)}, b_{k+1} = \frac{b_k \left(k^2+k-\lambda-\frac{3}{4}\right)}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve((1-z^2)*diff(y(z),z$2)-3*z*diff(y(z),z)+lambda*y(z)=0,y(z), singsol=all)
```

$$y(z) = \frac{c_1 \left(z+\sqrt{z^2-1}\right)^{\sqrt{\lambda+1}}}{\sqrt{z^2-1}} + \frac{c_2 \left(z+\sqrt{z^2-1}\right)^{-\sqrt{\lambda+1}}}{\sqrt{z^2-1}}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 54

```
DSolve[(1-z^2)*y''[z]-3*z*y'[z]+\[Lambda]*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to \frac{c_1 P_{\sqrt{\lambda+1}-\frac{1}{2}}^{\frac{1}{2}}(z) + c_2 Q_{\sqrt{\lambda+1}-\frac{1}{2}}^{\frac{1}{2}}(z)}{\sqrt[4]{z^2-1}}$$

2.215 problem 218

2.215.1 Maple step by step solution 2097

Internal problem ID [7705]
Internal file name [`OUTPUT/6638_Sunday_June_05_2022_05_03_00_PM_34796928/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 218.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4zy'' + 2(1-z)\,y' - y = 0}$$

Writing the ode as

$$4zy'' + (-2z+2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z \\ B &= -2z+2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = z^2 + 2z - 3$$
$$t = 16z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{8z} - \frac{3}{16z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\
&= \frac{1}{4}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}
$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{z^2 + 2z - 3}{16z^2} \\
&= Q + \frac{R}{16z^2} \\
&= \left(\frac{1}{16}\right) + \left(\frac{2z - 3}{16z^2}\right) \\
&= \frac{1}{16} + \frac{2z - 3}{16z^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{8}\right) - (0) \\
&= \frac{1}{8}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{z^2 + 2z - 3}{16z^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{z - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4z} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4} + \frac{1}{4z} \\
&= \frac{z+1}{4z}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(z) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2 + 2z - 3}{16z^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \mathrm{e}^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\
&= z^{\frac{1}{4}} \mathrm{e}^{\frac{z}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dz} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2z+2}{4z} \, dz} \\
&= z_1 e^{\frac{z}{4} - \frac{\ln(z)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{z}{4}}}{z^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\, dz}}{(y_1)^2}\, dz \\
&= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\, dz \\
&= y_1 \left(\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{z}{2}}\right) + c_2\left(\mathrm{e}^{\frac{z}{2}}\left(\sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{z}{2}} + c_2 \mathrm{e}^{\frac{z}{2}} \sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{z}{2}} + c_2 \mathrm{e}^{\frac{z}{2}} \sqrt{\pi}\,\sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\,\sqrt{z}}{2}\right)$$

Verified OK.

2.215.1 Maple step by step solution

Let's solve

$4zy'' + (-2z+2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{4z} + \frac{(z-1)y'}{2z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(z-1)y'}{2z} - \frac{y}{4z} = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

    $(z \cdot P_2(z))\Big|_{z=0} = \frac{1}{2}$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

    $(z^2 \cdot P_3(z))\Big|_{z=0} = 0$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $4zy'' + (-2z+2)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

    $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,z^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$$

- Convert $z \cdot y''$ to series expansion

$$z \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$$

- Shift index using $k- >k+1$

$$z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(a_{k+1}(k+1+r) - \frac{a_k}{2}\right)\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{2(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{2(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k}{2(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{2(k+1)}, b_{k+1}=\frac{b_k}{2(k+\frac{3}{2})}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(4*z*diff(y(z),z$2)+2*(1-z)*diff(y(z),z)-y(z)=0,y(z), singsol=all)
```

$$y(z)=c_1 e^{\frac{z}{2}}+c_2 e^{\frac{z}{2}}\left(\int \frac{e^{-\frac{z}{2}}}{\sqrt{z}} dz\right)$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 34

```
DSolve[4*z*y''[z]+2*(1-z)*y'[z]-y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z)\to e^{z/2}\left(c_1-\sqrt{2}c_2\Gamma\left(\frac{1}{2},\frac{z}{2}\right)\right)$$

2.216 problem 219

2.216.1 Maple step by step solution 2107

Internal problem ID [7706]
Internal file name [OUTPUT/6639_Sunday_June_05_2022_05_03_04_PM_86317946/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 219.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{f'' + 2(z-1) f' + 4f = 0}$$

Writing the ode as

$$f'' + (2z-2) f' + 4f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2z-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = fe^{\int \frac{B}{2A} dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 - 2z - 2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = z^2 - 2z - 2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(z) = (z^2 - 2z - 2)\, z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$f = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{1} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\ &= z - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{z^2 - 2z - 2}{1} \\
&= Q + \frac{R}{1} \\
&= (z^2 - 2z - 2) + (0) \\
&= z^2 - 2z - 2
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$
\begin{aligned}
b &= (-2) - (1) \\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= z - 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-3}{1} - 1\right) = -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-3}{1} - 1\right) = 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = z^2 - 2z - 2
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z - 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)(z - 1) \\
&= 1 - z \\
&= 1 - z
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(1 - z)(1) + \left((-1) + (1 - z)^2 - \left(z^2 - 2z - 2\right)\right) &= 0 \\
2 + 2a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z - 1) \, \mathrm{e}^{\int (1 - z) dz} \\
&= (z - 1) \, \mathrm{e}^{z - \frac{1}{2} z^2} \\
&= (z - 1) \, \mathrm{e}^{-\frac{z(z-2)}{2}}
\end{aligned}$$

The first solution to the original ode in f is found from

$$
\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\,dz} \\
&= z_1 e^{z-\frac{1}{2}z^2} \\
&= z_1\left(\mathrm{e}^{-\frac{z(z-2)}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
f_1 = (z-1)\,\mathrm{e}^{-z(z-2)}
$$

The second solution f_2 to the original ode is found using reduction of order

$$
f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{f_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\,dz}}{(f_1)^2}\,dz \\
&= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\,dz \\
&= f_1\left(\frac{-i\sqrt{\pi}\,(z-1)\,\mathrm{e}^{-1}\,\mathrm{erf}\,(i(z-1)) - \mathrm{e}^{z(z-2)}}{z-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1\left((z-1)\,\mathrm{e}^{-z(z-2)}\right) + c_2\left((z-1)\,\mathrm{e}^{-z(z-2)}\left(\frac{-i\sqrt{\pi}\,(z-1)\,\mathrm{e}^{-1}\,\mathrm{erf}\,(i(z-1)) - \mathrm{e}^{z(z-2)}}{z-1}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
f = c_1(z-1)\,\mathrm{e}^{-z(z-2)} + c_2\left(-1 - i(z-1)\sqrt{\pi}\ \mathrm{erf}\,(i(z-1))\,\mathrm{e}^{-(z-1)^2}\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
f = c_1(z-1)\,\mathrm{e}^{-z(z-2)} + c_2\left(-1 - i(z-1)\sqrt{\pi}\ \mathrm{erf}\,(i(z-1))\,\mathrm{e}^{-(z-1)^2}\right)
$$

Verified OK.

2.216.1 Maple step by step solution

Let's solve

$f'' + (2z - 2) f' + 4f = 0$

- Highest derivative means the order of the ODE is 2

f''

- Assume series solution for f

$f = \sum_{k=0}^{\infty} a_k z^k$

□ Rewrite DE with series expansions

◦ Convert $z^m \cdot f'$ to series expansion for $m = 0..1$

$z^m \cdot f' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, z^{k-1+m}$

◦ Shift index using $k- >k+1-m$

$z^m \cdot f' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, z^k$

◦ Convert f'' to series expansion

$f'' = \sum_{k=2}^{\infty} a_k k(k-1)\, z^{k-2}$

◦ Shift index using $k- >k+2$

$f'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, z^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)\right) z^k = 0$

- Each term in the series must be 0, giving the recursion relation

$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2})\, k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

$\left[f = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(a_k k - a_{k+1} k + 2a_k - a_{k+1})}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(diff(f(z),z$2)+2*(z-1)*diff(f(z),z)+4*f(z)=0,f(z), singsol=all)
```

$$f(z) = c_1 e^{-z^2+2z}(z-1) + c_2 e^{-z^2+2z}\left(-e^{-1}\sqrt{\pi}\,\operatorname{erf}(iz-i)\,z + e^{-1}\sqrt{\pi}\,\operatorname{erf}(iz-i) + i e^{z^2-2z}\right)$$

✓ Solution by Mathematica

Time used: 0.176 (sec). Leaf size: 72

```
DSolve[f''[z]+2*(z-a)*f'[z]+4*f[z]==0,f[z],z,IncludeSingularSolutions -> True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi}c_2\sqrt{(a-z)^2}\operatorname{erfi}\left(\sqrt{(a-z)^2}\right) + c_2 e^{(a-z)^2} - 2ac_1 + 2c_1 z\right)$$

2.217 problem 220

2.217.1 Maple step by step solution 2115

Internal problem ID [7707]
Internal file name [`OUTPUT/6640_Sunday_June_05_2022_05_03_07_PM_4284865/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 220.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{zy'' - 2y' + zy = 0}$$

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A} dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -z^2 + 2$$
$$t = z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\
&= i
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-z^2 + 2}{z^2} \\
&= Q + \frac{R}{z^2} \\
&= (-1) + \left(\frac{2}{z^2}\right) \\
&= -1 + \frac{2}{z^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2}{z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{z} + (-)\,(i) \\ &= -\frac{1}{z} - i \\ &= -\frac{1}{z} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2 + 2}{z^2}\right)\right) = 0$$
$$\frac{2ia_0 - 2}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z - i)\, \mathrm{e}^{\int \left(-\frac{1}{z} - i\right) dz} \\
&= (z - i)\, \mathrm{e}^{-iz - \ln(z)} \\
&= \frac{(z - i)\, \mathrm{e}^{-iz}}{z}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dz} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{z} \, dz} \\
&= z_1 e^{\ln(z)} \\
&= z_1(z)
\end{aligned}$$

Which simplifies to

$$y_1 = (z - i)\, \mathrm{e}^{-iz}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\, dz}}{(y_1)^2}\, dz \\ &= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\, dz \\ &= y_1 \left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((z-i)\,\mathrm{e}^{-iz}\right) + c_2\left((z-i)\,\mathrm{e}^{-iz}\left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(z-i)\,\mathrm{e}^{-iz} - \frac{c_2(iz-1)\,\mathrm{e}^{iz}}{2} \tag{1}$$

Verification of solutions

$$y = c_1(z-i)\,\mathrm{e}^{-iz} - \frac{c_2(iz-1)\,\mathrm{e}^{iz}}{2}$$

Verified OK.

2.217.1 Maple step by step solution

Let's solve

$zy'' - 2y' + zy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2y'}{z} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{z} + y = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $(z \cdot P_2(z))\bigg|_{z=0} = -2$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $(z^2 \cdot P_3(z))\bigg|_{z=0} = 0$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $zy'' - 2y' + zy = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $z \cdot y$ to series expansion

  $z \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

  ◦ Shift index using $k- > k-1$

  $z \cdot y = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

  ◦ Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r) z^{k+r-1}$

  ◦ Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$$

- Convert $z \cdot y''$ to series expansion

$$z \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$$

- Shift index using $k->k+1$

$$z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2} = -\frac{a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k z^k\right) + \left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0, b_{k+2} = -\frac{b_k}{(k+5)(k+2)}, 4b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(z*diff(y(z),z$2)-2*diff(y(z),z)+z*y(z)=0,y(z), singsol=all)
```

$$y(z) = c_1(\cos(z) z - \sin(z)) + c_2(\cos(z) + \sin(z) z)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 39

```
DSolve[z*y''[z]-2*y'[z]+z*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to -\sqrt{\frac{2}{\pi}}((c_1 z + c_2)\cos(z) + (c_2 z - c_1)\sin(z))$$

2.218 problem 221

2.218.1 Maple step by step solution 2126

Internal problem ID [7708]
Internal file name [`OUTPUT/6641_Sunday_June_05_2022_05_03_10_PM_91979529/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 221.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{zy'' + (2z-3)\,y' + \frac{4y}{z} = 0}$$

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4z^2 - 12z - 1 \\ t &= 4z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 419: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{3}{z} - \frac{1}{4z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i
\end{aligned}
\tag{8}
$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4z^2 - 12z - 1}{4z^2} \\ &= Q + \frac{R}{4z^2} \\ &= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\ &= 1 + \frac{-12z - 1}{4z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be

found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 0\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 0\right) &&= \frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4z^2 - 12z - 1}{4z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{z - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2z} + (-)\,(1) \\
&= \frac{1}{2z} - 1 \\
&= \frac{1}{2z} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(z) = z + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right)\right) = 0
$$

$$
\frac{1 + 2a_0}{z} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{2}\right\}
$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$
p(z) = z - \frac{1}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \left(z - \frac{1}{2}\right) \mathrm{e}^{\int \left(\frac{1}{2z} - 1\right) dz} \\
&= \left(z - \frac{1}{2}\right) \mathrm{e}^{-z + \frac{\ln(z)}{2}} \\
&= \frac{(-1 + 2z)\sqrt{z}\,\mathrm{e}^{-z}}{2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2z-3}{z}\,dz} \\ &= z_1 e^{-z+\frac{3\ln(z)}{2}} \\ &= z_1\left(z^{\frac{3}{2}}\mathrm{e}^{-z}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z}\,dz}}{(y_1)^2}\,dz \\ &= y_1 \int \frac{e^{-2z+3\ln(z)}}{(y_1)^2}\,dz \\ &= y_1\left(\frac{(-8z+4)\,\mathrm{expIntegral}_1(-2z)-4\,\mathrm{e}^{2z}}{-1+2z}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}\right) + c_2\left(\frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}\left(\frac{(-8z+4)\,\mathrm{expIntegral}_1(-2z)-4\,\mathrm{e}^{2z}}{-1+2z}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(-1+2z)\,z^2\mathrm{e}^{-2z}}{2} - 4c_2\left(\frac{1}{2}+\left(z-\frac{1}{2}\right)\mathrm{expIntegral}_1(-2z)\,\mathrm{e}^{-2z}\right)z^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1(-1+2z)\, z^2 e^{-2z}}{2} - 4c_2\left(\frac{1}{2} + \left(z - \frac{1}{2}\right) \mathrm{expIntegral}_1\left(-2z\right) e^{-2z}\right) z^2$$

Verified OK.

2.218.1 Maple step by step solution

Let's solve

$zy'' + (2z-3)\, y' + \frac{4y}{z} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{z^2} - \frac{(2z-3)y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2z-3)y'}{z} + \frac{4y}{z^2} = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(z) = \frac{2z-3}{z}, P_3(z) = \frac{4}{z^2}\right]$

  - $z \cdot P_2(z)$ is analytic at $z = 0$

    $\left(z \cdot P_2(z)\right)\Big|_{z=0} = -3$

  - $z^2 \cdot P_3(z)$ is analytic at $z = 0$

    $\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 4$

  - $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $y''z^2 + (2z-3)\, y'z + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

- Rewrite ODE with series expansions
  - Convert $z^m \cdot y'$ to series expansion for $m = 1..2$

    $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$
  - Convert $z^2 \cdot y''$ to series expansion

    $z^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2\, z^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 2a_{k-1}(k+r-1)\right) z^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 + 2a_{k-1}(k+r-1) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(z*diff(y(z),z$2)+(2*z-3)*diff(y(z),z)+4/z*y(z)=0,y(z), singsol=all)
```

$$y(z) = c_1 z^2 \mathrm{e}^{-2z}(2z-1) + c_2 z^2 \left(2\,\mathrm{expIntegral}_1\left(-2z\right) z - \mathrm{expIntegral}_1\left(-2z\right) + \mathrm{e}^{2z}\right) \mathrm{e}^{-2z}$$

✓ Solution by Mathematica

Time used: 0.614 (sec). Leaf size: 47

```
DSolve[z*y''[z]+(2*z-3)*y'[z]+4/z*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to -\frac{1}{2} e^{-2z} z^2 \left(4c_2(1-2z)\,\mathrm{ExpIntegralEi}(2z) - 2c_1 z + 4c_2 e^{2z} + c_1\right)$$

2.219 problem 222

2.219.1 Maple step by step solution 2135

Internal problem ID [7709]
Internal file name [OUTPUT/6642_Sunday_June_05_2022_05_03_13_PM_61243334/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 222.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

[_erf]

$$\boxed{y'' + 2xy' + 4y = 0}$$

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 2x \tag{3}$$
$$C = 4$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 - 3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{x^2-3}{1}\\ &= Q+\frac{R}{1}\\ &= (x^2-3)+(0)\\ &= x^2-3\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$\begin{aligned}b &= (-3)-(0)\\ &= -3\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= x\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-3}{1}-1\right) &&= -2\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-3}{1}-1\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2-3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^-\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\,(x) \\
&= -x \\
&= -x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - \left(x^2 - 3\right)\right) &= 0 \\
2a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\,\mathrm{e}^{\int -x dx} \\
&= (x)\,\mathrm{e}^{-\frac{x^2}{2}} \\
&= x\,\mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{\pi}\,\operatorname{erfi}(x)\,x - \mathrm{e}^{x^2}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x^2}\right) + c_2 \left(x\,\mathrm{e}^{-x^2} \left(\frac{\sqrt{\pi}\,\operatorname{erfi}(x)\,x - \mathrm{e}^{x^2}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x^2} + c_2 \left(\sqrt{\pi}\,\operatorname{erfi}(x)\,x\,\mathrm{e}^{-x^2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-x^2} + c_2 \left(\sqrt{\pi}\,\operatorname{erfi}(x)\,x\,\mathrm{e}^{-x^2} - 1\right)
$$

Verified OK.

2.219.1 Maple step by step solution

Let's solve

$y'' + 2xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\, \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \left(-\sqrt{\pi}\, \operatorname{erfi}(x)\, x + \mathrm{e}^{x^2}\right)$$

✓ Solution by Mathematica
Time used: 0.043 (sec). Leaf size: 51

```
DSolve[y''[x]+2*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2} \left(-\sqrt{\pi} c_2 \sqrt{x^2} \operatorname{erfi}\left(\sqrt{x^2}\right) + c_2 e^{x^2} + 2c_1 x\right)$$

2.220 problem 223

2.220.1 Maple step by step solution 2143

Internal problem ID [7710]
Internal file name [OUTPUT/6643_Sunday_June_05_2022_05_03_16_PM_91207297/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 223.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 3y = 0}$$

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{5}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2-1\right) e^{\int -\frac{x}{2} dx} \\ &= \left(x^2-1\right) e^{-\frac{x^2}{4}} \\ &= \left(x^2-1\right) e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2-1\right) e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x^2-1\right) e^{-\frac{x^2}{2}}\right) + c_2 \left(\left(x^2-1\right) e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}} + c_2(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}} + c_2(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.220.1 Maple step by step solution

Let's solve

$y'' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right) a_{k+2}+a_k(k+3)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1 \mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)+c_2 \mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x-1)^2(x+1)^2} d x\right)$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 65

```
DSolve[y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)+4c_1(x^2-1)-2c_2e^{\frac{x^2}{2}}x\right)$$

2.221 problem 224

2.221.1 Maple step by step solution 2152

Internal problem ID [7711]
Internal file name [`OUTPUT/6644_Sunday_June_05_2022_05_03_19_PM_43933576/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 224.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - x^2 y' - 3yx = 0}$$

Writing the ode as

$$y'' - x^2 y' - 3yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3 + 8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 + 8) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 + 8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$
$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int \frac{x^2}{2} dx} \\ &= (x) \, \mathrm{e}^{\frac{x^3}{6}} \\ &= x \, \mathrm{e}^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x \, \mathrm{e}^{\frac{x^3}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{\frac{x^3}{3}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)
$$

Verified OK.

2.221.1 Maple step by step solution

Let's solve

$y'' - x^2 y' - 3yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+1}$$

- Shift index using $k- > k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1) \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2) \, (k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2) \, (k+1) - a_{k-1}(k+2)) \, x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2) \, (k a_{k+2} - a_{k-1} + a_{k+2}) = 0$$

- Shift index using $k- > k+1$

$$(k+3) \, ((k+1) \, a_{k+3} - a_k + a_{k+3}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x^3}{3}} x - \frac{9 c_2 \mathrm{e}^{\frac{x^3}{3}} 3^{\frac{2}{3}} \mathrm{e}^{-\frac{x^3}{6}} \left(x^6 \,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) + 5 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right) x^3 + 10 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right)\right)}{10 x^3 \left(x^3\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 51

```
DSolve[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9 c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.222 problem 225

2.222.1 Maple step by step solution 2161

Internal problem ID [7712]
Internal file name [OUTPUT/6645_Sunday_June_05_2022_05_03_22_PM_10258994/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 225.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-4x^2+1\right)y''-20xy'-16y=0}$$

Writing the ode as

$$\left(-4x^2+1\right)y''-20xy'-16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^2+1 \\ B &= -20x \\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 6}{(4x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^2 + 6 \\ t &= (4x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 6}{(4x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2-1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x+\frac{1}{2}\right)^2} + \frac{7}{8\left(x+\frac{1}{2}\right)} + \frac{5}{16\left(x-\frac{1}{2}\right)^2} - \frac{7}{8\left(x-\frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2+6}{(4x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4x^2+6}{(4x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)} \\
&= -\frac{2x}{4x^2 - 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x - \frac{1}{2}\right)^2} + \frac{1}{4\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)}\right)^2 -
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(-\frac{1}{4\left(x - \frac{1}{2}\right)} - \frac{1}{4\left(x + \frac{1}{2}\right)}\right) dx} \\
&= (x)\, e^{-\frac{\ln(2x-1)}{4} - \frac{\ln(2x+1)}{4}} \\
&= \frac{x}{(2x - 1)^{\frac{1}{4}} (2x + 1)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(4x^2-1\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(4x^2-1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(4x^2-1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}\right) + c_2\left(\frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}\left(\frac{2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}}{x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(2\ln\left(2x+\sqrt{4x^2-1}\right)x - \sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(2\ln\left(2x+\sqrt{4x^2-1}\right)x - \sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}}$$

Verified OK.

2.222.1 Maple step by step solution

Let's solve

$(-4x^2+1)\,y'' - 20xy' - 16y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{20xy'}{4x^2-1} - \frac{16y}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{20xy'}{4x^2-1} + \frac{16y}{4x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{20x}{4x^2-1}, P_3(x) = \frac{16}{4x^2-1}\right]$

  ◦ $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = \frac{5}{2}$

  ◦ $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

  ◦ $x = -\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $y''(4x^2-1)+20xy'+16y=0$

- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(20u-10)\left(\frac{d}{du}y(u)\right)+16y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(3+2r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+5+2r)+4a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4a_k(k+r+2)^2-4\left(k+r+\frac{5}{2}\right)a_{k+1}(k+1+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r+2)^2}{(2k+5+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}$

- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}, b_{k+1} = \frac{2b_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve((1-4*x^2)*diff(y(x),x$2)-20*x*diff(y(x),x)-16*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2 \left(2\ln\left(2x+\sqrt{4x^2-1}\right)x - \sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 73

```
DSolve[(1-4*x^2)*y''[x]-20*x*y'[x]-16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_2 x \arctan\left(\frac{\sqrt{1-4x^2}}{2x+1}\right) - c_2\sqrt{1-4x^2} + c_1 x}{\sqrt[4]{1-4x^2}\,(4x^2-1)^{5/4}}$$

2.223 problem 226

2.223.1 Maple step by step solution 2170

Internal problem ID [7713]
Internal file name [OUTPUT/6646_Sunday_June_05_2022_05_03_24_PM_7223719/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 226.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''-6xy'+12y=0}$$

Writing the ode as

$$(x^2-1)\,y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(1+x)^2} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)} + \frac{15}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} + (-)(0) \\ &= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} \\ &= \frac{x-4}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(1+x)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{5}{2}}}{(x-1)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1} \, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(1+x)}{2}} \\ &= z_1 \left((x-1)^{\frac{3}{2}} (1+x)^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3\ln(x-1) + 3\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{x(x^2+1)}{(1+x)^4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^4\right) + c_2\left((1+x)^4\left(-\frac{x(x^2+1)}{(1+x)^4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)^4 - c_2 x(x^2+1) \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^4 - c_2 x(x^2+1)
$$

Verified OK.

2.223.1 Maple step by step solution

Let's solve

$(x^2-1)\, y'' - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -3$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\, y''-6xy'+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- ○ Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-3)+a_k(k+r-3)\,(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-3)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r-4)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-2a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{3a_0}{2}$

- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$

- Express in terms of a_0

  $a_3=-\frac{a_0}{2}$

- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{8}$

- Express in terms of a_0

  $a_4=\frac{a_0}{16}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$

- Revert the change of variables $u=1+x$

  $\left[y=\frac{a_0(x-1)^4}{16}\right]$

- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k k}{2(k+5)}$

- Solution for $r=4$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u=1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1} = \frac{a_k k}{2(k+5)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 (x-1)^4}{16} + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+4}\right), b_{k+1} = \frac{b_k k}{2(k+5)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^3 + x) + c_2(x^4 + 6x^2 + 1)$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 45

```
DSolve[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1) + c_1(x-1)^4)}{\sqrt{1-x^2}}$$

2.224 problem 227

2.224.1 Maple step by step solution 2180

Internal problem ID [7714]
Internal file name [`OUTPUT/6647_Sunday_June_05_2022_05_03_27_PM_81148867/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 227.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + (x + 2) y = 0}$$

Writing the ode as

$$y'' + xy' + (x + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\
&= \frac{1}{4}x^2 - x - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (1) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} - 1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) &= 0 \\ (x+2)\,a_1 + 4x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{\int \left(1-\frac{x}{2}\right) dx} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{x-\frac{1}{4}x^2} \\
&= \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(x-4)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(x-2)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(x-2)}{2}} \right) + c_2 \left(\left(x^2 - 4x + 3\right) \mathrm{e}^{-\frac{x(x-2)}{2}} \left(\int \frac{\mathrm{e}^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-4x+3)\,e^{-\frac{x(x-2)}{2}} + c_2(x^2-4x+3)\,e^{-\frac{x(x-2)}{2}}\left(\int \frac{e^{\frac{x(x-4)}{2}}}{(x^2-4x+3)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-4x+3)\,e^{-\frac{x(x-2)}{2}} + c_2(x^2-4x+3)\,e^{-\frac{x(x-2)}{2}}\left(\int \frac{e^{\frac{x(x-4)}{2}}}{(x^2-4x+3)^2}dx\right)$$

Verified OK.

2.224.1 Maple step by step solution

Let's solve

$y'' + xy' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2) + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 + 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{1}{2}x^2+x}(x^2-4x+3) + c_2 e^{-\frac{1}{2}x^2+x}(x^2-4x+3)\left(\int \frac{e^{\frac{1}{2}x^2-2x}}{(x-1)^2(x-3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.526 (sec). Leaf size: 94

```
DSolve[y''[x]+x*y'[x]+(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{x^2}{2}+x-\frac{9}{2}}\left(e^{5/2}\sqrt{2\pi}c_2\left(x^2-4x+3\right)\operatorname{erfi}\left(\frac{x-2}{\sqrt{2}}\right)\right.$$
$$\left.+4e^{9/2}c_1\left(x^2-4x+3\right)-2c_2e^{\frac{1}{2}(x-3)^2+x}(x-2)\right)$$

2.225 problem 228

Internal problem ID [7715]
Internal file name [`OUTPUT/6648_Sunday_June_05_2022_05_03_31_PM_33046832/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 228.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+1)\,y''+7xy'+2y=0}$$

Writing the ode as

$$(2x^2+1)\,y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2 + 6 \\ t &= 4\left(2x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x+\frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x-\frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x+\frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right) \\
&= \frac{5}{4}-\left(\frac{1}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x)\left(4x^2 + 2\right)^{\frac{1}{8}} \\
&= x\left(4x^2 + 2\right)^{\frac{1}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1\left(\frac{1}{\left(2x^2+1\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\right) + c_2\left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve((1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{(2x^2+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 66

```
DSolve[(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{(2x^2+1)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{(2x^2+1)^{3/4}\,\mathrm{Gamma}\left(\frac{1}{4}\right)}$$

2.226 problem 229

2.226.1 Maple step by step solution 2198

Internal problem ID [7716]
Internal file name [`OUTPUT/6649_Sunday_June_05_2022_05_03_35_PM_60177689/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 229.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{4y'' + xy' + 4y = 0}$$

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 56 \\ t &= 64 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 434: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 56}{64} \\
&= Q + \frac{R}{64} \\
&= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\
&= \frac{x^2}{64} - \frac{7}{8}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{7}{8}\right) - (0) \\
&= -\frac{7}{8}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{8} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= -4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= 3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{64} - \frac{7}{8}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x + 2a_2) + 2\left(-\frac{x}{8}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) &= 0 \\ 6x + 2a_2 + \frac{1}{4}a_2x^2 + \frac{1}{2}a_1x + \frac{3}{4}a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^3 - 12x\right) \mathrm{e}^{\int -\frac{x}{8} dx} \\
&= \left(x^3 - 12x\right) \mathrm{e}^{-\frac{x^2}{16}} \\
&= x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{16}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{4} dx} \\
&= z_1 e^{-\frac{x^2}{16}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{16}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{8}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4} dx}}{\left(y_1\right)^2} \,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{8}}}{\left(y_1\right)^2} \,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{8}}\right) + c_2 \left(x\left(x^2 - 12\right) \mathrm{e}^{-\frac{x^2}{8}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x(x^2-12)\,\mathrm{e}^{-\frac{x^2}{8}} + c_2 x(x^2-12)\,\mathrm{e}^{-\frac{x^2}{8}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2\,(x^2-12)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(x^2-12)\,\mathrm{e}^{-\frac{x^2}{8}} + c_2 x(x^2-12)\,\mathrm{e}^{-\frac{x^2}{8}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2\,(x^2-12)^2}dx\right)$$

Verified OK.

2.226.1 Maple step by step solution

Let's solve

$4y'' + xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{4} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{4} + y = 0$

- Multiply by denominators

  $4y'' + xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} (4a_{k+2}(k+2)(k+1) + a_k(k+4)) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2+3k+2) a_{k+2} + a_k(k+4) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(4*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^2}{8}} x(x^2 - 12) + c_2 \mathrm{e}^{-\frac{x^2}{8}} x(x^2 - 12) \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{(x^2 - 12)^2 x^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 122

```
DSolve[4*y''[x]+x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}} \left(\sqrt{2\pi} c_2 (x^2 - 12) x^2 \operatorname{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right) + 4\sqrt{x^2} \left(2\sqrt{2} c_1 x^3 - c_2 e^{\frac{x^2}{8}} x^2 + 8 c_2 e^{\frac{x^2}{8}} - 24\sqrt{2} c_1 x\right)\right)}{32\sqrt{x^2}}$$

2.227 problem 230

Internal problem ID [7717]
Internal file name [OUTPUT/6650_Sunday_June_05_2022_05_03_38_PM_72503412/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 230.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' - 4y = 0}$$

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+18}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{9}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{9}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{9}{2}\right) - (0) \\
&= \frac{9}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &&= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}} - 1\right) &&= -5
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} + \frac{9}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 4$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^{+} \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{x}{2}\right) \\
&= \frac{x}{2} \\
&= \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{2}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0 \\
-a_3x^3 + (-2a_2 + 12)\,x^2 + (-3a_1 + 6a_3)\,x - 4a_0 + 2a_2 = 0
\end{gathered}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^4 + 6x^2 + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 6x^2 + 3\right) e^{\int \frac{x}{2} dx} \\
&= \left(x^4 + 6x^2 + 3\right) e^{\frac{x^2}{4}} \\
&= \left(x^4 + 6x^2 + 3\right) e^{\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^4 + 6x^2 + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 6x^2 + 3\right) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{e^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1(x^4 + 6x^2 + 3) + c_2(x^4 + 6x^2 + 3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^4 + 6x^2 + 3) + c_2(x^4 + 6x^2 + 3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 6x^2 + 3) + c_2(x^4 + 6x^2 + 3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 43

```
DSolve[y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \text{HermiteH}\left(-5, \frac{x}{\sqrt{2}}\right) + \frac{1}{3}c_2(x^4 + 6x^2 + 3)$$

2.228 problem 231

Internal problem ID [7718]
Internal file name [`OUTPUT/6651_Sunday_June_05_2022_05_03_42_PM_29456375/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 231.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4xy'' - xy' + 2y = 0}$$

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32 + x}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32 + x \\ t &= 64x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32 + x}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i\\ &= \frac{1}{8}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-32 + x}{64x} \\ &= Q + \frac{R}{64x} \\ &= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\ &= \frac{1}{64} - \frac{1}{2x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32 + x}{64x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (-)\left(\frac{1}{8}\right) \\ &= \frac{1}{x} - \frac{1}{8} \\ &= \frac{1}{x} - \frac{1}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{-32 + x}{64x}\right)\right) = 0$$

$$\frac{8 + a_0}{4x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 8) \, \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx} \\
&= (x - 8) \, \mathrm{e}^{-\frac{x}{8} + \ln(x)} \\
&= (x - 8) \, x \, \mathrm{e}^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{4x} \, dx} \\
&= z_1 e^{\frac{x}{8}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x - 8) \, x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(-x^2+8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right) - 4\,\mathrm{e}^{\frac{x}{4}}(x-4)}{128\,(x-8)\,x} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1((x-8)\,x) + c_2\left((x-8)\,x \left(\frac{(-x^2+8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right) - 4\,\mathrm{e}^{\frac{x}{4}}(x-4)}{128\,(x-8)\,x} \right) \right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1(x-8)\,x + c_2\left(\frac{(-x^2+8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}(x-4)}{32} \right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1(x-8)\,x + c_2\left(\frac{(-x^2+8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}(x-4)}{32} \right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(4*x*diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 - 8x) + c_2\left(\frac{\text{expIntegral}_1\left(-\frac{x}{4}\right)x^2}{128} + \frac{e^{\frac{x}{4}}x}{32} - \frac{\text{expIntegral}_1\left(-\frac{x}{4}\right)x}{16} - \frac{e^{\frac{x}{4}}}{8}\right)$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 43

```
DSolve[4*x*y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{128}c_2\left((x-8)x\,\text{ExpIntegralEi}\left(\frac{x}{4}\right) - 4e^{x/4}(x-4)\right) + c_1(x-8)x$$

2.229 problem 232

2.229.1 Maple step by step solution 2224

Internal problem ID [7719]
Internal file name [`OUTPUT/6652_Sunday_June_05_2022_05_03_45_PM_11864265/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 232.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2y'' + x(1+18x)\, y' + (12x+1)\, y = 0}$$

Writing the ode as

$$6x^2y'' + \left(18x^2 + x\right) y' + (12x+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 12x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 324x^2 - 252x - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 438: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{7}{4x} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{324x^2 - 252x - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\ &= \frac{9}{4} + \frac{-252x - 35}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{4}\right) - (0) \\ &= -\frac{7}{4} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= -\frac{7}{12} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= \frac{7}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{324x^2 - 252x - 35}{144x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{12} - \left(\frac{7}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\
&= \frac{7}{12x} - \frac{3}{2} \\
&= \frac{7}{12x} - \frac{3}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\
&= x^{\frac{7}{12}} e^{-\frac{3x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{18x^2 + x}{6x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \frac{\ln(x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{3x}{2}}}{x^{\frac{1}{12}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-3x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}}dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-3x} + c_2\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-3x} + c_2\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}}dx\right)$$

Verified OK.

2.229.1 Maple step by step solution

Let's solve

$6x^2y'' + (18x^2 + x)\, y' + (12x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(12x+1)y}{6x^2} - \frac{(1+18x)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+18x)y'}{6x} + \frac{(12x+1)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{12x+1}{6x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{6}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{6}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' + x(1 + 18x)\, y' + (12x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$6\left(k+r-\frac{1}{3}\right)\left(\left(k+r-\frac{1}{2}\right) a_k + 3a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

  $$6\left(k+\frac{2}{3}+r\right)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + 3a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{6a_k}{2k+1+2r}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = -\frac{6a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = -\frac{6a_k}{2k+2}, b_{k+1} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(6*x^2*diff(y(x),x$2)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,e^{-3x} + c_2\sqrt{x}\,e^{-3x}\left(\int \frac{e^{3x}}{x^{\frac{7}{6}}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.237 (sec). Leaf size: 47

```
DSolve[6*x^2*y''[x]+x*(1+18*x)*y'[x]+(1+12*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6},-3x\right)}{(-x)^{5/6}}+c_1\sqrt{x}\right)$$

2.230 problem 233

2.230.1 Maple step by step solution 2235

Internal problem ID [7720]
Internal file name [OUTPUT/6653_Sunday_June_05_2022_05_03_48_PM_50253346/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 233.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' - x(x+8)\,y' + 6y = 0}$$

Writing the ode as

$$3x^2y'' + \left(-x^2 - 8x\right)y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2 - 8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 16x + 40}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 16x + 40 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 16x + 40}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 440: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{4}{9x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{2}{3}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 16x + 40}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\ &= \frac{1}{36} + \frac{16x + 40}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= \frac{4}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &= -\frac{4}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 16x + 40}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{4}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{4}{3} - \left(-\frac{2}{3}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{3x} + \left(\frac{1}{6}\right) \\
&= -\frac{2}{3x} + \frac{1}{6} \\
&= \frac{x-4}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 16x + 40}{36x^2}\right)\right) = 0
$$

$$
\frac{(-a_1 - 2)x - 2a_0 - 4a_1}{3x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4, a_1 = -2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2x + 4
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2x + 4\right) e^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 - 2x + 4\right) e^{\frac{x}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{\left(x^2 - 2x + 4\right) e^{\frac{x}{6}}}{x^{\frac{2}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-8x}{3x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}+\frac{4\ln(x)}{3}} \\
&= z_1\left(x^{\frac{4}{3}}\mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-8x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}+\frac{8\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}}\right) + c_2\left(\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}}\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}} + c_2\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}}\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1(x^2 - 2x + 4)\,\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}} + c_2(x^2 - 2x + 4)\,\mathrm{e}^{\frac{x}{3}} x^{\frac{2}{3}} \left(\int \frac{x^{\frac{4}{3}} \mathrm{e}^{-\frac{x}{3}}}{(x^2 - 2x + 4)^2} dx\right)$$

Verified OK.

2.230.1 Maple step by step solution

Let's solve

$3x^2y'' + (-x^2 - 8x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2} + \frac{(x+8)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+8)y'}{3x} + \frac{2y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x+8}{3x}, P_3(x) = \frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{8}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' - x(x+8)\,y' + 6y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+3r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-2)\,(k+r-3) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+3r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{3, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3\left(k+r-\frac{2}{3}\right)(k+r-3)\, a_k - a_{k-1}(k+r-1) = 0$$

- Shift index using $k->k+1$

  $$3\left(k+\frac{1}{3}+r\right)(k-2+r)\, a_{k+1} - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r)}{(3k+1+3r)(k-2+r)}$$

- Recursion relation for $r = 3$

  $$a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}\right]$$

- Recursion relation for $r = \frac{2}{3}$

  $$a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{2}{3}}\right), a_{k+1}=\frac{a_k(k+3)}{(3k+10)(k+1)}, b_{k+1}=\frac{b_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(3*x^2*diff(y(x),x$2)-x*(x+8)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{2}{3}} \mathrm{e}^{\frac{x}{3}} \left(x^2 - 2x + 4\right) + c_2 x^{\frac{2}{3}} \mathrm{e}^{\frac{x}{3}} \left(x^2 - 2x + 4\right) \left(\int \frac{x^{\frac{4}{3}} \mathrm{e}^{-\frac{x}{3}}}{\left(x^2 - 2x + 4\right)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 1.152 (sec). Leaf size: 79

```
DSolve[3*x^2*y''[x]-x*(x+8)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x/3} x^{2/3} \left(x^2 - 2x + 4\right) - \frac{c_2 e^{x/3} x^{2/3} (x^2 - 2x + 4) \, \Gamma\left(\frac{1}{3}, \frac{x}{3}\right)}{6 \; 3^{2/3}} + \frac{1}{6} c_2 (x - 4) x$$

2.231 problem 234

2.231.1 Maple step by step solution 2246

Internal problem ID [7721]
Internal file name [OUTPUT/6654_Sunday_June_05_2022_05_03_52_PM_22932001/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 234.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - x(2x+1)\, y' + 2(4x-1)\, y = 0}$$

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right) y' + (8x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 60x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 442: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{15}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2-60x+21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x+21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x+21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{15}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &&= \frac{15}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 60x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{15}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{15}{4} - \left(\frac{7}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{7}{4x} - \frac{1}{2} \\
&= \frac{7}{4x} - \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right)\right) = 0
$$

$$
\frac{2(9 + a_1)\, x + 4a_0 + 7a_1}{2x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{63}{4}, a_1 = -9\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 9x + \frac{63}{4}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{\int \left(\frac{7}{4x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{-\frac{x}{2} + \frac{7\ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{\frac{7}{4}} e^{-\frac{x}{2}}}{4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{2x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{4}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}} x^{\frac{1}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^4 - 9x^3 + \frac{63}{4}x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-x}{2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\ &= y_1\left(\int \frac{16\,\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + c_2\left(x^4 - 9x^3 + \frac{63}{4}x^2\left(\int \frac{16\,\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + 16c_2\left(\int \frac{\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right)\left(x^2 - 9x + \frac{63}{4}\right)x^2 \tag{1}$$

Verification of solutions

$$y = c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + 16c_2\left(\int \frac{e^x}{x^{\frac{7}{2}}\left(4x^2 - 36x + 63\right)^2}dx\right)\left(x^2 - 9x + \frac{63}{4}\right)x^2$$

Verified OK.

2.231.1 Maple step by step solution

Let's solve

$2x^2y'' + (-2x^2 - x)\,y' + (8x - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-1)y}{x^2} + \frac{(2x+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{2x} + \frac{(4x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x+1}{2x}, P_3(x) = \frac{4x-1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - x(2x + 1)\,y' + (8x - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1 + 2r)\,(-2 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k + 2r + 1)\,(k + r - 2) - 2a_{k-1} (k - 5 + r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{2, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k + r + \frac{1}{2}\right)(k + r - 2)\, a_k - 2a_{k-1}(k - 5 + r) = 0$

- Shift index using $k- >k + 1$

  $2\left(k + \frac{3}{2} + r\right)(k + r - 1)\, a_{k+1} - 2a_k (k + r - 4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k (k+r-4)}{(2k+3+2r)(k+r-1)}$

- Recursion relation for $r = 2$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k (k-2)}{(2k+7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{4a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{9}$

- Express in terms of a_0

  $a_2 = \frac{4a_0}{63}$

- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right)$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(2*x^2*diff(y(x),x$2)-x*(1+2*x)*diff(y(x),x)+2*(4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 (4x^2 - 36x + 63) + c_2 x^2 (4x^2 - 36x + 63) \left(\int \frac{e^x}{(4x^2 - 36x + 63)^2 x^{\frac{7}{2}}} dx \right)$$

✓ Solution by Mathematica

Time used: 1.738 (sec). Leaf size: 89

```
DSolve[2*x^2*y''[x]-x*(1+2*x)*y'[x]+2*(4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1\left(x^4 - 9x^3 + \frac{63x^2}{4}\right) - \frac{4c_2\left(\sqrt{\pi}(-4x^2+36x-63)\,x^{5/2}\operatorname{erfi}\left(\sqrt{x}\right) + 2e^x(2x^4-17x^3+24x^2+6x+3)\right)}{945\sqrt{x}}$$

2.232 problem 235

2.232.1 Maple step by step solution 2257

Internal problem ID [7722]
Internal file name [`OUTPUT/6655_Sunday_June_05_2022_05_03_56_PM_72286017/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 235.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4x^2y' + (2x+1)\,y = 0}$$

Writing the ode as

$$4x^2y'' - 4x^2y' + (2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 444: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x - 1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x - 1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^x}{(y_1)^2}\, dx \\
&= y_1(-\operatorname{expIntegral}_1(-x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{expIntegral}_1(-x))\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} - c_2\sqrt{x}\ \operatorname{expIntegral}_1(-x) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} - c_2\sqrt{x}\ \operatorname{expIntegral}_1(-x)$$

Verified OK.

2.232.1 Maple step by step solution

Let's solve

$4x^2y'' - 4x^2y' + (2x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = y' - \frac{(2x+1)y}{4x^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - y' + \frac{(2x+1)y}{4x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$\left[P_2(x) = -1, P_3(x) = \frac{2x+1}{4x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$4x^2y'' - 4x^2y' + (2x+1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k- >k - 1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2 + (-4k+6-4r)\, a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{4a_k k}{(2k+2)^2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 20

```
dsolve(4*x^2*diff(y(x),x$2)-4*x^2*diff(y(x),x)+(1+2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\,\text{expIntegral}_1(-x)$$

✓ Solution by Mathematica
Time used: 0.044 (sec). Leaf size: 19

```
DSolve[4*x^2*y''[x]-4*x^2*y'[x]+(1+2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2\,\text{ExpIntegralEi}(x) + c_1)$$

2.233 problem 236

2.233.1 Maple step by step solution 2268

Internal problem ID [7723]
Internal file name [OUTPUT/6656_Sunday_June_05_2022_05_03_59_PM_643868/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 236.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-2x+3)\,y' + (1-2x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right)y' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 446: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-1 - 4x}{4x^2}\right) \\ &= 1 + \frac{-1 - 4x}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(1) \\
&= \frac{1}{2x} - 1 \\
&= \frac{1}{2x} - 1
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - 1\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 + 3x}{x^2} \, dx} \\
&= z_1 e^{x - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{2x-3\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1(-\operatorname{expIntegral}_1(-2x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{expIntegral}_1(-2x))\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1}{x} - \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1}{x} - \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x}$$

Verified OK.

2.233.1 Maple step by step solution

Let's solve

$x^2y'' + (-2x^2 + 3x)\, y' + (1 - 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{x^2} + \frac{(2x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-3)y'}{x} - \frac{(2x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(2x - 3)\, y' + (1 - 2x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)^2 - 2a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -1$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+1)^2 - 2a_{k-1}(k+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)^2 - 2a_k(k+r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+1)}{(k+2+r)^2}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{2a_k k}{(k+1)^2}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*(3-2*x)*diff(y(x),x)+(1-2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x}$$

✓ Solution by Mathematica
Time used: 0.05 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]+x*(3-2*x)*y'[x]+(1-2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \operatorname{ExpIntegralEi}(2x) + c_1}{x}$$

2.234 problem 237

2.234.1 Maple step by step solution 2278

Internal problem ID [7724]
Internal file name [OUTPUT/6657_Sunday_June_05_2022_05_04_02_PM_76387852/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 237.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(x+3)\,y' + (-x+4)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + (-x+4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= -x + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 448: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+10x-1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{10x-1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{10x-1}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 10x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2}\right)\right) = 0
$$

$$
\frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2, a_1 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 4x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 4x + 2\right) \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 + 4x + 2\right) \mathrm{e}^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 + 4x + 2\right) \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+4x+2\right) x^2 \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{(x+3)\,\mathrm{e}^{-x} - \left(x^2+4x+2\right) \mathrm{expIntegral}_1\left(x\right)}{4x^2+16x+8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+4x+2\right) x^2 \mathrm{e}^x\right) \\
&\quad + c_2\left(\left(x^2+4x+2\right) x^2 \mathrm{e}^x \left(\frac{(x+3)\,\mathrm{e}^{-x} - \left(x^2+4x+2\right) \mathrm{expIntegral}_1\left(x\right)}{4x^2+16x+8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+4x+2\right) x^2 \mathrm{e}^x - \frac{c_2 x^2 \left(\left(x^2+4x+2\right) \mathrm{e}^x \,\mathrm{expIntegral}_1\left(x\right) - x - 3\right)}{4} \tag{1}
$$

Verification of solutions

$$y = c_1 (x^2 + 4x + 2) x^2 e^x - \frac{c_2 x^2 ((x^2 + 4x + 2) e^x \text{expIntegral}_1(x) - x - 3)}{4}$$

Verified OK.

2.234.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 3x) y' + (-x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x+3) y' + (-x + 4) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)^2 - a_{k-1}(k+r) = 0$
- Shift index using $k- > k + 1$

  $a_{k+1}(k+r-1)^2 - a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+1)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k (k+3)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
dsolve(x^2*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 (x^2 + 4x + 2) - \frac{c_2 x^2 (-x^2 \operatorname{expIntegral}_1(x) + e^{-x} x - 4 \operatorname{expIntegral}_1(x) x + 3 e^{-x} - 2 \operatorname{expIntegral}_1(x)) e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 52

```
DSolve[x^2*y''[x]-x*(3+x)*y'[x]+(4-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} x^2 (c_2 e^x (x^2 + 4x + 2) \operatorname{ExpIntegralEi}(-x) + 4 c_1 e^x (x^2 + 4x + 2) + c_2 (x + 3))$$

2.235 problem 238

2.235.1 Maple step by step solution 2288

Internal problem ID [7725]
Internal file name [OUTPUT/6658_Sunday_June_05_2022_05_04_06_PM_29511323/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 238.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(3-x)\,y' + y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 + 3x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 450: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-6x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right)\right) &= 0 \\
\frac{1 + a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1) \, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (x-1) \, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (x-1) \sqrt{x} \, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x-1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - \mathrm{e}^x}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x-1}{x}\right) + c_2\left(\frac{x-1}{x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - \mathrm{e}^x}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-1)}{x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - \mathrm{e}^x)}{x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x-1)}{x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - e^x)}{x}$$

Verified OK.

2.235.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2+3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2} + \frac{(-3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(-3+x)y'}{x} + \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{-3+x}{x}, P_3(x) = \frac{1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  - $x=0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(-3+x)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x$2)+x*(3-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x-1)}{x} + \frac{c_2(\text{expIntegral}_1(-x)\,x - \text{expIntegral}_1(-x) + e^x)}{x}$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]+x*(3-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2 e^x}{x}$$

2.236 problem 239

2.236.1 Maple step by step solution 2294

Internal problem ID [7726]
Internal file name [`OUTPUT/6659_Sunday_June_05_2022_05_04_08_PM_89597898/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 239.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2} y'' - \left(2\sqrt{5}-1\right) x y' + \left(\frac{19}{4}-3x^{2}\right) y = 0}$$

Writing the ode as

$$x^{2} y'' + \left(-2\sqrt{5}\, x + x\right) y' + \left(\frac{19}{4}-3x^{2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2} \\ B &= -2\sqrt{5}\, x + x \\ C &= \frac{19}{4}-3x^{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 452: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\sqrt{3}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2\sqrt{5}\, x+x}{x^2}\, dx} \\ &= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}} \\ &= z_1 \left(x^{\sqrt{5}-\frac{1}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-\sqrt{3}\, x} x^{\sqrt{5}-\frac{1}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2\sqrt{5}\, x+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\left(2\sqrt{5}-1\right)\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{3}\, e^{2\sqrt{3}\, x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}}\right) + c_2 \left(\mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}} \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}} + \frac{c_2 \sqrt{3}\,x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{\sqrt{3}\,x}}{6} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}} + \frac{c_2 \sqrt{3}\,x^{\sqrt{5}-\frac{1}{2}} \mathrm{e}^{\sqrt{3}\,x}}{6}
$$

Verified OK.

2.236.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-2\sqrt{5}\,x + x\right) y' + \left(\frac{19}{4} - 3x^2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(12x^2-19)y}{4x^2} + \frac{\left(2\sqrt{5}-1\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{\left(2\sqrt{5}-1\right)y'}{x} - \frac{(12x^2-19)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2\sqrt{5}-1}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1 - 2\sqrt{5}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{19}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4\left(2\sqrt{5} - 1\right) xy' + \left(-12x^2 + 19\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r}$

  Rewrite ODE with series expansions

  $\left(1 + 2\sqrt{5} - 2r\right) \left(-1 + 2\sqrt{5} - 2r\right) a_0 x^r + \left(-1 + 2\sqrt{5} - 2r\right) \left(-3 + 2\sqrt{5} - 2r\right) a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} ((-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\left(1 + 2\sqrt{5} - 2r\right) \left(-1 + 2\sqrt{5} - 2r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} + \sqrt{5}, \sqrt{5} - \frac{1}{2}\right\}$

- Each term must be 0

$$\left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right)a_1 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-8a_k(k+r)\sqrt{5}+\left(4k^2+8kr+4r^2+19\right)a_k-12a_{k-2} = 0$$

- Shift index using $k->k+2$

$$-8a_{k+2}(k+2+r)\sqrt{5}+\left(4(k+2)^2+8(k+2)\,r+4r^2+19\right)a_{k+2}-12a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\,r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$$

- Recursion relation for $r = \frac{1}{2}+\sqrt{5}$

$$a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$$

- Solution for $r = \frac{1}{2}+\sqrt{5}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1 = 0\right]$$

- Recursion relation for $r = \sqrt{5}-\frac{1}{2}$

$$a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}$$

- Solution for $r = \sqrt{5}-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}-\frac{1}{2}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\sqrt{5}-\frac{1}{2}}\right), a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right.}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 37

```
dsolve(x^2*diff(y(x),x$2)-(2*sqrt(5)-1)*x*diff(y(x),x)+(19/4-3*x^2)*y(x)=0,y(x), singsol=all
```

$$y(x) = \frac{c_1 x^{\sqrt{5}} \sinh\left(\sqrt{3}\, x\right)}{\sqrt{x}} + \frac{c_2 x^{\sqrt{5}} \cosh\left(\sqrt{3}\, x\right)}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.1 (sec). Leaf size: 53

```
DSolve[x^2*y''[x]-(2*Sqrt[5]-1)*x*y'[x]+(19/4-3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{6} e^{-\sqrt{3}x} x^{\sqrt{5}-\frac{1}{2}} \left(\sqrt{3} c_2 e^{2\sqrt{3}x} + 6c_1\right)$$

2.237 problem 240

2.237.1 Maple step by step solution 2304

Internal problem ID [7727]
Internal file name [OUTPUT/6660_Sunday_June_05_2022_05_04_11_PM_17249137/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 240.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-3+x)\,y' + (-x+4)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (-x+4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= -x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 454: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{0} a_i x^i$$
$$= \frac{1}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2}\, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{expIntegral}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x} x^2\right) + c_2\left(\mathrm{e}^{-x} x^2(-\operatorname{expIntegral}_1(-x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x} c_1 x^2 - c_2 \mathrm{e}^{-x} x^2 \operatorname{expIntegral}_1(-x) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x} c_1 x^2 - c_2 \mathrm{e}^{-x} x^2 \operatorname{expIntegral}_1(-x)$$

Verified OK.

2.237.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 3x)\, y' + (-x + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-4)y}{x^2} - \frac{(-3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-3+x)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{-3+x}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(-3+x)\, y' + (-x+4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-2)^2 + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\left(a_k (k+r-2) + a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

$$(k+r-1)\left(a_{k+1}(k+r-1) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+r-1}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(x^2*diff(y(x),x$2)+x*(x-3)*diff(y(x),x)+(4-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} c_1 x^2 + c_2 x^2 \mathrm{e}^{-x} \operatorname{expIntegral}_1(-x)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*(x-3)*y'[x]+(4-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} x^2 (c_2 \operatorname{ExpIntegralEi}(x) + c_1)$$

2.238 problem 241

2.238.1 Maple step by step solution 2315

Internal problem ID [7728]
Internal file name [OUTPUT/6661_Sunday_June_05_2022_05_04_14_PM_27245086/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 241.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x^2y' - (x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + x^2y' + (-x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 456: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{x+2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1\left(\left(x^2 - 2x + 2\right) e^{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\left(x^2 - 2x + 2\right) e^{x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2(x^2 - 2x + 2)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2(x^2 - 2x + 2)}{x}$$

Verified OK.

2.238.1 Maple step by step solution

Let's solve

$x^2 y'' + x^2 y' + (-x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{(x+2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 1, P_3(x) = -\frac{x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x^2 y' + (-x - 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\,(a_k(k+r+1) + a_{k-1}) = 0$$

- Shift index using $k->k+1$

  $$(k-1+r)\,(a_{k+1}(k+2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r=-1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r=-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

  $$a_{k+1} = -\frac{a_k}{k+4}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x^2-2x+2)}{x}+\frac{c_2\mathrm{e}^{-x}}{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]+x^2*y'[x]-(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{e^{-x}(c_2e^x(x^2-2x+2)+c_1)}{x}$$

2.239 problem 242

2.239.1 Maple step by step solution 2325

Internal problem ID [7729]
Internal file name [OUTPUT/6662_Sunday_June_05_2022_05_04_19_PM_64642738/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 242.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 458: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = 1 - \frac{1}{x} + \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2-4x+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{x^2}\, dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x}} - \frac{c_2(2x+1)\,\mathrm{e}^{-2x}}{4\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} - \frac{c_2(2x+1)\,\mathrm{e}^{-2x}}{4\sqrt{x}}$$

Verified OK.

2.239.1 Maple step by step solution

Let's solve

$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-3)y}{4x^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{(4x-3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 8x^2y' + (4x - 3) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- Shift index using $k- >k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r+\frac{1}{2}\right) a_k + 8a_{k-1}\left(k-\frac{1}{2}+r\right) = 0$

- Shift index using $k- >k+1$

  $4\left(k-\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+1} + 8a_k\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(2k+2r+1)}{(2k-1+2r)(2k+3+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{8a_k k}{(2k-2)(2k+2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = -\frac{8a_k k}{(2k-2)(2k+2)}$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{4a_k(2k+4)}{(2k+2)(2k+6)}$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4a_k(2k+4)}{(2k+2)(2k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x-3/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 \mathrm{e}^{-2x}(2x+1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]+2*x^2*y'[x]+(x-3/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 - c_2 e^{-2x}(2x+1)}{4\sqrt{x}}$$

2.240 problem 243

2.240.1 Maple step by step solution 2334

Internal problem ID [7730]
Internal file name [`OUTPUT/6663_Sunday_June_05_2022_05_04_22_PM_8548427/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 243.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^{2}(1+x)\,y''+x^{2}y'-2y=0}$$

Writing the ode as

$$x^{2}(1+x)\,y''+x^{2}y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}(1+x) \\ B &= x^{2} \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 8x + 8$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 460: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} - \frac{1}{4(1+x)^2} + \frac{2}{x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+8x+8}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x + 2} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{2x + 2} - \frac{1}{x} \\
&= -\frac{x + 2}{2x(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+2} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2x+2} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-2+a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2)\, e^{\int \left(\frac{1}{2x+2} - \frac{1}{x}\right) dx} \\ &= (x+2)\, e^{\frac{\ln(1+x)}{2} - \ln(x)} \\ &= \frac{(x+2)\sqrt{1+x}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\, dx} \\ &= z_1 e^{-\frac{\ln(1+x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{1+x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4}{x+2} + \ln(1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x+2}{x}\right) + c_2 \left(\frac{x+2}{x} \left(\frac{4}{x+2} + \ln(1+x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x+2)}{x} + \frac{c_2(4 + \ln(1+x)(x+2))}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x+2)}{x} + \frac{c_2(4 + \ln(1+x)(x+2))}{x}$$

Verified OK.

2.240.1 Maple step by step solution

Let's solve

$x^2(1+x)\, y'' + x^2 y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{1+x} + \frac{2y}{x^2(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{1+x} - \frac{2y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{1+x}, P_3(x) = -\frac{2}{x^2(1+x)}\right]$

- ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

- ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\, y'' + x^2 y' - 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 2u + 1)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 u^{-1+r} + \left(a_1 (1+r)^2 - 2a_0 (r^2+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k(k^2+2kr+r^2+1) + \right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term must be 0

  $a_1(1+r)^2 - 2a_0(r^2+1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - 2a_k(k^2+1) + a_{k-1}(k-1)^2 = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 - 2a_{k+1}\left((k+1)^2+1\right) + k^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-4k a_{k+1}-4a_{k+1}}{(k+2)^2}, a_1-2a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-4k a_{k+1}-4a_{k+1}}{(k+2)^2}, a_1-2a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x^2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x+2)}{x}+\frac{c_2(\ln(x+1)x+2\ln(x+1)+4)}{x}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 30

```
DSolve[x^2*(1+x)*y''[x]+x^2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{c_1(x+2)+c_2(x+2)\log(x+1)+4c_2}{x}$$

2.241 problem 244

2.241.1 Maple step by step solution 2343

Internal problem ID [7731]
Internal file name [`OUTPUT/6664_Sunday_June_05_2022_05_04_24_PM_85347076/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 244.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(x^2+6)\, y' + 6y = 0}$$

Writing the ode as

$$x^2y'' + (x^3+6x)\, y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3+6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 14 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 462: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+14}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{7}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{7}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{7}{2}\right) - (0) \\
&= \frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= 3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &&= -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$, and since there are no poles, then

$$\begin{aligned}d &= \alpha_\infty^+\\ &= 3\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty\\ &= 0 + \left(\frac{x}{2}\right)\\ &= \frac{x}{2}\\ &= \frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(6x + 2a_2) + 2\left(\frac{x}{2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) = 0\\ -a_2x^2 + (-2a_1 + 6)\,x - 3a_0 + 2a_2 = 0\end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 + 3x\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^3 + 3x\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= x\left(x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3+6x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^3} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 + 3}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2} - 6\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 \left(x^2 + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+3}{x^2}\right) + c_2 \left(\frac{x^2+3}{x^2}\left(\int \frac{e^{-\frac{x^2}{2}}}{x^2 \left(x^2+3\right)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{x^2(x^2+3)^2} dx\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{x^2(x^2+3)^2} dx\right)}{x^2}
$$

Verified OK.

2.241.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 + 6x) y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y}{x^2} - \frac{(x^2+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+6)y'}{x} + \frac{6y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(x^2+6)\, y' + 6y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+3)(k+r+2)+a_{k-2}(k-2+r)=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+5+r)(k+4+r)+a_k(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$$

- Recursion relation for $r=-3$

$$a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}$$

- Solution for $r=-3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}, a_1=0\right]$$

- Recursion relation for $r=-2$; series terminates at $k=2$

$$a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k(k-2)}{(k+3)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*(6+x^2)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{x^2(x^2+3)^2}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.532 (sec). Leaf size: 65

```
DSolve[x^2*y''[x]+x*(6+x^2)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2\pi}c_2x(x^2+3)\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) - 12c_1x(x^2+3) + 2c_2e^{-\frac{x^2}{2}}(x^2+2)}{12x^3}$$

2.242 problem 245

2.242.1 Maple step by step solution 2355

Internal problem ID [7732]
Internal file name [`OUTPUT/6665_Sunday_June_05_2022_05_04_28_PM_11464479/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 245.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(1-x)\,y' - y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 + x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 464: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 2x + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-2x + 3}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-(1+x)\,e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-(1+x)\,e^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x-1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x-1)}{x}$$

Verified OK.

2.242.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

  $$(k+r)\,(a_{k+1}(k+2+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+1)}{x} + \frac{c_2 e^x}{x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]+x*(1-x)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 e^x - c_1(x+1)}{x}$$

2.243 problem 246

2.243.1 Maple step by step solution 2365

Internal problem ID [7733]
Internal file name [OUTPUT/6666_Sunday_June_05_2022_05_04_31_PM_15242828/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 246.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+3)\,y' + 4y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 466: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 6x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{6x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right) \right) &= 0 \\
\frac{1 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1 + x)\, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1 + x) \sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\ &= z_1\left(x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)\,x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{-x}+(-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((1+x)\,x^2\mathrm{e}^x\right) + c_2\left((1+x)\,x^2\mathrm{e}^x\left(\frac{\mathrm{e}^{-x}+(-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(1+x)\,x^2\mathrm{e}^x - c_2\left(-1+\mathrm{e}^x(1+x)\,\mathrm{expIntegral}_1(x)\right)x^2 \tag{1}$$

Verification of solutions

$$y = c_1(1+x)\,x^2\mathrm{e}^x - c_2\left(-1+\mathrm{e}^x(1+x)\,\mathrm{expIntegral}_1(x)\right)x^2$$

Verified OK.

2.243.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 - 3x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x+3)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 38

```
dsolve(x^2*diff(y(x),x$2)-x*(x+3)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 (x+1) + c_2 x^2 \left(-\operatorname{expIntegral}_1(x)\, x - \operatorname{expIntegral}_1(x) + e^{-x}\right) e^x$$

✓ Solution by Mathematica
Time used: 0.065 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]-x*(x+3)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 (c_2 e^x (x+1) \operatorname{ExpIntegralEi}(-x) + c_1 e^x (x+1) + c_2)$$

2.244 problem 247

2.244.1 Maple step by step solution 2375

Internal problem ID [7734]
Internal file name [`OUTPUT/6667_Sunday_June_05_2022_05_04_34_PM_53533248/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 247.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x^2y' - 2y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 468: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\
&= \frac{1}{4} + \frac{2}{x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) \quad = \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) \quad = 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) = 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{x+2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+8}{4x^2}\right)\right) = 0
$$

$$
\frac{-2 + a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-\frac{x}{2} - \ln(x)} \\
&= \frac{(x+2)\, \mathrm{e}^{-\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x-2)\,\mathrm{e}^x}{x+2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x+2}{x}\right) + c_2 \left(\frac{x+2}{x}\left(\frac{(x-2)\,\mathrm{e}^x}{x+2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x+2)}{x} + \frac{c_2(x-2)\,\mathrm{e}^x}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x+2)}{x} + \frac{c_2(x-2)\,\mathrm{e}^x}{x}
$$

Verified OK.

2.244.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x^2y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r = -1$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$$

- Apply recursion relation for $k = 0$

$$a_1 = \frac{a_0}{2}$$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

$$y = a_0 \cdot \left(\frac{x}{2} + 1\right)$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(\frac{x}{2}+1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+2)}{x} + \frac{c_2 e^x (x-2)}{x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 72

```
DSolve[x^2*y''[x]-x^2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2e^{x/2}\left((c_1x+2ic_2)\cosh\left(\frac{x}{2}\right) - (ic_2x+2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}$$

2.245 problem 248

2.245.1 Maple step by step solution 2385

Internal problem ID [7735]
Internal file name [OUTPUT/6668_Sunday_June_05_2022_05_04_37_PM_75102089/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 248.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x^2y' - (3x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 470: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{12x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned}b &= (3) - (0) \\ &= 3\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &= 3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &= -3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 12x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{2}{x} + \frac{1}{2} \\
&= \frac{4+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2+12x+8}{4x^2}\right)\right) &= 0 \\
\frac{4-a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, e^{\int \left(\frac{2}{x}+\frac{1}{2}\right) dx} \\
&= (4+x)\, e^{\frac{x}{2}+2\ln(x)} \\
&= (4+x)\, e^{\frac{x}{2}} x^2
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (4+x)\,\mathrm{e}^{x} x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-x^3-3x^2+2x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\,x^3(4+x)}{24\,(4+x)\,x^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)\,\mathrm{e}^{x} x^2\right) \\
&\quad + c_2\left((4+x)\,\mathrm{e}^{x} x^2\left(\frac{(-x^3-3x^2+2x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\,x^3(4+x)}{24\,(4+x)\,x^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(4+x)\,\mathrm{e}^{x} x^2 + \frac{c_2\left(\mathrm{e}^{x} x^3(4+x)\,\mathrm{expIntegral}_1(x) - x^3 - 3x^2 + 2x - 2\right)}{24x} \tag{1}
$$

Verification of solutions

$$y = c_1(4+x)\,\mathrm{e}^{x}x^{2}+\frac{c_2(\mathrm{e}^{x}x^{3}(4+x)\,\mathrm{expIntegral}_1\left(x\right)-x^{3}-3x^{2}+2x-2)}{24x}$$

Verified OK.

2.245.1 Maple step by step solution

Let's solve

$x^{2}y''-x^{2}y'+\left(-3x-2\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(3x+2)y}{x^{2}}+y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-y'-\frac{(3x+2)y}{x^{2}}=0$

- □ Check to see if $x_0=0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x)=-1, P_3(x)=-\frac{3x+2}{x^{2}}\right]$

  - ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=0$

  - ◦ $x^{2}\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^{2}\cdot P_3(x)\right)\bigg|_{x=0}=-2$

  - ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $x^{2}y''-x^{2}y'+\left(-3x-2\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+2+r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 70

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-(3*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 e^x (x+4) - \frac{c_2(-\operatorname{expIntegral}_1(x)\, x^4 + e^{-x} x^3 - 4 \operatorname{expIntegral}_1(x)\, x^3 + 3x^2 e^{-x} - 2 e^{-x} x + 2 e^{-x})\, e^x}{24x}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]-x^2*y'[x]-(3*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{24} c_2 e^x (x+4) x^2 \operatorname{ExpIntegralEi}(-x) + c_1 e^x (x+4) x^2 - \frac{c_2(x^3 + 3x^2 - 2x + 2)}{24x}$$

2.246 problem 249

2.246.1 Maple step by step solution 2395

Internal problem ID [7736]
Internal file name [OUTPUT/6669_Sunday_June_05_2022_05_04_40_PM_38924108/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 249.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' + x(5-x)\,y' + 4y = 0}$$

Writing the ode as

$$x^2 y'' + \left(-x^2 + 5x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + 5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 472: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$[\sqrt{r}]_\infty = \sum_{i=0}^{0} a_i x^i$$
$$= \frac{1}{2} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-10x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right)\right) &= 0 \\
\frac{(a_1 + 4)\, x + 2a_0 + a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2, a_1 = -4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 4x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 2\right) e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x + 2\right) e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 4x + 2\right) \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+5x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{5\ln(x)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-4x+2}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-x^2+4x-2)\,\mathrm{expIntegral}_1(-x)-\mathrm{e}^{x}(-3+x)}{4x^2-16x+8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2-4x+2}{x^2}\right) \\
&\quad + c_2\left(\frac{x^2-4x+2}{x^2}\left(\frac{(-x^2+4x-2)\,\mathrm{expIntegral}_1(-x)-\mathrm{e}^{x}(-3+x)}{4x^2-16x+8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2-4x+2)}{x^2} + \frac{c_2((-x^2+4x-2)\,\text{expIntegral}_1(-x) - e^x(-3+x))}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-4x+2)}{x^2} + \frac{c_2((-x^2+4x-2)\,\text{expIntegral}_1(-x) - e^x(-3+x))}{4x^2}$$

Verified OK.

2.246.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2+5x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-5)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x-5}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$x^2 y'' - x(x-5)\, y' + 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 (2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(2+r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = -2$

- Each term in the series must be 0, giving the recursion relation

$a_k (k+r+2)^2 - a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

$a_{k+1}(k+3+r)^2 - a_k (k+r) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k (k+r)}{(k+3+r)^2}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

$a_{k+1} = \frac{a_k (k-2)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

$a_1 = -2a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve(x^2*diff(y(x),x$2)+x*(5-x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 4x + 2)}{x^2} + \frac{c_2\left(\frac{x^2 \operatorname{expIntegral}_1(-x)}{4} + \frac{x\,e^x}{4} - \operatorname{expIntegral}_1(-x)\,x - \frac{3\,e^x}{4} + \frac{\operatorname{expIntegral}_1(-x)}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.197 (sec). Leaf size: 48

```
DSolve[x^2*y''[x]+x*(5-x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_2(x^2-4x+2)\,\text{ExpIntegralEi}(x)+4c_1(x^2-4x+2)-c_2e^x(x-3)}{4x^2}$$

2.247 problem 250

2.247.1 Maple step by step solution 2406

Internal problem ID [7737]
Internal file name [OUTPUT/6670_Sunday_June_05_2022_05_04_43_PM_3458057/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 250.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4x(1-x)\,y' + (2x-9)\,y = 0}$$

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right)y' + (2x-9)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 474: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-4x+8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x+8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x+8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x-2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2+4x}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^x}{x^{\frac{3}{2}}}\right) + c_2\left(\frac{\mathrm{e}^x}{x^{\frac{3}{2}}}\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^x}{x^{\frac{3}{2}}} - \frac{c_2(x^2+2x+2)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^x}{x^{\frac{3}{2}}} - \frac{c_2(x^2+2x+2)}{x^{\frac{3}{2}}}$$

Verified OK.

2.247.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 + 4x)\,y' + (2x - 9)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} - \frac{(2x-9)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} + \frac{(2x-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{9}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4x(x - 1)\,y' + (2x - 9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r+\frac{3}{2}\right) a_k - a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

$$4\left(k-\frac{1}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right) a_{k+1} - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{2k+5+2r}$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+2}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+8}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+8}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 + 2x + 2)}{x^{\frac{3}{2}}} + \frac{c_2 e^x}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 30

```
DSolve[4*x^2*y''[x]+4*x*(1-x)*y'[x]+(2*x-9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x^{3/2}}$$

2.248 problem 251

Internal problem ID [7738]
Internal file name [`OUTPUT/6671_Sunday_June_05_2022_05_04_46_PM_97961411/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 251.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + 2x(x+2)\,y' + 2(1+x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(2x^2 + 4x\right)y' + (2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + 4x \\ C &= 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+2}{x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x + 2$$
$$t = x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+2}{x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+2}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{1} - 0\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+2}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + (1) \\
&= 1 + \frac{1}{x} \\
&= 1 + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{x+2}{x}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(1+\frac{1}{x}\right) dx} \\ &= x \, \mathrm{e}^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^2+4x}{x^2} \, dx} \\ &= z_1 e^{-x-2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(\frac{2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x}}{x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2(2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x})}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2(2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x})}{x^2}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)+2*x*(2+x)*diff(y(x),x)+2*(1+x)*y(x)=0,y(x), singsol=all)
```

$$
y(x) = \frac{c_1}{x} + \frac{c_2(2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x})}{x^2}
$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+2*x*(2+x)*y'[x]+2*(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2c_2 x \operatorname{ExpIntegralEi}(-2x) + c_1 x - c_2 e^{-2x}}{x^2}$$

2.249 problem 252

2.249.1 Maple step by step solution 2423

Internal problem ID [7739]
Internal file name [`OUTPUT/6672_Sunday_June_05_2022_05_04_50_PM_45934552/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 252.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(1-x)\,y' + (1-x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 - x \\ C &= 1 - x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2x} \\
&= \frac{1 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2} + \frac{1}{2x}\right)^2 - \left(\frac{x^2 + 2x - 1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2x}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{expIntegral}_1(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(-\operatorname{expIntegral}_1(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x - \operatorname{expIntegral}_1(x)\, c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x - \operatorname{expIntegral}_1(x)\, c_2 x$$

Verified OK.

2.249.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - x)\, y' + (1 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(x-1) y' + (1-x) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k k}{(k+1)^2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x)*diff(y(x),x)+(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + \text{expIntegral}_1(x)\, x c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[x^2*y''[x]-x*(1-x)*y'[x]+(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2 \, \text{ExpIntegralEi}(-x) + c_1)$$

2.250 problem 253

2.250.1 Maple step by step solution 2430

Internal problem ID [7740]
Internal file name [`OUTPUT/6673_Sunday_June_05_2022_05_04_53_PM_82125412/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 253.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4x(2x+1)\,y' + (4x-1)\,y = 0}$$

Writing the ode as

$$4x^2y'' + \left(8x^2 + 4x\right)y' + (4x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2 + 4x \\ C &= 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^{-2x}}{\sqrt{x}}\right) + c_2\left(\frac{e^{-2x}}{\sqrt{x}}\left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-2x}}{\sqrt{x}} + \frac{c_2}{2\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-2x}}{\sqrt{x}} + \frac{c_2}{2\sqrt{x}}$$

Verified OK.

2.250.1 Maple step by step solution

Let's solve

$4x^2y'' + (8x^2 + 4x)\, y' + (4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-1)y}{4x^2} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} + \frac{(4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{4x-1}{4x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4x(2x+1)\,y' + (4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m\cdot y$ to series expansion for $m = 0..1$

    $x^m\cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k->k-m$

    $x^m\cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m\cdot y'$ to series expansion for $m = 1..2$

    $x^m\cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k->k+1-m$

    $x^m\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right)x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r+\frac{1}{2}\right) a_k + 2a_{k-1}\right)\left(k+r-\frac{1}{2}\right) = 0$

- Shift index using $k->k+1$

  $4\left(\left(k+\frac{3}{2}+r\right) a_{k+1} + 2a_k\right)\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{2k+3+2r}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+2}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{4a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{4a_k}{2k+2}, b_{k+1} = -\frac{4b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)+(4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 e^{-2x}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 26

```
DSolve[4*x^2*y''[x]+4*x*(1+2*x)*y'[x]+(4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

2.251 problem 254

Internal problem ID [7741]
Internal file name [`OUTPUT/6674_Sunday_June_05_2022_05_04_55_PM_9185653/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 254.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(4+x)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+4x\right)y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4+x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4+x \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4+x}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4+x}{4x} \\
&= Q + \frac{R}{4x} \\
&= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\
&= \frac{1}{4} + \frac{1}{x}
\end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4+x}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{x} \\
&= \frac{1}{2} + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4+x}{4x}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\ &= x\, \mathrm{e}^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2+4x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x - 4\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{expIntegral}_1(x)\, x - \mathrm{e}^{-x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{\operatorname{expIntegral}_1(x)\,x - \mathrm{e}^{-x}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2(\operatorname{expIntegral}_1(x)\,x - \mathrm{e}^{-x})}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2(\operatorname{expIntegral}_1(x)\,x - \mathrm{e}^{-x})}{x^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x^2*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2+x)*y(x)=0,y(x), singsol=all)
```

$$
y(x) = \frac{c_1}{x} + \frac{c_2(-\operatorname{expIntegral}_1(x)\,x + \mathrm{e}^{-x})}{x^2}
$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+x*(4+x)*y'[x]+(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2 x \operatorname{ExpIntegralEi}(-x) + c_1 x - c_2 e^{-x}}{x^2}$$

2.252 problem 255

2.252.1 Maple step by step solution 2449

Internal problem ID [7742]
Internal file name [`OUTPUT/6675_Sunday_June_05_2022_05_04_58_PM_77565417/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 255.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 482: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2+2}{x^2}\right)\right) = 0$$

$$\frac{2ia_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= (x-i)\,e^{\int \left(-\frac{1}{x}-i\right)dx} \\ &= (x-i)\,e^{-ix-\ln(x)} \\ &= \frac{(x-i)\,e^{-ix}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}} - \frac{c_2 (ix-1)\,\mathrm{e}^{ix}}{2x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x-i)\,e^{-ix}}{x^{\frac{3}{2}}} - \frac{c_2(ix-1)\,e^{ix}}{2x^{\frac{3}{2}}}$$

Verified OK.

2.252.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-9)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-9}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{9}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2-9)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+2r)(-3+2r)\, x^r + a_1(5+2r)(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)(2k+2r-3) + 4a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+2r)(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(5+2r)(-1+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 9) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 9\right) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+4k-8}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+28k+40}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+28k+40}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+28k+40}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-9/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{ix}(x+i)}{x^{\frac{3}{2}}} + \frac{c_2 e^{-ix}(x-i)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 44

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-9/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}((c_1x+c_2)\cos(x)+(c_2x-c_1)\sin(x))}{x^{3/2}}$$

2.253 problem 256

2.253.1 Maple step by step solution 2456

Internal problem ID [7743]
Internal file name [`OUTPUT/6676_Sunday_June_05_2022_05_05_03_PM_64607862/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 256.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 484: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.253.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - ◦ Shift index using $k- >k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - ◦ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

  - ◦ Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - ◦ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

  - ◦ Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.254 problem 257

2.254.1 Maple step by step solution 2467

Internal problem ID [7744]
Internal file name [`OUTPUT/6677_Sunday_June_05_2022_05_05_05_PM_69585205/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 257.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2xy'' + 5(1-2x)\,y' - 5y = 0}$$

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 100x^2 - 60x + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 486: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} - \frac{15}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{5}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{100x^2 - 60x + 5}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\
&= \frac{25}{4} + \frac{-60x + 5}{16x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{15}{4}\right) - (0) \\
&= -\frac{15}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{5}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= -\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= \frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{100x^2 - 60x + 5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{3}{4} - \left(-\frac{1}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\ &= -\frac{1}{4x} - \frac{5}{2} \\ &= -\frac{1}{4x} - \frac{5}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right)\right) = 0$$

$$\frac{-1 + 10a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{10}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{10}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{10}\right) e^{\int \left(-\frac{1}{4x} - \frac{5}{2}\right) dx} \\ &= \left(x + \frac{1}{10}\right) e^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\ &= \frac{(1 + 10x)\, e^{-\frac{5x}{2}}}{10x^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x+5}{2x}\,dx} \\
&= z_1 e^{\frac{5x}{2}-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{5x}{2}}}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1+10x}{10x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5x-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{100\sqrt{x}\,\mathrm{e}^{5x}}{(1+10x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1+10x}{10x^{\frac{3}{2}}}\right) + c_2\left(\frac{1+10x}{10x^{\frac{3}{2}}}\left(\int \frac{100\sqrt{x}\,\mathrm{e}^{5x}}{(1+10x)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(1+10x)}{10x^{\frac{3}{2}}} + \frac{c_2(100x+10)\left(\int \frac{\sqrt{x}\,\mathrm{e}^{5x}}{(1+10x)^2}dx\right)}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+10x)}{10x^{\frac{3}{2}}} + \frac{c_2(100x+10)\left(\int \frac{\sqrt{x}\,e^{5x}}{(1+10x)^2}dx\right)}{x^{\frac{3}{2}}}$$

Verified OK.

2.254.1 Maple step by step solution

Let's solve

$2xy'' + (-10x+5)\,y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y}{2x} + \frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5(2x-1)y'}{2x} - \frac{5y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-10x+5)\,y' - 5y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{5}{2}+r\right)(k+1+r)\, a_{k+1} - 10\left(k+r+\frac{1}{2}\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{5(2k+2r+1)a_k}{(2k+5+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}\right]$$

- Recursion relation for $r = -\frac{3}{2}$; series terminates at $k = 1$

$a_{k+1} = \frac{5(2k-2)a_k}{(2k+2)\left(k-\frac{1}{2}\right)}$

- Apply recursion relation for $k = 0$

  $a_1 = 10a_0$

- Terminating series solution of the ODE for $r = -\frac{3}{2}$. Use reduction of order to find the second

  $y = a_0 \cdot (1 + 10x)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot (1 + 10x), a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(2*x*diff(y(x),x$2)+5*(1-2*x)*diff(y(x),x)-5*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(10x+1)}{x^{\frac{3}{2}}} + \frac{c_2(10x+1)\left(\int \frac{\sqrt{x}\,\mathrm{e}^{5x}}{(10x+1)^2}dx\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 40

```
DSolve[2*x*y''[x]+5*(1-2*x)*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x) + \frac{c_1(10x+1)}{10\sqrt{5}x^{3/2}}$$

2.255 problem 258

2.255.1 Maple step by step solution 2474

Internal problem ID [7745]
Internal file name [OUTPUT/6678_Sunday_June_05_2022_05_05_09_PM_95856848/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 258.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 488: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}$$

Verified OK.

2.255.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.256 problem 259

2.256.1 Maple step by step solution 2485

Internal problem ID [7746]
Internal file name [OUTPUT/6679_Sunday_June_05_2022_05_05_11_PM_20413407/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 259.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (x+n)\,y' + (n+1)\,y = 0}$$

Writing the ode as

$$\begin{align*} xy'' + (x+n)\,y' + (n+1)\,y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= x \\ B &= x+n \tag{3} \\ C &= n+1 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= n^2 - 2xn + x^2 - 2n - 4x \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 490: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{n}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{n}{2} + 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{3n^6}{2x^7} - \frac{3n^5}{2x^6} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \frac{53n^4}{2x^6} - \frac{67n^3}{4x^5} - \frac{37n^2}{4x^4} - \frac{4n}{x^3} - \frac{1075n^4}{4x^7} - \frac{491n^3}{4x^6} - \frac{93n^2}{2x^5} - \frac{13n}{x^4} \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)\,x + n^2 - 2n}{4x^2}\right) \\ &= \frac{1}{4} + \frac{(-2n-4)\,x + n^2 - 2n}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n - 4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2} - 1$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{n}{2} - 1\right) - (0) \\ &= -\frac{n}{2} - 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &&= -\frac{n}{2} - 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{n}{2}-1}{\frac{1}{2}} - 0\right) &&= \frac{n}{2} + 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{n}{2}$	$-\frac{n}{2}+1$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{n}{2}-1$	$\frac{n}{2}+1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{n}{2} + 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{n}{2} + 1 - \left(\frac{n}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n-x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{n}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{n}{2x^2}\right) + \left(\frac{n}{2x} - \frac{1}{2}\right)^2 - \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right)\right) = 0
$$

$$
\frac{n + a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -n\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - n
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-n)\, e^{\int \left(\frac{n}{2x} - \frac{1}{2}\right) dx} \\
&= (x-n)\, e^{-\frac{x}{2} + \frac{n \ln(x)}{2}} \\
&= -(n-x)\, x^{\frac{n}{2}} e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}} \\
&= z_1\left(x^{-\frac{n}{2}}\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-n)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+n}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-n\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right) + c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-n)\,\mathrm{e}^{-x} - c_2(n-x)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-n)\,\mathrm{e}^{-x} - c_2(n-x)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)
$$

Verified OK.

2.256.1 Maple step by step solution

Let's solve

$xy'' + (x+n)\,y' + (n+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{(n+1)y}{x} - \frac{(x+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x+n)y'}{x} + \frac{(n+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = \frac{x+n}{x}, P_3(x) = \frac{n+1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = n$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (x+n)\,y' + (n+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(n-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(n+k+r) + a_k(n+k+r+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(n-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -n+1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(n+k+r) + a_k(n+k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(n+k+r+1)}{(k+1+r)(n+k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}\right]$$

- Recursion relation for $r = -n+1$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}$$

- Solution for $r = -n+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-n+1}, a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n+1}\right), a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}, b_{k+1} = -\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x$2)+(x+n)*diff(y(x),x)+(n+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(-x+n) + c_2 e^{-x}(-x+n)\left(\int \frac{e^x x^{-n}}{(-x+n)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 1.051 (sec). Leaf size: 48

```
DSolve[x*y''[x]+(x+n)*y'[x]+(n+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(n-x)\left(c_2\int_1^x \frac{e^{K[1]}K[1]^{-n}}{(n-K[1])^2}dK[1]+c_1\right)$$

2.257 problem 260

Internal problem ID [7747]
Internal file name [`OUTPUT/6680_Sunday_June_05_2022_05_05_15_PM_98276218/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 260.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^4y'' + xy' + y = 0}$$

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-10x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -10x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{5}{2x^4}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= -1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= 4 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-10x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha^{\pm}_{\infty}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha^{-}_{\infty} = 1$ then

$$
\begin{aligned}
d &= \alpha^{-}_{\infty} - \left(\alpha^{+}_{c_1}\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha^{+}_{c_1}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} - \frac{1}{x} \\
&= \frac{1}{2x^3} - \frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2}\right) + \left(\frac{1}{2x^3} - \frac{1}{x}\right)^2 - \left(\frac{-10x^2+1}{4x^6}\right)\right) &= 0 \\
\frac{(2a_0+2)\,x + a_1}{x^3} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2} - \ln(x)} \\
&= \frac{\left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^4} \, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{4x^2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{1}{2x^2}} x^2}{\left(x^2 - 1\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^2-1}{x}\right) + c_2\left(\frac{x^2-1}{x}\left(\int \frac{e^{\frac{1}{2x^2}}x^2}{(x^2-1)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{e^{\frac{1}{2x^2}}x^2}{(x^2-1)^2}dx\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{e^{\frac{1}{2x^2}}x^2}{(x^2-1)^2}dx\right)}{x}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve(x^4*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{x^2 e^{\frac{1}{2x^2}}}{(x+1)^2(x-1)^2}dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.218 (sec). Leaf size: 61

```
DSolve[x^4*y''[x]+x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2\pi}c_2(x^2-1)\operatorname{erfi}\left(\frac{1}{\sqrt{2}x}\right)-4c_1(x^2-1)+2c_2e^{\frac{1}{2x^2}}x}{4x}$$

2.258 problem 261

2.258.1 Maple step by step solution 2505

Internal problem ID [7748]
Internal file name [OUTPUT/6681_Sunday_June_05_2022_05_05_19_PM_41773691/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 261.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + (2x^2 + x)\, y' - 4y = 0}$$

Writing the ode as

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{1}{x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = 1
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 + 4x + 15}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (1) + \left(\frac{4x + 15}{4x^2}\right) \\
&= 1 + \frac{4x + 15}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$
\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)(1) \\ &= -\frac{3}{2x} - 1 \\ &= -\frac{3}{2x} - 1 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right)\right) = 0$$
$$\frac{-3 + 2a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x + \frac{3}{2}\right) e^{\int \left(-\frac{3}{2x} - 1\right) dx} \\
&= \left(x + \frac{3}{2}\right) e^{-x - \frac{3\ln(x)}{2}} \\
&= \frac{(3 + 2x)\, e^{-x}}{2x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2 + x}{x^2}\, dx} \\
&= z_1 e^{-x - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-x}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}(2x^2-4x+3)}{4x+6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}\right) + c_2 \left(\frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}\left(\frac{\mathrm{e}^{2x}(2x^2-4x+3)}{4x+6}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3+2x)\,\mathrm{e}^{-2x}}{2x^2} + \frac{c_2(2x^2-4x+3)}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3+2x)\,\mathrm{e}^{-2x}}{2x^2} + \frac{c_2(2x^2-4x+3)}{4x^2}$$

Verified OK.

2.258.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^2 + x)\,y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{x^2} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} - \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = -\frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x+1)\,y' - 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1) = 0$$

- Shift index using $k- > k+1$

  $$a_{k+1}(k+3+r)\,(k+r-1) + 2a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $$a_{k+1} = -\frac{2a_k(k-2)}{(k+1)(k-3)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{4a_0}{3}$$

- Apply recursion relation for $k = 1$

  $$a_2 = -\frac{a_1}{2}$$

- Express in terms of a_0

  $$a_2 = \frac{2a_0}{3}$$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $$y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right)$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+5)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x$2)+(x+2*x^2)*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x^2 - 4x + 3)}{x^2} + \frac{c_2 e^{-2x}(2x + 3)}{2x^2}$$

✓ Solution by Mathematica
Time used: 0.362 (sec). Leaf size: 44

```
DSolve[x^2*y''[x]+(x+2*x^2)*y'[x]-4*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\frac{2c_1 e^{-2x}(2x + 3)}{x^2} + \frac{c_2(2x^2 - 4x + 3)}{x^2} - 2\right)$$

2.259 problem 262

2.259.1 Maple step by step solution 2514

Internal problem ID [7749]
Internal file name [OUTPUT/6682_Sunday_June_05_2022_05_05_21_PM_54634142/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 262.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4x^3 - 14x^2 - 2x)\, y'' - (6x^2 - 7x + 1)\, y' + (6x - 1)\, y = 0}$$

Writing the ode as

$$(4x^3 - 14x^2 - 2x)\, y'' + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3 - 14x^2 - 2x \\ B &= -6x^2 + 7x - 1 \\ C &= 6x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -12x^4 + 156x^3 + 297x^2 - 78x - 3 \\ t &= 16\left(2x^3 - 7x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 495: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{4x} - \frac{3}{16x^2} + \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$[\sqrt{r}]_c = 0$$
$$\alpha_c^+ = \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4}$$
$$\alpha_c^- = \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}$$

For the pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$[\sqrt{r}]_c = 0$$
$$\alpha_c^+ = \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}$$
$$\alpha_c^- = \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}$$

For the pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$[\sqrt{r}]_c = 0$$
$$\alpha_c^+ = \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2}$$
$$\alpha_c^- = \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4} + \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4} - \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} + (-)(0) \\
&= \frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} \\
&= \frac{-6x^2 + 7x - 1}{8x^3 - 28x^2 - 4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} \right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x - \frac{7}{4} + \right.} \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\,\mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right) dx} \\
&= (x-1)\,\mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln\left(4x-7-\sqrt{57}\right)}{2} - \frac{\ln\left(4x-7+\sqrt{57}\right)}{2}} \\
&= \frac{(x-1)\,x^{\frac{1}{4}}}{\sqrt{4x-7-\sqrt{57}}\,\sqrt{4x-7+\sqrt{57}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x^2+7x-1}{4x^3-14x^2-2x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln\left(2x^2-7x-1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{2x^2-7x-1}}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-1)\sqrt{2}}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2} + \ln(2x^2-7x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(32x+16)\sqrt{x}}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2\left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{(32x+16)\sqrt{x}}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-1)\sqrt{2}}{4} + c_2(8x+4)\sqrt{2}\,\sqrt{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x-1)\sqrt{2}}{4} + c_2(8x+4)\sqrt{2}\,\sqrt{x}
$$

Verified OK.

2.259.1 Maple step by step solution

Let's solve

$(4x^3 - 14x^2 - 2x)\,y'' + (-6x^2 + 7x - 1)\,y' + (6x - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(6x-1)y}{2x(2x^2-7x-1)} + \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)} + \frac{(6x-1)y}{2x(2x^2-7x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x(2x^2 - 7x - 1) + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-2+m}$

  - Shift index using $k- >k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1 + 2r)\, x^{-1+r} + (-a_1(1 + r)\,(1 + 2r) - a_0(14r^2 - 21r + 1))\, x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k + 1 + r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, \frac{1}{2}\}$

- Each term must be 0

  $-a_1(1+r)(1+2r) - a_0(14r^2 - 21r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-14a_k + 4a_{k-1} - 2a_{k+1})k^2 + ((-28a_k + 8a_{k-1} - 4a_{k+1})r + 21a_k - 18a_{k-1} - 3a_{k+1})k + (-14a$

- Shift index using $k- >k+1$

  $(-14a_{k+1} + 4a_k - 2a_{k+2})(k+1)^2 + ((-28a_{k+1} + 8a_k - 4a_{k+2})r + 21a_{k+1} - 18a_k - 3a_{k+2})(k+$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} + 8kra_k - 28kra_{k+1} + 4r^2a_k - 14r^2a_{k+1} - 10ka_k - 7ka_{k+1} - 10ra_k - 7ra_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 4kr + 2r^2 + 7k + 7r + 6}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}, -3a_1 + 6a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0, b_k\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve((4*x^3-14*x^2-2*x)*diff(y(x),x$2)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x)=0,y(x), sin
```

$$y(x) = c_1(x-1) + c_2\sqrt{x}\,(2x+1)$$

✓ Solution by Mathematica
Time used: 6.298 (sec). Leaf size: 26

```
DSolve[(4*x^3-14*x^2-2*x)*y''[x]-(6*x^2-7*x+1)*y'[x]+(6*x-1)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \to c_1(x-1) - 2c_2\sqrt{x}(2x+1)$$

2.260 problem 263

2.260.1 Maple step by step solution 2525

Internal problem ID [7750]
Internal file name [`OUTPUT/6683_Sunday_June_05_2022_05_05_24_PM_21407444/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 263.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x^2y' + (x - 2)\, y = 0}$$

Writing the ode as

$$x^2y'' + x^2y' + (x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{x} + \frac{1}{2} \\ &= \frac{x-2}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x}+\frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2}\, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} - \frac{c_2 \mathrm{e}^{-x}\left(x^2+2x+2\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} - \frac{c_2 \mathrm{e}^{-x}\left(x^2+2x+2\right)}{x}$$

Verified OK.

2.260.1 Maple step by step solution

Let's solve

$x^2y'' + x^2y' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{(x-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{(x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 1, P_3(x) = \frac{x-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x^2y' + (x-2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k(k+3)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x}+\frac{c_2 e^{-x}(x^2+2x+2)}{x}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 29

```
DSolve[x^2*y''[x]+x^2*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{c_1-c_2 e^{-x}(x^2+2x+2)}{x}$$

2.261 problem 264

2.261.1 Maple step by step solution 2535

Internal problem ID [7751]
Internal file name [OUTPUT/6684_Sunday_June_05_2022_05_05_27_PM_55189330/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 264.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x^2y' + (x-2)\,y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}b &= (-1) - (0) \\ &= -1\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x-2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^x}{x}\right) + c_2\left(\frac{\mathrm{e}^x}{x}\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^x}{x} + \frac{c_2(-x^2-2x-2)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^x}{x} + \frac{c_2(-x^2-2x-2)}{x}$$

Verified OK.

2.261.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' + (x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{(x-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{(x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x^2y' + (x-2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- Shift index using $k->k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r+1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k-1+r)\,(a_{k+1}(k+2+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+4}$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x^2+2x+2)}{x}+\frac{c_2 e^x}{x}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]-x^2*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{c_1 e^x-c_2(x^2+2x+2)}{x}$$

2.262 problem 265

2.262.1 Maple step by step solution 2544

Internal problem ID [7752]
Internal file name [`OUTPUT/6685_Sunday_June_05_2022_05_05_31_PM_30994879/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 265.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-4x)\,y''-\frac{xy'}{2}-\frac{3yx}{4}=0}$$

Writing the ode as

$$\left(-4x^3+x^2\right)y''-\frac{xy'}{2}-\frac{3yx}{4}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^3+x^2 \\ B &= -\frac{x}{2} \\ C &= -\frac{3x}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -48x^2 - 20x + 5 \\ t &= 16\left(4x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4x} + \frac{5}{16x^2} - \frac{3}{16\left(x - \frac{1}{4}\right)^2} - \frac{5}{4\left(x - \frac{1}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-1, 2, 5\}
\end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{1, 2, 3\}
\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-48x^2 - 20x + 5}{16\left(4x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{1, 2, 3\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$
$\frac{1}{4}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{1, 2, 3\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$
e_1 = 2,\ e_2 = 1,\ e_\infty = 3
$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(3 - (2 + (1))) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{2}{(x-(0))} + \frac{1}{(x-(\frac{1}{4}))}\right) \\
&= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{144x^2 - 12x - 5}{16x^2 (4x - 1)^2} = 0$$

Solving for ω gives

$$\omega = \frac{12x - 2 + 3\sqrt{1 - 4x}}{4x (4x - 1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{12x - 2 + 3\sqrt{1-4x}}{4x(4x-1)} dx} \\ &= \frac{(4x - 1)^{\frac{1}{4}} \sqrt{x}\, \sqrt{2} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{3}{2}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{x}{2}}{-4x^3 + x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(4x-1)}{4} + \frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{x^{\frac{1}{4}}}{(4x - 1)^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{x}{2}}{-4x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(4x-1)}{2}+\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{16\sqrt{1-4x}\, x^{\frac{3}{2}}}{3\sqrt{4x-1}\,\left(\sqrt{1-4x}+1\right)^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\right) \\
&\quad + c_2 \left(\frac{x^{\frac{1}{4}}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4}\left(\frac{16\sqrt{1-4x}\, x^{\frac{3}{2}}}{3\sqrt{4x-1}\,\left(\sqrt{1-4x}+1\right)^3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4} + \frac{4c_2 x^{\frac{7}{4}}\sqrt{2}\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}\sqrt{1-4x}}{3\left(\sqrt{1-4x}+1\right)^2\sqrt{4x-1}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}\sqrt{2}\left(\sqrt{1-4x}+1\right)\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}}{4} + \frac{4c_2 x^{\frac{7}{4}}\sqrt{2}\sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}\sqrt{1-4x}}{3\left(\sqrt{1-4x}+1\right)^2\sqrt{4x-1}}$$

Verified OK.

2.262.1 Maple step by step solution

Let's solve

$\left(-4x^3+x^2\right)y''-\frac{xy'}{2}-\frac{3yx}{4}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y}{4x(4x-1)}-\frac{y'}{2x(4x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{2x(4x-1)}+\frac{3y}{4x(4x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{2x(4x-1)},P_3(x)=\frac{3}{4x(4x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4y''x(4x-1)+2y'+3y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(4k+4r-1)\,(4k+4r-3$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-4\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 16a_k\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(4k+4r-3)(4k+4r-1)}{2(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+1} = \frac{a_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{a_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{a_k(4k-3)(4k-1)}{2(2k-1)(k+1)}, b_{k+1}=\frac{b_k(4k+3)(4k+5)}{2(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 131

```
dsolve(x^2*(1-4*x)*diff(y(x),x$2)+((1-(3/2))*x-(6-4*(3/2))*x^2)*diff(y(x),x)+(3/2)*(1-(3/2))
```

$$y(x)=c_1\sqrt{3x-1}\left(\frac{4\sqrt{(-1+4x)\,x}\,x+8x^2-2\sqrt{(-1+4x)\,x}-5x+1}{5x-1+2\sqrt{(-1+4x)\,x}}\right)^{\frac{1}{4}}+c_2\sqrt{3x-1}\left(\frac{5x-1+2\sqrt{(-1+4x)\,x}}{4\sqrt{(-1+4x)\,x}\,x+8x^2-2\sqrt{(-1+4x)\,x}-5x+1}\right)^{\frac{1}{4}}$$

✓ Solution by Mathematica

Time used: 0.385 (sec). Leaf size: 111

```
DSolve[x^2*(1-4*x)*y''[x]+((1-(3/2))*x-(6-4*(3/2))*x^2)*y'[x]+(3/2)*(1-(3/2))*x*y[x]==0,y[x]
```

$$y(x)\to\frac{\sqrt[4]{x}\sqrt[4]{4x-1}\left(6c_1\left(\sqrt{4x-1}-i\right)^{3/2}+ic_2\left(\sqrt{4x-1}+i\right)^{3/2}\right)}{6\sqrt[4]{1-4x}\sqrt[4]{\sqrt{4x-1}-i}\sqrt[4]{\sqrt{4x-1}+i}}$$

2.263 problem 266

2.263.1 Maple step by step solution 2554

Internal problem ID [7753]
Internal file name [`OUTPUT/6686_Sunday_June_05_2022_05_05_34_PM_68320460/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 266.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + (x^2 + x)\,y' + (-9 + x)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^2 + x)\,y' + (-9 + x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= -9 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 35$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 35}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 35}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\
&= -\frac{5}{2x} + \frac{1}{2} \\
&= \frac{x - 5}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right)\right) = 0
$$

$$
\frac{(-a_1 - 8)\, x - 2a_0 - 5a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 20, a_1 = -8\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 8x + 20
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 8x + 20\right) e^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 - 8x + 20\right) e^{\frac{x}{2} - \frac{5 \ln(x)}{2}} \\
&= \frac{\left(x^2 - 8x + 20\right) e^{\frac{x}{2}}}{x^{\frac{5}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-8x+20}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^2-8x+20}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-8x+20}{x^3}\right) + c_2 \left(\frac{x^2-8x+20}{x^3}\left(-\frac{(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^2-8x+20}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2-8x+20)}{x^3} - \frac{c_2(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2 - 8x + 20)}{x^3} - \frac{c_2(x^3 + 9x^2 + 36x + 60)\,e^{-x}}{x^3}$$

Verified OK.

2.263.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\,y' + (-9 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-9+x)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{(-9+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{-9+x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1 + x)\,y' + (-9 + x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r))\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(k-5)}$
- Apply recursion relation for $k = 0$

$a_1 = -\frac{2a_0}{5}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{8}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{20}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+7)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(x^2*diff(y(x),x$2)+(x+x^2)*diff(y(x),x)+(x-9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 8x + 20)}{x^3} + \frac{c_2 \mathrm{e}^{-x}(x^3 + 9x^2 + 36x + 60)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.209 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]+(x+x^2)*y'[x]+(x-9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1((x-8)x+20) - c_2 e^{-x}(x^3 + 9x^2 + 36x + 60)}{x^3}$$

2.264 problem 267

2.264.1 Maple step by step solution 2565

Internal problem ID [7754]
Internal file name [OUTPUT/6687_Sunday_June_05_2022_05_05_38_PM_94792420/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 267.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(1+x)\,y' + (3x-1)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+x\right)y' + (3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= 3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 10x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x + 3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2x} - \frac{1}{2} \\
&= -\frac{-3 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x + 3}{4x^2}\right)\right) = 0$$

$$\frac{3 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -3 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-3 + x)\, e^{\int \left(\frac{3}{2x} - \frac{1}{2}\right) dx} \\
&= (-3 + x)\, e^{-\frac{x}{2} + \frac{3 \ln(x)}{2}} \\
&= (-3 + x)\, x^{\frac{3}{2}} e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (-3+x)\,x\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(-x^3+3x^2)\,\mathrm{expIntegral}_1(-x) - \mathrm{e}^x(x^2-2x-1)}{6x^2\,(-3+x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((-3+x)\,x\,\mathrm{e}^{-x}\big) \\
&\quad + c_2\left((-3+x)\,x\,\mathrm{e}^{-x}\left(\frac{(-x^3+3x^2)\,\mathrm{expIntegral}_1(-x) - \mathrm{e}^x(x^2-2x-1)}{6x^2\,(-3+x)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(-3+x)\,x\,\mathrm{e}^{-x} + \frac{c_2(-x^2\mathrm{e}^{-x}(-3+x)\,\mathrm{expIntegral}_1(-x) - x^2 + 2x + 1)}{6x} \tag{1}
$$

Verification of solutions

$$y = c_1(-3+x)\,x\,\mathrm{e}^{-x} + \frac{c_2(-x^2\mathrm{e}^{-x}(-3+x)\,\mathrm{expIntegral}_1\,(-x) - x^2 + 2x + 1)}{6x}$$

Verified OK.

2.264.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2+x)\,y' + (3x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-1)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{(3x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{3x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1+x)\,y' + (3x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1}(k+2+r)\,(k+r) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(x^2*diff(y(x),x$2)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\, \mathrm{e}^{-x}(x-3) + \frac{c_2(\mathrm{expIntegral}_1(-x)\, x^3 + \mathrm{e}^x x^2 - 3x^2\, \mathrm{expIntegral}_1(-x) - 2x\, \mathrm{e}^x - \mathrm{e}^x)\, \mathrm{e}^{-x}}{6x}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 66

```
DSolve[x^2*y''[x]+x*(x+1)*y'[x]+(3*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_2(x-3)x^2\, \mathrm{ExpIntegralEi}(x) + 6c_1x^3 - x^2(c_2e^x + 18c_1) + 2c_2e^x x + c_2e^x)}{6x}$$

2.265 problem 268

2.265.1 Maple step by step solution 2575

Internal problem ID [7755]
Internal file name [OUTPUT/6688_Sunday_June_05_2022_05_05_41_PM_40099810/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 268.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - (x^2+4x)\,y' + 4y = 0}$$

Writing the ode as

$$x^2y'' + (-x^2-4x)\,y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{8x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned}b &= (2) - (0) \\ &= 2\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_{\infty} &= \frac{1}{2} \\
\alpha_{\infty}^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\
\alpha_{\infty}^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 8x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{2}{x} \\
&= \frac{4 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{2}{x}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{1}{2} + \frac{2}{x}\right)^2 - \left(\frac{x^2 + 8x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{2}{x}\right) dx} \\
&= \mathrm{e}^{\frac{x}{2}} x^2
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 - 4x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} + 2\ln(x)} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}} x^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^4 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{(-x^2+x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\, x^3}{6x^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^4 \mathrm{e}^x) + c_2 \left(x^4 \mathrm{e}^x \left(\frac{(-x^2+x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\, x^3}{6x^3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x x^4 + \frac{c_2 x(\mathrm{expIntegral}_1(x)\, x^3 \mathrm{e}^x - x^2 + x - 2)}{6} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x x^4 + \frac{c_2 x(\mathrm{expIntegral}_1(x)\, x^3 \mathrm{e}^x - x^2 + x - 2)}{6}$$

Verified OK.

2.265.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 4x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(4+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4+x)y'}{x} + \frac{4y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{4+x}{x}, P_3(x) = \frac{4}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(4+x) y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 4\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-4) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k-3+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k-3+r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{k-2}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = \frac{a_k}{k-2}$$

- Recursion relation for $r = 4$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 4$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(x^2*diff(y(x),x$2)-(x^2+4*x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^x x^4 - \frac{c_2 x\, \mathrm{e}^x (-\operatorname{expIntegral}_1(x)\, x^3 + x^2 \mathrm{e}^{-x} - \mathrm{e}^{-x} x + 2\,\mathrm{e}^{-x})}{6}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]-(x^2+4*x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^x x^4 - \frac{1}{6} c_1 x \left(e^x x^3 \operatorname{ExpIntegralEi}(-x) + x^2 - x + 2\right)$$

2.266 problem 269

Internal problem ID [7756]
Internal file name [`OUTPUT/6689_Sunday_June_05_2022_05_05_44_PM_78618439/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 269.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0}$$

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2 - \frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2 - \frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 36x + 4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 36x + 4$$
$$t = 16x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{9}{4x^3} + \frac{1}{4x^4} + \frac{5}{16x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x^2} - \frac{5}{4x} + (-)(0) \\
&= -\frac{1}{2x^2} - \frac{5}{4x} \\
&= \frac{-5x-2}{4x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2 + 36x + 4}{16x^4}\right)\right) = 0$$

$$\frac{-2 + 5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{5}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{5}\right) \mathrm{e}^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right) dx} \\
&= \left(x + \frac{2}{5}\right) \mathrm{e}^{\frac{1}{2x} - \frac{5 \ln(x)}{4}} \\
&= \frac{(5x + 2)\, \mathrm{e}^{\frac{1}{2x}}}{5x^{\frac{5}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x-2}{2x^2} \, dx} \\
&= z_1 e^{-\frac{1}{2x} + \frac{3 \ln(x)}{4}} \\
&= z_1 \left(x^{\frac{3}{4}} \mathrm{e}^{-\frac{1}{2x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{5x + 2}{5\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{1}{x}+\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{25x^{\frac{5}{2}}\mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{5x+2}{5\sqrt{x}}\right) + c_2 \left(\frac{5x+2}{5\sqrt{x}} \left(\int \frac{25x^{\frac{5}{2}}\mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(5x+2)}{5\sqrt{x}} + \frac{c_2(25x+10)\left(\int \frac{x^{\frac{5}{2}}\mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(5x+2)}{5\sqrt{x}} + \frac{c_2(25x+10)\left(\int \frac{x^{\frac{5}{2}}\mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(2*x^2*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)+(2*x-1)/x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(5x+2)}{\sqrt{x}} + \frac{c_2(5x+2)\left(\int \frac{x^{\frac{5}{2}} \mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.225 (sec). Leaf size: 70

```
DSolve[2*x^2*y''[x]-(3*x+2)*y'[x]+(2*x-1)/x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2\sqrt{\pi}c_2(5x+2)\operatorname{erf}\left(\frac{1}{\sqrt{x}}\right)}{3\sqrt{x}} + \frac{2}{3}c_2e^{-1/x}\left(x^2-4x-2\right) + \frac{c_1(5x+2)}{5\sqrt{x}}$$

2.267 problem 270

2.267.1 Maple step by step solution 2593

Internal problem ID [7757]
Internal file name [OUTPUT/6690_Sunday_June_05_2022_05_05_50_PM_90205707/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 270.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\,y''+\left(-2x+\frac{3}{2}\right)y'-\frac{y}{4}=0}$$

Writing the ode as

$$\left(-x^{2}+x\right)y''+\left(-2x+\frac{3}{2}\right)y'-\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^{2}+x \\ B &= -2x+\frac{3}{2} \\ C &= -\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 4x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^2 + 4x - 3$$
$$t = 16\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 4x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 510: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} - \frac{3}{16(x-1)^2} - \frac{3}{16x^2} + \frac{1}{8x-8}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-4x^2 + 4x - 3}{16\left(x^2 - x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + \frac{1}{4x-4} + (-)(0) \\
&= \frac{1}{4x} + \frac{1}{4x-4} \\
&= \frac{2x-1}{4x(x-1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4x} + \frac{1}{4x-4}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4(x-1)^2}\right) + \left(\frac{1}{4x} + \frac{1}{4x-4}\right)^2 - \left(\frac{-4x^2+4x-3}{16(x^2-x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{4x} + \frac{1}{4x-4}\right) dx} \\
&= x^{\frac{1}{4}} (x-1)^{\frac{1}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x+\frac{3}{2}}{-x^2+x}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+\frac{3}{2}}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{\ln(x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x(x-1)}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}}\right) + c_2\left(\frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x(x-1)}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}} + \frac{c_2(x(x-1))^{\frac{1}{4}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x(x-1)}\right)\right)}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}} + \frac{c_2(x(x-1))^{\frac{1}{4}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x(x-1)}\right)\right)}{x^{\frac{3}{4}}(x-1)^{\frac{1}{4}}}$$

Verified OK.

2.267.1 Maple step by step solution

Let's solve

$\left(-x^2+x\right) y''+\left(-2x+\frac{3}{2}\right) y'-\frac{y}{4}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x-3)y'}{2x(x-1)}-\frac{y}{4x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x-3)y'}{2x(x-1)}+\frac{y}{4x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{4x-3}{2x(x-1)}, P_3(x)=\frac{1}{4x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{3}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $4y''x(x-1)+(8x-6)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $$x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_{0}r(1+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(2k+3+2r)+a_{k}(2k+2r+1)^{2}\right)x^{k+r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-2r(1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,-\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k}(2k+2r+1)^{2}-4a_{k+1}\left(k+\frac{3}{2}+r\right)(k+1+r)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{a_{k}(2k+2r+1)^{2}}{2(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{a_{k}(2k+1)^{2}}{2(2k+3)(k+1)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+1}=\frac{a_{k}(2k+1)^{2}}{2(2k+3)(k+1)}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

  $$a_{k+1}=\frac{2a_{k}k^{2}}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r=-\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-\frac{1}{2}},a_{k+1}=\frac{2a_{k}k^{2}}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = \frac{a_k(2k+1)^2}{2(2k+3)(k+1)}, b_{k+1} = \frac{2b_k k^2}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*(1-x)*diff(y(x),x$2)+(3/2-2*x)*diff(y(x),x)-1/4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 \ln\left(x - \frac{1}{2} + \sqrt{x(x-1)}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 51

```
DSolve[x*(1-x)*y''[x]+(3/2-2*x)*y'[x]-1/4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x}} - \frac{2c_2\sqrt{x-1}\log\left(\sqrt{x-1} - \sqrt{x}\right)}{\sqrt{-((x-1)x)}}$$

2.268 problem 271

Internal problem ID [7758]
Internal file name [`OUTPUT/6691_Sunday_June_05_2022_05_05_53_PM_35930984/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 271.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$\left(-2x^2+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-2x^2+2x\\B&=x\\C&=-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x + 8}{16x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x + 8$$
$$t = 16x(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x + 8}{16x(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 512: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} + \frac{5}{16(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x+8}{16x(x-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x+8}{16x\left(x-1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{3}{4}-\left(\frac{3}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{4(x-1)} + (0) \\
&= \frac{1}{x} - \frac{1}{4(x-1)} \\
&= \frac{1}{x} - \frac{1}{4x-4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{x} - \frac{1}{4(x-1)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(x-1)}\right)^2 - \left(\frac{-3x+8}{16x(x-1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4(x-1)}\right) dx} \\
&= \frac{x}{(x-1)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{-2x^2+2x} \, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{4}} \\
&= z_1 \left((x-1)^{\frac{1}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}}{x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2\left(\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2\left(\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(2*x*(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2\left(\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}\right)$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 43

```
DSolve[2*x*(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[4]{2}\left(c_2 x \operatorname{arctanh}\left(\sqrt{1-x}\right) + c_1 x - c_2\sqrt{1-x}\right)$$

2.269 problem 272

2.269.1 Maple step by step solution 2609

Internal problem ID [7759]
Internal file name [`OUTPUT/6692_Sunday_June_05_2022_05_05_56_PM_81047170/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 272.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{2x(1-x)\,y''+(1-11x)\,y'-10y=0}$$

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^2+2x \\ B &= 1-11x \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 + 66x - 3 \\ t &= 16\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 513: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x} + \frac{15}{4(x-1)^2} - \frac{3}{16x^2} - \frac{15}{4(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2+66x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{3}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{3}{2\left(x-1\right)} \\
&= -\frac{3(1+x)}{4x\left(x-1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4x} - \frac{3}{2(x-1)}\right)(1) + \left(\left(-\frac{3}{4x^2} + \frac{3}{2(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{3}{2(x-1)}\right)^2 - \left(\frac{-3x^2 + 66x - 3}{16(x^2 - x)^2}\right)\right) = \frac{-3 + 3a_0}{2x(x-1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(\frac{3}{4x} - \frac{3}{2(x-1)}\right) dx} \\
&= (1 + x)\, e^{\frac{3\ln(x)}{4} - \frac{3\ln(x-1)}{2}} \\
&= \frac{(1 + x)\, x^{\frac{3}{4}}}{(x-1)^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-11x}{-2x^2+2x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} - \frac{5\ln(x-1)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{4}} (x-1)^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)\sqrt{x}}{(x-1)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-11x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-5\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2x^2+12x+2}{(1+x)\sqrt{x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)\sqrt{x}}{(x-1)^4}\right) + c_2 \left(\frac{(1+x)\sqrt{x}}{(x-1)^4} \left(\frac{2x^2+12x+2}{(1+x)\sqrt{x}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(1+x)\sqrt{x}}{(x-1)^4} + \frac{c_2(2x^2+12x+2)}{(x-1)^4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(1+x)\sqrt{x}}{(x-1)^4} + \frac{c_2(2x^2+12x+2)}{(x-1)^4}$$

Verified OK.

2.269.1 Maple step by step solution

Let's solve

$(-2x^2+2x)\, y''+(1-11x)\, y'-10y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-1+11x)y'}{2x(x-1)}-\frac{5y}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-1+11x)y'}{2x(x-1)}+\frac{5y}{x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{-1+11x}{2x(x-1)}, P_3(x)=\frac{5}{x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-1+11x)\, y'+10y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y'$ to series expansion for $m=0..1$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r+5)\,(k+r+2)\right) x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2\left(k+r+\frac{5}{2}\right) a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(2k+2r+5)a_k(k+r+2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = \frac{(2k+6)a_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{(2k+6)a_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=\frac{(2k+5)a_k(k+2)}{(2k+1)(k+1)}, b_{k+1}=\frac{(2k+6)b_k\left(k+\frac{5}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(2*x*(1-x)*diff(y(x),x$2)+(1-11*x)*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x^2+6x+1)}{(x-1)^4}+\frac{c_2\sqrt{x}\,(x+1)}{(x-1)^4}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 35

```
DSolve[2*x*(1-x)*y''[x]+(1-11*x)*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{c_1\sqrt{x}(x+1)-2c_2(x^2+6x+1)}{(x-1)^4}$$

2.270 problem 273

2.270.1 Maple step by step solution 2618

Internal problem ID [7760]
Internal file name [OUTPUT/6693_Sunday_June_05_2022_05_05_58_PM_15972581/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 273.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

[_Jacobi]

$$\boxed{x(1-x)\, y''+\frac{(1-2x)\, y'}{3}+\frac{20y}{9}=0}$$

Writing the ode as

$$\left(-x^{2}+x\right) y''+\left(-\frac{2x}{3}+\frac{1}{3}\right) y'+\frac{20y}{9}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^{2}+x \\ B &= -\frac{2x}{3}+\frac{1}{3} \\ C &= \frac{20}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 72x^2 - 72x - 5$$
$$t = 36\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 515: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{41}{18x} - \frac{5}{36\left(x-1\right)^2} - \frac{5}{36x^2} + \frac{41}{18\left(x-1\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{72x^2 - 72x - 5}{36\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + \frac{5}{6\left(x-1\right)} + (0) \\
&= \frac{1}{6x} + \frac{5}{6\left(x-1\right)} \\
&= \frac{6x-1}{6x\left(x-1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{5}{6\left(x-1\right)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6\left(x-1\right)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6\left(x-1\right)}\right)^2 - \left(\frac{72x^2-72x-5}{36\left(x^2-x\right)^2}\right)\right) = \frac{-1-6a_0}{3x\left(x-1\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{6}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{1}{6}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x - \frac{1}{6}\right) e^{\int \left(\frac{1}{6x} + \frac{5}{6(x-1)}\right) dx} \\
&= \left(x - \frac{1}{6}\right) e^{\frac{\ln(x)}{6} + \frac{5\ln(x-1)}{6}} \\
&= \left(x - \frac{1}{6}\right) x^{\frac{1}{6}} \left(x-1\right)^{\frac{5}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x-1))}{6}} \\
&= z_1 \left(\frac{1}{(x(x-1))^{\frac{1}{6}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x(x-1))^{\frac{1}{6}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x-1))}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(-324x+270)\,x^{\frac{2}{3}}}{(x-1)^{\frac{2}{3}}\,(30x-5)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x(x-1))^{\frac{1}{6}}}\right) + c_2 \left(\frac{(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x(x-1))^{\frac{1}{6}}}\left(\frac{(-324x+270)\,x^{\frac{2}{3}}}{(x-1)^{\frac{2}{3}}\,(30x-5)}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x(x-1))^{\frac{1}{6}}} - \frac{9c_2(6x-5)\,(x-1)^{\frac{1}{6}}\,x^{\frac{5}{6}}}{5\,(x(x-1))^{\frac{1}{6}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x\,(x-1))^{\frac{1}{6}}} - \frac{9c_2(6x-5)\,(x-1)^{\frac{1}{6}}\,x^{\frac{5}{6}}}{5\,(x\,(x-1))^{\frac{1}{6}}}$$

Verified OK.

2.270.1 Maple step by step solution

Let's solve

$(-x^2+x)\,y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(2x-1)y'}{3x(x-1)}+\frac{20y}{9x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x-1)y'}{3x(x-1)}-\frac{20y}{9x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2x-1}{3x(x-1)}, P_3(x)=-\frac{20}{9x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=\frac{1}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9y''x(x-1)+(6x-3)\,y'-20y=0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ○ Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$
  - ○ Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0 r(-2+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)(3k+1+3r) + a_k(3k+3r+4)(3k+3r-5$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(-2+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{2}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-9\left(k+\frac{1}{3}+r\right)(k+1+r)\, a_{k+1} + 9\left(k+r-\frac{5}{3}\right)\left(k+r+\frac{4}{3}\right) a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(3k+3r-5)(3k+3r+4)a_k}{3(3k+1+3r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}\right]$
- Recursion relation for $r = \frac{2}{3}$; series terminates at $k = 1$

$a_{k+1} = \frac{(3k-3)(3k+6)a_k}{3(3k+3)\left(k+\frac{5}{3}\right)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{6a_0}{5}$

- Terminating series solution of the ODE for $r = \frac{2}{3}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(-\frac{6x}{5} + 1\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(-\frac{6x}{5} + 1\right), a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*(1-x)*diff(y(x),x$2)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(6x-5)\,x^{\frac{2}{3}} + c_2(6x-1)\,(x-1)^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 51

```
DSolve[x*(1-x)*y''[x]+1/3*(1-2*x)*y'[x]+20/9*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_2 \sqrt[3]{-((x-1)x)} Q_1^{\frac{2}{3}}(2x-1) + \frac{c_1 x^{2/3}(6x-5)}{3\,\text{Gamma}\left(\frac{4}{3}\right)}$$

2.271 problem 274

2.271.1 Maple step by step solution 2628

Internal problem ID [7761]
Internal file name [OUTPUT/6694_Sunday_June_05_2022_05_06_01_PM_22895960/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 274.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y''+\frac{3(-x^2+2)\,y}{(1-x^2)^2}=0}$$

Writing the ode as

$$4y''+\frac{(-3x^2+6)\,y}{(x^2-1)^2}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4\\B&=0\\C&=\frac{-3x^2+6}{(x^2-1)^2}\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int\frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 6$$
$$t = 4(x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 517: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(1+x)^2} - \frac{3}{16\,(x-1)^2} + \frac{9}{16\,(x-1)} - \frac{9}{16\,(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6}{4\,(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3x^2-6}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^+\right)\\
&= \frac{3}{2}-\left(\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{3}{4(x-1)} + \frac{3}{4(1+x)} + (0) \\ &= \frac{3}{4(x-1)} + \frac{3}{4(1+x)} \\ &= \frac{3x}{2x^2-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right)(0) + \left(\left(-\frac{3}{4(x-1)^2} - \frac{3}{4(1+x)^2}\right) + \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right)^2 - \left(\frac{3}{4(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right)dx} \\ &= (x-1)^{\frac{3}{4}}(1+x)^{\frac{3}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= (x-1)^{\frac{3}{4}}(1+x)^{\frac{3}{4}}\end{aligned}$$

Which simplifies to

$$y_1 = (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \int \frac{1}{(x-1)^{\frac{3}{2}} (1+x)^{\frac{3}{2}}}\, dx \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}\right) + c_2\left((x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} - c_2 (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}} x \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} - c_2 (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}} x$$

Verified OK.

2.271.1 Maple step by step solution

Let's solve

$4y'' + \frac{(-3x^2+6)y}{(x^2-1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x^2-2)y}{4(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(x^2-2)y}{4(x^2-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 0$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = \frac{3}{16}$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4y''(x^2-1)^2 + (-3x^2+6)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^4 - 16u^3 + 16u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-3u^2 + 6u + 3)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 2..4$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+4r)(-3+4r) u^r + \left(a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3)\right) u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r -$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+4r)(-3+4r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$$

- Each term must be 0

  $$a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$$

- Each term in the series must be 0, giving the recursion relation

  $$4(4a_k + a_{k-2} - 4a_{k-1}) k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1}) r - 4a_k - 5a_{k-2} + 12a_{k-1}) k + 4(4a_k + a_{k-2}$$

- Shift index using $k- > k + 2$

  $$4(4a_{k+2} + a_k - 4a_{k+1}) (k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1}) r - 4a_{k+2} - 5a_k + 12a_{k+1}) (k+2) + 4$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2 a_k - 16r^2 a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2 + 32kr + 16r^2 + 48k + 48r + 35}$$

- Recursion relation for $r = \frac{1}{4}$

  $$a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48}$$

- Solution for $r = \frac{1}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}$$

- Solution for $r = \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15 a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48},\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(4*diff(y(x),x$2)+3*(2-x^2)/(1-x^2)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 (x^2 - 1)^{\frac{1}{4}} x + c_2 (x^2 - 1)^{\frac{3}{4}}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 51

```
DSolve[4*y''[x]+3*(2-x^2)/(1-x^2)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2 - 1} \left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x) + \frac{\sqrt{\frac{2}{\pi}} c_1 x}{\sqrt[4]{1 - x^2}} \right)$$

2.272 problem 275

2.272.1 Maple step by step solution 2639

Internal problem ID [7762]
Internal file name [OUTPUT/6695_Sunday_June_05_2022_05_06_04_PM_80394104/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 275.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' - \frac{2u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 519: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-xa-1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right)\right) = 0
$$

$$
\frac{2aa_0 - 2}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-xa - \ln(x)} \\
&= \frac{(xa+1)\, e^{-xa}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2}{x}}{1}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa+1)\,\mathrm{e}^{-xa}}{a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1u_1 + c_2u_2 \\
&= c_1\left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a}\right) + c_2\left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a}\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2} \tag{1}
$$

Verification of solutions

$$
u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2}
$$

Verified OK.

2.272.1 Maple step by step solution

Let's solve

$u'' - \frac{2u'}{x} - a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  u''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2 u x + u'' x - 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k - 1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert u' to series expansion

$$u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

$$x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(k+r-1) - a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$

- Solution for $r = 3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(u(x),x$2)-2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = c_1 e^{ax}(ax - 1) + \frac{c_2 e^{-ax}(ax + 1)}{a}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 68

```
DSolve[u''[x]-2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

2.273 problem 276

2.273.1 Maple step by step solution 2646

Internal problem ID [7763]
Internal file name [OUTPUT/6696_Sunday_June_05_2022_05_06_08_PM_14447855/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 276.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{2u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 521: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{e^{ax\, \mathrm{csgn}(a)}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1 \left(-\frac{\mathrm{csgn}(a)\, e^{-2ax\, \mathrm{csgn}(a)}}{2a}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{e^{ax \operatorname{csgn}(a)}}{x} \right) + c_2 \left(\frac{e^{ax \operatorname{csgn}(a)}}{x} \left(-\frac{\operatorname{csgn}(a)\, e^{-2ax \operatorname{csgn}(a)}}{2a} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are tl

Simplifying the solution $u = \frac{c_1 e^{ax \operatorname{csgn}(a)}}{x} - \frac{c_2 \operatorname{csgn}(a) e^{-ax \operatorname{csgn}(a)}}{2ax}$ to $u = \frac{c_1 e^{xa}}{x} - \frac{c_2 e^{-xa}}{2ax}$

Verification of solutions

$$
u = \frac{c_1 e^{xa}}{x} - \frac{c_2 e^{-xa}}{2ax}
$$

Verified OK.

2.273.1 Maple step by step solution

Let's solve

$u'' + \frac{2u'}{x} - a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + u''x + 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

  - Convert $x \cdot u''$ to series expansion

    $x \cdot u'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)\,(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) - a^2 a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$

  $a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}$
- Solution for $r = -1$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}$
- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sinh(ax)}{x} + \frac{c_2 \cosh(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 35

```
DSolve[u''[x]+2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{2ac_1 e^{-ax} + c_2 e^{ax}}{2ax}$$

2.274 problem 277

2.274.1 Maple step by step solution 2653

Internal problem ID [7764]
Internal file name [`OUTPUT/6697_Sunday_June_05_2022_05_06_10_PM_17867104/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 277.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} + a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -a^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx
$$

Substituting gives

$$u_2 = u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1} dx}}{(u_1)^2} dx$$

$$= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2} dx$$

$$= u_1 \left(\frac{\sqrt{-a^2}\, e^{-2\sqrt{-a^2}\, x}}{2a^2} \right)$$

Therefore the solution is

$$u = c_1 u_1 + c_2 u_2$$

$$= c_1 \left(\frac{e^{\sqrt{-a^2}\, x}}{x} \right) + c_2 \left(\frac{e^{\sqrt{-a^2}\, x}}{x} \left(\frac{\sqrt{-a^2}\, e^{-2\sqrt{-a^2}\, x}}{2a^2} \right) \right)$$

Summary

The solution(s) found are the following

$$u = \frac{c_1 e^{\sqrt{-a^2}\, x}}{x} + \frac{c_2 \sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2 x} \tag{1}$$

Verification of solutions

$$u = \frac{c_1 e^{\sqrt{-a^2}\, x}}{x} + \frac{c_2 \sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x}}{2a^2 x}$$

Verified OK.

2.274.1 Maple step by step solution

Let's solve

$u'' + \frac{2u'}{x} + a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  u''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2ux + u''x + 2u' = 0$

- Assume series solution for u

  $$u = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot u$ to series expansion

  $$x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

  $$u' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$u' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

  $$x \cdot u'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)\,(2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k+2+r) + a^2 a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)\,(k+3+r) + a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[u = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a^2 a_k}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k+1)(k+2)}, 0 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sin(ax)}{x} + \frac{c_2 \cos(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 42

```
DSolve[u''[x]+2/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2 e^{2iax}}{a}\right)}{2x}$$

2.275 problem 278

2.275.1 Maple step by step solution 2664

Internal problem ID [7765]
Internal file name [`OUTPUT/6698_Sunday_June_05_2022_05_06_13_PM_77515274/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 278.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{4u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 525: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{x} + (-)(a) \\ &= -\frac{1}{x} - a \\ &= \frac{-xa - 1}{x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2 + 2}{x^2}\right)\right) = 0$$

$$\frac{2aa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{a}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{a}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\ &= \left(x + \frac{1}{a}\right) e^{-xa - \ln(x)} \\ &= \frac{(xa + 1)\, e^{-xa}}{ax}\end{aligned}$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}\right) + c_2 \left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1 (xa+1)\,\mathrm{e}^{-xa}}{a\,x^3} + \frac{c_2 (xa-1)\,\mathrm{e}^{xa}}{2a^2 x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2x^3}$$

Verified OK.

2.275.1 Maple step by step solution

Let's solve

$u'' + \frac{4u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
- Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$
- Shift index using $k->k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$
- Convert $x \cdot u''$ to series expansion

  $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$
- Shift index using $k->k+1$

  $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) (k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r) x^{-1+r} + a_1 (1+r) (4+r) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1) (k+4+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$
- Each term must be 0

  $a_1 (1+r) (4+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1) (k+4+r) - a^2 a_{k-1} = 0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r) (k+5+r) - a^2 a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$
- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

$$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 e^{ax}(ax-1)}{x^3} + \frac{c_2 e^{-ax}(ax+1)}{x^3 a}$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 68

```
DSolve[u''[x]+4/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

2.276 problem 279

2.276.1 Maple step by step solution 2675

Internal problem ID [7766]
Internal file name [`OUTPUT/6699_Sunday_June_05_2022_05_06_16_PM_18009554/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 279.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{u'' + \frac{4u'}{x} + a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{4u'}{x} + a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 527: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2 \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= ia\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &&= 0\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= 0 - (-1)\\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(ia) \\
&= -\frac{1}{x} - ia \\
&= -\frac{1}{x} - ia
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2iaa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{i}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{i}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) \mathrm{e}^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) \mathrm{e}^{-ixa - \ln(x)} \\
&= \frac{(xa - i)\, \mathrm{e}^{-ixa}}{xa}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa-i)\,\mathrm{e}^{-ixa}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(ixa-1)\,\mathrm{e}^{2ixa}}{2a\,(-xa+i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(xa-i)\,\mathrm{e}^{-ixa}}{x^3 a}\right) + c_2 \left(\frac{(xa-i)\,\mathrm{e}^{-ixa}}{x^3 a}\left(\frac{(ixa-1)\,\mathrm{e}^{2ixa}}{2a\,(-xa+i)}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
u = \frac{c_1 (xa-i)\,\mathrm{e}^{-ixa}}{x^3 a} - \frac{c_2 (ixa-1)\,\mathrm{e}^{ixa}}{2a^2 x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(xa - i)\,\mathrm{e}^{-ixa}}{x^3 a} - \frac{c_2(ixa - 1)\,\mathrm{e}^{ixa}}{2a^2x^3}$$

Verified OK.

2.276.1 Maple step by step solution

Let's solve

$u'' + \frac{4u'}{x} + a^2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- □ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 4$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

- Convert $x \cdot u''$ to series expansion

  $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, x^{-1+r} + a_1 (1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)\,(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) + a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = -\frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 41

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1(\cos(ax)\,ax - \sin(ax))}{x^3} + \frac{c_2(\cos(ax) + \sin(ax)\,ax)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 57

```
DSolve[u''[x]+4/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1x + c_2)\cos(ax) + (ac_2x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

2.277 problem 280

2.277.1 Maple step by step solution 2686

Internal problem ID [7767]
Internal file name [OUTPUT/6700_Sunday_June_05_2022_05_06_19_PM_91803771/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 280.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - a^2 y - \frac{6y}{x^2} = 0}$$

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2x^2 + 6}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2x^2 + 6 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2x^2 + 6}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 529: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= \left(a^2\right) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= a\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right)\\
&= 0 - (-2)\\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(a) \\
&= -\frac{2}{x} - a \\
&= \frac{-xa - 2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2 + 6}{x^2}\right)\right) &= 0 \\
\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-xa - 2\ln(x)} \\
&= \frac{(a^2x^2 + 3xa + 3)\, e^{-xa}}{a^2x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3xa+3)^2\mathrm{e}^{-2xa}}{a^4x^4}}\,dx \\
&= \frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2} \left(\frac{(a^2x^2 - 3xa + 3)\,\mathrm{e}^{2xa}}{2a\,(a^2x^2 + 3xa + 3)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2}\right) + c_2\left(\frac{(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2}\left(\frac{(a^2x^2 - 3xa + 3)\,\mathrm{e}^{2xa}}{2a\,(a^2x^2 + 3xa + 3)}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2} + \frac{c_2\mathrm{e}^{xa}(a^2x^2 - 3xa + 3)}{2a^3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(a^2x^2 + 3xa + 3)\,\mathrm{e}^{-xa}}{a^2x^2} + \frac{c_2\mathrm{e}^{xa}(a^2x^2 - 3xa + 3)}{2a^3x^2}$$

Verified OK.

2.277.1 Maple step by step solution

Let's solve

$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^2+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -6$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + (-a^2x^2 - 6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-3+r) x^r + a_1(3+r)(-2+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-3) - a^2 a_{k-2}) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) - a^2 a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+4+r)(k+r-1) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-3)}$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-3)}, a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{a^2 a_k}{(k+7)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+7)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-3)}, b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+7)(k+2)}, c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-a^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-ax}(a^2x^2 + 3ax + 3)}{x^2a^2} + \frac{c_2 e^{ax}(a^2x^2 - 3ax + 3)}{3x^2}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 90

```
DSolve[y''[x]-a^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}\left(\left(a^2c_2x^2 - 3iac_1x + 3c_2\right)\cosh(ax) + i\left(c_1\left(a^2x^2+3\right) + 3iac_2x\right)\sinh(ax)\right)}{a^2x^{3/2}\sqrt{-iax}}$$

2.278 problem 281

2.278.1 Maple step by step solution 2697

Internal problem ID [7768]
Internal file name [`OUTPUT/6701_Sunday_June_05_2022_05_06_23_PM_26510423/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 281.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + n^2 y - \frac{6y}{x^2} = 0}$$

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -n^2x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 531: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -n^2 + \frac{6}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= in \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{in} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-n^2x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(in) \\
&= -\frac{2}{x} - in \\
&= -\frac{2}{x} - in
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{(2ina_1 - 6)\,x + 4ina_0 - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{\int \left(-\frac{2}{x} - in\right) dx} \\
&= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{-inx - 2\ln(x)} \\
&= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2 n^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2-3inx-3)^2\mathrm{e}^{-2inx}}{x^4n^4}}\, dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3xn - 3i)\,\mathrm{e}^{2inx}}{6n\left(-\frac{1}{3}n^2x^2 + inx + 1\right)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\right) \\ &\quad + c_2\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3xn - 3i)\,\mathrm{e}^{2inx}}{6n\left(-\frac{1}{3}n^2x^2 + inx + 1\right)}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3xn - 3i)}{2n^3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3xn - 3i)}{2n^3x^2}$$

Verified OK.

2.278.1 Maple step by step solution

Let's solve

$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(n^2x^2-6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(n^2x^2-6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -6$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$x^2y'' + (n^2x^2 - 6)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-3+r)\, x^r + a_1(3+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-3) + n^2 a_{k-2})\, x^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 3\}$$

- Each term must be 0

$$a_1(3+r)\,(-2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+2)\,(k+r-3) + n^2 a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}(k+4+r)\,(k+r-1) + n^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{n^2 a_k}{(k+4+r)(k+r-1)}$$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}$

- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0\right]$$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{n^2 b_k}{(k+7)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 67

```
dsolve(diff(y(x),x$2)+n^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(\cos(nx)\,x^2n^2 - 3\sin(nx)\,nx - 3\cos(nx))}{x^2} + \frac{c_2(\sin(nx)\,x^2n^2 + 3\cos(nx)\,nx - 3\sin(nx))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 79

```
DSolve[y''[x]+n^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)\,x^2 + 3c_1nx + 3c_2)\cos(nx) + (c_1(n^2x^2 - 3) + 3c_2nx)\sin(nx))}{(nx)^{5/2}}$$

2.279 problem 282

2.279.1 Maple step by step solution 2704

Internal problem ID [7769]
Internal file name [OUTPUT/6702_Sunday_June_05_2022_05_06_27_PM_66881080/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 282.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right) y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 533: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-x}}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{-x}}{\sqrt{x}} + \frac{c_2 e^{x}}{2\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^{-x}}{\sqrt{x}} + \frac{c_2 e^{x}}{2\sqrt{x}}
$$

Verified OK.

2.279.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(-x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (-4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k + 2)^2 + 8(k + 2)\, r + 4r^2 - 1\right) - 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = \frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{c_1 \sinh(x)}{\sqrt{x}} + \frac{c_2 \cosh(x)}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+x*y'[x]-(x^2+1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

2.280 problem 283

2.280.1 Maple step by step solution 2715

Internal problem ID [7770]
Internal file name [`OUTPUT/6703_Sunday_June_05_2022_05_06_29_PM_26849047/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 283.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{a^2 x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2a^2 - x^2$$
$$t = a^2 x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{a^2 x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 535: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = a^2x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{i}{a}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2a^2 - x^2}{a^2x^2} \\
&= Q + \frac{R}{a^2x^2} \\
&= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\
&= -\frac{1}{a^2} + \frac{2}{x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{a^2 x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\ &= -\frac{1}{x} - \frac{i}{a} \\ &= -\frac{ix + a}{xa}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{a^2 x^2}\right)\right) &= 0 \\ \frac{2ia_0 - 2a}{xa} &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -ia\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -ia + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (-ia + x)\, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\ &= (-ia + x)\, \mathrm{e}^{-\frac{ix}{a} - \ln(x)} \\ &= \frac{(-ia + x)\, \mathrm{e}^{-\frac{ix}{a}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia+x)}{2ix+2a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}\left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia+x)}{2ix+2a}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix-a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix-a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}}$$

Verified OK.

2.280.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(9a^2-4x^2)y}{4a^2x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(9a^2-4x^2)y}{4a^2x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y''a^2 + 4xy'a^2 - (9a^2-4x^2)\,y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)a^2+4\left(\frac{d}{dt}y(t)\right)a^2-\left(9a^2-4x^2\right)y(t)=0$$

- Simplify

$$4a^2\left(\frac{d^2}{dt^2}y(t)\right)-9y(t)\,a^2+4y(t)\,x^2=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=\frac{\left(9a^2-4x^2\right)y(t)}{4a^2}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)-\frac{\left(9a^2-4x^2\right)y(t)}{4a^2}=0$$

- Characteristic polynomial of ODE

$$r^2-\frac{9a^2-4x^2}{4a^2}=0$$

- Factor the characteristic polynomial

$$\frac{4r^2a^2-9a^2+4x^2}{4a^2}=0$$

- Roots of the characteristic polynomial

$$r=\left(\frac{\sqrt{9a^2-4x^2}}{2a},-\frac{\sqrt{9a^2-4x^2}}{2a}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$$

- Simplify

$$y=c_1x^{\frac{\sqrt{9a^2-4x^2}}{2a}}+c_2x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(4*x^2-9*a^2)/(4*a^2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \mathrm{e}^{\frac{ix}{a}}(-ix+a)}{x^{\frac{3}{2}}}+\frac{c_2 \mathrm{e}^{-\frac{ix}{a}}(ix+a)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 62

```
DSolve[x^2*y''[x]+x*y'[x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2+c_1x)\cos\left(\frac{x}{a}\right)+(c_2x-ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

2.281 problem 284

2.281.1 Maple step by step solution 2725

Internal problem ID [7771]
Internal file name [`OUTPUT/6704_Sunday_June_05_2022_05_06_34_PM_59610394/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 284.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 537: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) &= 0 \\
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) e^{-ix - 2\ln(x)} \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2\mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}$$

Verified OK.

2.281.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{25}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(5 + 2r)\,(-5 + 2r)\, x^r + a_1(7 + 2r)\,(-3 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 5)\,(2k + 2r - 5) + 4a\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(5 + 2r)\,(-5 + 2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

$$a_1(7 + 2r)\,(-3 + 2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k + 2r + 5)\,(2k + 2r - 5) + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$a_{k+2}(2k + 9 + 2r)\,(2k - 1 + 2r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$$

- Recursion relation for $r=-\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r=-\frac{5}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$$
- Recursion relation for $r=\frac{5}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r=\frac{5}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+14)(2k+4)}, a_1=0\right]$$
- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+14)(2k+4)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/4)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 e^{ix}(x^2+3ix-3)}{x^{\frac{5}{2}}}+\frac{c_2 e^{-ix}(x^2-3ix-3)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x))}{x^{5/2}}$$

2.282 problem 285

2.282.1 Maple step by step solution 2736

Internal problem ID [7772]
Internal file name [OUTPUT/6705_Sunday_June_05_2022_05_06_37_PM_82143151/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 285.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + qy' - \frac{2y}{x^2} = 0}$$

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2q^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2q^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2q^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 539: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{q^2}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3x^4} + \frac{16}{q^5x^6} - \frac{80}{q^7x^8} + \frac{448}{q^9x^{10}} - \frac{2688}{q^{11}x^{12}} + \frac{16896}{q^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2q^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2q^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx+2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{x^2q^2+8}{4x^2}\right)\right) = 0$$

$$\frac{qa_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{q}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{q}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\
&= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\
&= \frac{(qx+2)\, e^{-\frac{qx}{2}}}{qx}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{q}{1}\,dx} \\
&= z_1 e^{-\frac{qx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{qx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-qx}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\right) + c_2 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2 x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2x}$$

Verified OK.

2.282.1 Maple step by step solution

Let's solve

$y'' + qy' - \frac{2y}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $qy'x^2 + x^2y'' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+1}$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k-1+r) + q a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{q a_k (k+r)}{(k+2+r)(k-1+r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{q a_k (k-1)}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{q a_0}{2}$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k (k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(-\frac{qx}{2}+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+q*diff(y(x),x)=2*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(qx-2)}{x}+\frac{c_2 \mathrm{e}^{-qx}(qx+2)}{qx}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 80

```
DSolve[y''[x]+q*y'[x]==2*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{qx^{3/2}e^{-\frac{qx}{2}}\left(2(ic_2qx+2c_1)\sinh\left(\frac{qx}{2}\right)-2(c_1qx+2ic_2)\cosh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi}(-iqx)^{5/2}}$$

2.283 problem 286

2.283.1 Maple step by step solution 2745

Internal problem ID [7773]
Internal file name [OUTPUT/6706_Sunday_June_05_2022_05_06_40_PM_34009668/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 286.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 3y' + 4yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -16x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 541: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{x} \, dx} \\ &= z_1 e^{-\frac{3 \ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-ix^2}}{x^2} - \frac{i c_2 \mathrm{e}^{ix^2}}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-ix^2}}{x^2} - \frac{i c_2 \mathrm{e}^{ix^2}}{4x^2}$$

Verified OK.

2.283.1 Maple step by step solution

Let's solve

$xy'' + 3y' + 4yx^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + 4x^2 y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $xy'' + 3y' + 4yx^3 = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - Shift index using $k- >k - 3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$
  - Shift index using $k- >k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r) x^{-1+r} + a_1(1+r)(3+r) x^r + a_2(2+r)(4+r) x^{1+r} + a_3(3+r)(5+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+3) + 4a_{k-3} = 0$$

- Shift index using $k->k+3$

$$a_{k+4}(k+4+r)(k+6+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$$a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin \left(x^2\right)}{x^2}+\frac{c_2 \cos \left(x^2\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 41

```
DSolve[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.284 problem 287

2.284.1 Maple step by step solution 2754

Internal problem ID [7774]
Internal file name [`OUTPUT/6707_Sunday_June_05_2022_05_06_43_PM_73186776/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 287.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2x)\,y''-2(1+x)\,y'+2y=0}$$

Writing the ode as

$$(x^2+2x)\,y''+(-2x-2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= -2x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 543: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2} + \frac{3}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} \\
&= \frac{x+3}{x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x^2+2x} \, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1 \left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{3}{2}} \sqrt{x(x+2)}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-x-1}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\right) + c_2\left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\left(\frac{-x-1}{x^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}
$$

Verified OK.

2.284.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y''+(-2x-2)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x+2)}+\frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{2(1+x)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = -\frac{2(1+x)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$$

- $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $$\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = -1$$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $$x_0 = -2$$

- Multiply by denominators

  $$y''x(x+2) + (-2x-2)\, y' + 2y = 0$$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $$(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 a_{0} r \left(-2+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(-2 a_{k +1} \left(k +1+r \right) \left(k +r -1\right)+a_{k} \left(k +r -1\right) \left(k +r -2\right)\right) u^{k +r}\right)$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2 r \left(-2+r \right) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, 2\right\}$
- Each term in the series must be 0, giving the recursion relation

  $\left(k +r -1\right) \left(\left(-2 k -2 r -2\right) a_{k +1}+a_{k} \left(k +r -2\right)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k +1} = \frac{a_{k} \left(k +r -2\right)}{2 \left(k +1+r \right)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k +1} = \frac{a_{k} \left(k -2\right)}{2 \left(k +1\right)}$
- Apply recursion relation for $k = 0$

  $a_{1} = -a_{0}$
- Apply recursion relation for $k = 1$

  $a_{2} = -\frac{a_{1}}{4}$
- Express in terms of a_{0}

  $a_{2} = \frac{a_{0}}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y \left(u \right) = a_{0} \cdot \left(1-u +\frac{1}{4} u^{2}\right)$
- Revert the change of variables $u = x +2$

  $\left[y = \frac{a_{0} x^{2}}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k +1} = \frac{a_{k} k}{2 \left(k +3\right)}$
- Solution for $r = 2$

  $\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k +2}, a_{k +1} = \frac{a_{k} k}{2 \left(k +3\right)}\right]$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2 x^2$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 19

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

2.285 problem 288

2.285.1 Maple step by step solution 2763

Internal problem ID [7775]
Internal file name [`OUTPUT/6708_Sunday_June_05_2022_05_06_46_PM_85980721/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 288.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 + 2x)\, y'' - 2(1 + x)\, y' + 2y = 0}$$

Writing the ode as

$$\left(x^2 + 2x\right) y'' + (-2x - 2)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 2x \\ B &= -2x - 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 545: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2} + \frac{3}{4x^2}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} \\ &= \frac{x+3}{x\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+2\right)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2\left(x+2\right)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2\left(x+2\right)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{\left(x^2+2x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x^2+2x} \, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1 \left(\sqrt{x\left(x+2\right)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{3}{2}} \sqrt{x\left(x+2\right)}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-x-1}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\right) + c_2\left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\left(\frac{-x-1}{x^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}
$$

Verified OK.

2.285.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (-2x-2)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x+2)} + \frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2(1+x)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2(1+x)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$

- $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = -1$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

- $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (-2x-2)\, y' + 2y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

Rewrite ODE with series expansions

$-2a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\, a_{k+1} + a_k(k+r-2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2x^2$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 19

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x^2 - c_2(x+1)$$

2.286 problem 289

Internal problem ID [7776]
Internal file name [`OUTPUT/6709_Sunday_June_05_2022_05_06_48_PM_85929379/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 289.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{(x^2+1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= (x^2+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 547: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\
&= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1\left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x - c_1(x - i)^2$$

2.287 problem 290

Internal problem ID [7777]
Internal file name [`OUTPUT/6710_Sunday_June_05_2022_05_06_51_PM_47423142/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 290.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2+1)\, y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\, x}{(x-i)^2\,(x+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\, x}{(x-i)^2\,(x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

2.288 problem 291

2.288.1 Maple step by step solution 2784

Internal problem ID [7778]
Internal file name [`OUTPUT/6711_Sunday_June_05_2022_05_06_53_PM_97739219/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 291.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 549: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\, dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x^2} + c_2 x\, \mathrm{e}^{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x^2} + c_2 x\, \mathrm{e}^{x^2}$$

Verified OK.

2.288.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x \,e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.289 problem 292

2.289.1 Maple step by step solution 2790

Internal problem ID [7779]
Internal file name [OUTPUT/6712_Sunday_June_05_2022_05_06_55_PM_38184233/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 292.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\,y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 551: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{x^2}\right) + c_2 \left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2}
$$

Verified OK.

2.289.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x\, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.290 problem 293

2.290.1 Maple step by step solution 2800

Internal problem ID [7780]
Internal file name [OUTPUT/6713_Sunday_June_05_2022_05_06_57_PM_1674529/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 293.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x-3)\,y''-xy'+y=0}$$

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x-3 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 8x + 18$$
$$t = 4(2x - 3)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 553: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{33}{64\left(x-\frac{3}{2}\right)^2} - \frac{5}{16\left(x-\frac{3}{2}\right)}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= -\frac{5}{8} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &&= \frac{5}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{8} - \left(-\frac{3}{8}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x - 6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4}\right)(1) + \left(\left(\frac{3}{8\left(x - \frac{3}{2}\right)^2}\right) + \left(-\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}\right)\right) = 0$$

$$\frac{a_0}{2x - 3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{4}-\frac{3\ln(2x-3)}{8}} \\
&= \frac{x\, \mathrm{e}^{-\frac{x}{4}}}{(2x-3)^{\frac{3}{8}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\, dx} \\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}} \\
&= z_1 \left((2x-3)^{\frac{3}{8}}\, \mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x-3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x}{2}} (2x-3)^{\frac{3}{4}}}{x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\int \frac{e^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2 x\left(\int \frac{e^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x + c_2 x\left(\int \frac{e^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right)
$$

Verified OK.

2.290.1 Maple step by step solution

Let's solve

$(2x-3)\,y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x-3} + \frac{xy'}{2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{2x-3} + \frac{y}{2x-3} = 0$

□ Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{2x-3}, P_3(x) = \frac{1}{2x-3}\right]$

  - $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x = \frac{3}{2}$

    $\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}} = -\frac{3}{4}$

  - $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x = \frac{3}{2}$

$$\left(\left(x-\frac{3}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{3}{2}} = 0$$

- $x = \frac{3}{2}$ is a regular singular point

  Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  $x_0 = \frac{3}{2}$

- Multiply by denominators

  $(2x-3)\, y'' - xy' + y = 0$

- Change variables using $x = u + \frac{3}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + \left(-u - \frac{3}{2}\right)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{a_0 r(-7+4r)u^{-1+r}}{2} + \left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(4k-3+4r)}{2} - a_k(k+r-1)\right)u^{k+r}\right) = 0$

- a_0 cannot be 0 by assumption, giving the indicial equation

  $\frac{r(-7+4r)}{2} = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{7}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{3}{4}+r\right)\left(k+1+r\right)a_{k+1}-a_{k}\left(k+r-1\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_{k}\left(k+r-1\right)}{\left(4k-3+4r\right)\left(k+1+r\right)}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{2a_{k}\left(k-1\right)}{\left(4k-3\right)\left(k+1\right)}$

- Apply recursion relation for $k=0$

  $a_{1}=\frac{2a_{0}}{3}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_{0}\cdot\left(1+\frac{2u}{3}\right)$

- Revert the change of variables $u=x-\frac{3}{2}$

  $\left[y=\frac{2a_{0}x}{3}\right]$

- Recursion relation for $r=\frac{7}{4}$

  $a_{k+1}=\frac{2a_{k}\left(k+\frac{3}{4}\right)}{\left(4k+4\right)\left(k+\frac{11}{4}\right)}$

- Solution for $r=\frac{7}{4}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+\frac{7}{4}}, a_{k+1}=\frac{2a_{k}\left(k+\frac{3}{4}\right)}{\left(4k+4\right)\left(k+\frac{11}{4}\right)}\right]$

- Revert the change of variables $u=x-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}\left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1}=\frac{2a_{k}\left(k+\frac{3}{4}\right)}{\left(4k+4\right)\left(k+\frac{11}{4}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\frac{2a_{0}x}{3}+\left(\sum_{k=0}^{\infty}b_{k}\left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1}=\frac{2b_{k}\left(k+\frac{3}{4}\right)}{\left(4k+4\right)\left(k+\frac{11}{4}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve((2*x-3)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2x\left(\int \frac{(-3+2x)^{\frac{3}{4}} e^{\frac{x}{2}}}{x^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 63

```
DSolve[(2*x-3)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4}L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2}-\frac{3}{4}\right)+\frac{4\sqrt{2}c_1x}{2x-3}\right)$$

2.291 problem 294

2.291.1 Maple step by step solution 2811

Internal problem ID [7781]
Internal file name [`OUTPUT/6714_Sunday_June_05_2022_05_07_00_PM_8046336/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 294.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' - 3y = 0}$$

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 10$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 555: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\int \frac{x}{2}dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1\left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(\left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^2+1)\,e^{\frac{x^2}{2}} + c_2(x^2+1)\,e^{\frac{x^2}{2}}\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2+1)\,e^{\frac{x^2}{2}} + c_2(x^2+1)\,e^{\frac{x^2}{2}}\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)$$

Verified OK.

2.291.1 Maple step by step solution

Let's solve

$y'' - xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+3)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k^2+3k+2)\,a_{k+2}-a_k(k+3)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\mathrm{e}^{\frac{x^2}{2}}\left(x^2+1\right)+c_2\mathrm{e}^{\frac{x^2}{2}}\left(x^2+1\right)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 35

```
DSolve[y''[x]-x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2 e^{\frac{x^2}{2}} (x^2 + 1)$$

2.292 problem 295

Internal problem ID [7782]
Internal file name [`OUTPUT/6715_Sunday_June_05_2022_05_07_03_PM_37429158/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 295.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x^2+1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 6$$
$$t = 4\left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 557: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x^2-6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-i\right)^2} + \frac{1}{4\left(x+i\right)^2}\right) + \left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)^2 - \left(\frac{-}{4(\cdots}\right.\right.
$$

$$
\frac{(x^2+1\cdots}{(-x+i)^2(\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x)\, \frac{1}{\left(x^2+1\right)^{\frac{1}{4}}} \\
&= \frac{x}{\left(x^2+1\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\left(x^2+1\right)^{\frac{1}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\operatorname{arcsinh}\left(x\right) x - \sqrt{x^2+1}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{\operatorname{arcsinh}\left(x\right) x - \sqrt{x^2+1}}{x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2\left(\operatorname{arcsinh}\left(x\right) x - \sqrt{x^2+1}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 \left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2 + 1}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve((1+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2 + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 42

```
DSolve[(1+x^2)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_2 \sqrt{x^2 + 1} - c_2 x \log\left(\sqrt{x^2 + 1} - x\right) + c_1 x$$

2.293 problem 296

2.293.1 Maple step by step solution 2827

Internal problem ID [7783]
Internal file name [`OUTPUT/6716_Sunday_June_05_2022_05_07_06_PM_84304364/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 296.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' + 2y = 0}$$

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 558: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0 \\ a_1 x + 2a_0 + 2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2-1\right) + c_2 \left(x^2 - 1\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-1)+c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1)+c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.293.1 Maple step by step solution

Let's solve

$y''-xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty}(a_{k+2}(k+2)(k+1)-a_k(k-2))\, x^k=0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right)a_{k+2}-a_k(k-2)=0$$

- Recursion relation; series terminates at $k=2$

$$a_{k+2}=\frac{a_k(k-2)}{k^2+3k+2}$$

- Apply recursion relation for $k=0$

$$a_2=-a_0$$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

$$y=A_2x^2+A_1x-a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 - 1) + c_2(x^2 - 1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x-1)^2 (x+1)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 54

```
DSolve[y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2 - 1)\,\text{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2 - 1)$$

2.294 problem 297

2.294.1 Maple step by step solution 2836

Internal problem ID [7784]
Internal file name [`OUTPUT/6717_Sunday_June_05_2022_05_07_09_PM_45536023/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 297.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-x^2)\, y''-y'+y=0}$$

Writing the ode as

$$(1-x^2)\, y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 3$$
$$t = 4\left(x^2 - 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 3}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 560: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(1+x)^2} - \frac{3}{16(x-1)^2} + \frac{7}{16(x-1)} - \frac{7}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4(x^2-1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (-1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{-1}{(x-(-1))}\right) \\ &= \frac{1}{2x-2} - \frac{1}{2(1+x)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{(1+x)^2(x-1)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2\left(1+x\right)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2\left(1+x\right)}\right) w + \frac{-8x^3 - 4x^2 + 10x + 7}{4\left(x^2-1\right)^2\left(3+2x\right)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)}\,x + 2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)} + 2x^2 + 2x + 1}{2\left(3+2x\right)\left(x-1\right)\left(1+x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(1+x)}\,x+2\sqrt{5}\,\sqrt{(x-1)(1+x)}+2x^2+2x+1}{2(3+2x)(x-1)(1+x)} dx} \\ &= \frac{\left(x-1\right)^{\frac{1}{4}}\sqrt{3+2x}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{\frac{1}{4}}}{\left(1+x\right)^{\frac{1}{4}}\sqrt{\frac{5\sqrt{x^2-1}+(3x+2)\sqrt{5}}{\sqrt{x^2-1}\,\sqrt{-\frac{(3+2x)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1-x^2} \, dx} \\ &= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}} \\ &= z_1 \left(\sqrt{\frac{1+x}{\sqrt{1-x^2}}}\right)\end{aligned}$$

Which simplifies to

$$
y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{i\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{(3+2x)^2 \sqrt{5x-5}}dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} \right) \\
&\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} \left(\int \frac{i\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x+2\sqrt{5}+\right.}{(3+2x)^2 \sqrt{5x-5}} \right.\right.
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} + \frac{ic_2\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}\left(\int \frac{\sqrt{1+x}\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}}{(3+2x)^2\sqrt{5x-5}}dx\right)}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} + \frac{ic_2\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}\left(\int \frac{\sqrt{1+x}\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}}{(3+2x)^2\sqrt{5x-5}}dx\right)}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}}$$

Verified OK.

2.294.1 Maple step by step solution

Let's solve

$(1-x^2)\,y''-y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{x^2-1}+\frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{x^2-1}-\frac{y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -\frac{1}{2}$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + y' - y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + \frac{d}{du}y(u) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(-3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k-1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+r-\frac{1}{2}\right)(k+1+r)\,a_{k+1}+\left(k^2+(2r-1)\,k+r^2-r-1\right)a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(k^2+2kr+r^2-k-r-1)\,a_k}{(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(k^2-k-1)\,a_k}{(2k-1)(k+1)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k,a_{k+1}=\frac{(k^2-k-1)\,a_k}{(2k-1)(k+1)}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+1}=\frac{(k^2-k-1)\,a_k}{(2k-1)(k+1)}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+1}=\frac{(k^2+2k-\frac{1}{4})\,a_k}{(2k+2)(k+\frac{5}{2})}$$

- Solution for $r=\frac{3}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{3}{2}},a_{k+1}=\frac{(k^2+2k-\frac{1}{4})\,a_k}{(2k+2)(k+\frac{5}{2})}\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{3}{2}},a_{k+1}=\frac{(k^2+2k-\frac{1}{4})\,a_k}{(2k+2)(k+\frac{5}{2})}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k+\frac{3}{2}}\right),a_{k+1}=\frac{(k^2-k-1)\,a_k}{(2k-1)(k+1)},b_{k+1}=\frac{(k^2+2k-\frac{1}{4})\,b_k}{(2k+2)(k+\frac{5}{2})}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 177

```
dsolve((1-x^2)*diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{2x+3}\left(\frac{3\sqrt{5}\,x+2\sqrt{5}-5\sqrt{x^2-1}}{3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{5}} + c_2\sqrt{2x+3}\left(\frac{3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}}{3\sqrt{5}\,x+2\sqrt{5}-5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{-\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{5}}$$

✓ Solution by Mathematica

Time used: 30.669 (sec). Leaf size: 198

```
DSolve[(1-x^2)*y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[4]{x+1}\left(\sqrt{x+1}-\sqrt{x-1}\right)^{-1-\sqrt{5}}\left(-2x+2\sqrt{x-1}\sqrt{x+1}+\sqrt{5}-3\right)e^{-\operatorname{arctanh}\left(x-\sqrt{x-1}\sqrt{x+1}\right)}\left(c_2\int_1^x \frac{e^{2\operatorname{arct}\cdots}}{\cdots}\right.}{\sqrt[4]{1-x}}$$

2.295 problem 298

2.295.1 Maple step by step solution 2845

Internal problem ID [7785]
Internal file name [`OUTPUT/6718_Sunday_June_05_2022_05_07_13_PM_59096032/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 298.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x(1+x)^2 y'' + (1-x^2) y' + (x-1) y = 0}$$

Writing the ode as

$$x(1+x)^2 y'' + (1-x^2) y' + (x-1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(1+x)^2 \\ B &= 1-x^2 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 562: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-x^2}{x(1+x)^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} + \ln(1+x)} \\ &= z_1\left(\frac{1+x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1 + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1-x^2}{x(1+x)^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\,dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(1+x) + c_2(1 + x(\ln(x)))$$

Summary
The solution(s) found are the following

$$y = c_1(1+x) + c_2(1+x)\ln(x) \tag{1}$$

Verification of solutions

$$y = c_1(1+x) + c_2(1+x)\ln(x)$$

Verified OK.

2.295.1 Maple step by step solution

Let's solve
$x(1+x)^2 y'' + (1-x^2)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative
  $y'' = -\frac{(x-1)y}{x(1+x)^2} + \frac{(x-1)y'}{x(1+x)}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
  $y'' - \frac{(x-1)y'}{x(1+x)} + \frac{(x-1)y}{x(1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point
  - Define functions

$\left[P_2(x) = -\frac{x-1}{(1+x)x}, P_3(x) = \frac{x-1}{x(1+x)^2}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\Big|_{x=-1} = -2$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\Big|_{x=-1} = 2$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 y'' - (x-1)(1+x)y' + (x-1)y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2 + 2u)\left(\frac{d}{du}y(u)\right) + (u-2)y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- > k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+r)(-2+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2 = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+r)(k+r-1) + a_k(k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+r}$$

- Recursion relation for $r=1$

$$a_{k+1} = \frac{a_k k}{k+1}$$

- Solution for $r=1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k(k+1)}{k+2}$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = \frac{a_k (k+1)}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k (k+1)}{k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*(x+1)^2*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2(x+1)\ln(x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[x*(x+1)^2*y''[x]+(1-x^2)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x+1)(c_2 \log(x) + c_1)$$

2.296 problem 299

2.296.1 Maple step by step solution 2854

Internal problem ID [7786]
Internal file name [OUTPUT/6719_Sunday_June_05_2022_05_07_16_PM_74868184/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 299.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5 - 16x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 564: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)\\ &= -\frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\left(1+4x\right)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)(1+4x)}}{\sqrt{2\sqrt{-x}+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)(1+4x)} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\right) \\
&\quad + c_2\left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
& + \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
& + \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned}
$$

Verified OK.

2.296.1 Maple step by step solution

Let's solve

$2xy'' - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' - y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-3+2r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(2k-1+2r)+2a_k\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0,\frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r-\frac{1}{2}\right)(k+1+r)\,a_{k+1}+2a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=-\frac{2a_k}{(2k-1+2r)(k+1+r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}\right]$$

- Recursion relation for $r=\frac{3}{2}$

  $$a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$$

- Solution for $r=\frac{3}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}, b_{k+1}=-\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 75

```
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(2i\sqrt{x}-1\right)}{1+2i\sqrt{x}}} + c_2 \mathrm{e}^{-2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(1+2i\sqrt{x}\right)}{2i\sqrt{x}-1}}$$

✓ Solution by Mathematica
Time used: 0.21 (sec). Leaf size: 59

```
DSolve[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right) + \frac{1}{8} c_2 e^{-2i\sqrt{x}}\left(1+2i\sqrt{x}\right)$$

2.297 problem 300

Internal problem ID [7787]
Internal file name [`OUTPUT/6720_Sunday_June_05_2022_05_07_20_PM_68552865/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 300.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + xy' - 2y = 0}$$

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x+8 \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+8}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\ &= \frac{1}{4} + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+8}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{x} + \frac{1}{2} \\
&= \frac{1}{x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x+8}{4x}\right)\right) = 0$$

$$\frac{2 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2)\, e^{\int \left(\frac{1}{x} + \frac{1}{2}\right) dx} \\ &= (x+2)\, e^{\frac{x}{2} + \ln(x)} \\ &= (x+2)\, x\, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x} \, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x(x+2)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x} dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(-x-1) \mathrm{e}^{-x} + (x+2) \, x \ \mathrm{expIntegral}_1 (x)}{2 (x+2) \, x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x(x+2)) + c_2 \left(x(x+2) \left(\frac{(-x-1) \mathrm{e}^{-x} + (x+2) \, x \ \mathrm{expIntegral}_1 (x)}{2 (x+2) \, x} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x(x+2) + c_2 \left(\frac{(-x-1) \mathrm{e}^{-x}}{2} + \frac{(x+2) \, x \ \mathrm{expIntegral}_1 (x)}{2} \right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x(x+2) + c_2 \left(\frac{(-x-1) \mathrm{e}^{-x}}{2} + \frac{(x+2) \, x \ \mathrm{expIntegral}_1 (x)}{2} \right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 43

```
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 + 2x) + c_2\left(\frac{x^2 \operatorname{expIntegral}_1(x)}{2} - \frac{e^{-x}x}{2} + \operatorname{expIntegral}_1(x)\, x - \frac{e^{-x}}{2}\right)$$

✓ Solution by Mathematica
Time used: 0.131 (sec). Leaf size: 39

```
DSolve[x*y''[x]+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x(x+2) - \frac{1}{2} c_2 e^{-x}(e^x(x+2)x \operatorname{ExpIntegralEi}(-x) + x + 1)$$

2.298 problem 301

2.298.1 Maple step by step solution 2871

Internal problem ID [7788]
Internal file name [`OUTPUT/6721_Sunday_June_05_2022_05_07_23_PM_66307869/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 301.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x(x-1)^2 y''-2y=0}$$

Writing the ode as

$$\begin{align*} x(x-1)^2 y''-2y &= 0 \tag{1}\\ Ay''+By'+Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= x(x-1)^2\\ B &= 0 \tag{3}\\ C &= -2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x\left(x-1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x(x-1)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x\left(x-1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 567: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x} + \frac{2}{(x-1)^2} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x\,(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{x-1} + (0) \\
&= \frac{1}{x} - \frac{1}{x-1} \\
&= -\frac{1}{x\,(x-1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x\,(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}}\, dx \\ &= \frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{x-1}\right) + c_2 \left(\frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{x-1} + \frac{c_2(-2x\ln(x) + x^2 - 1)}{x-1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{x-1} + \frac{c_2(-2x\ln(x) + x^2 - 1)}{x-1}$$

Verified OK.

2.298.1 Maple step by step solution

Let's solve

$x(x-1)^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{2}{x(x-1)^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)^2 y'' - 2y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+r)\, x^{-1+r} + \left(a_1(1+r)\, r - 2a_0(r^2-r+1)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+r) - 2a_k(k^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r)\, r - 2a_0(r^2-r+1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 2a_k - 3a_{k-1} + a_{k+1})\, k + (-2a_k + a_{k-1} -$

- Shift index using $k->k+1$

  $(-2a_{k+1} + a_k + a_{k+2})(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 2a_{k+1} - 3a_k + a_{k+2})(k+1) + (-$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - k a_k - 2k a_{k+1} - r a_k - 2r a_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 6k a_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}, 2a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - k a_k - 2k a_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0, b_{k+2} = -\frac{k^2 b\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(x*(x-1)^2*diff(y(x),x$2)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x-1} + \frac{c_2(2x\ln(x) - x^2 + 1)}{x-1}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 33

```
DSolve[x*(x-1)^2*y''[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2 x^2 - c_1 x + 2c_2 x\log(x) + c_2}{x-1}$$

2.299 problem 302

2.299.1 Maple step by step solution 2878

Internal problem ID [7789]
Internal file name [OUTPUT/6722_Sunday_June_05_2022_05_07_26_PM_27221302/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 302.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + x^2y = 0}$$

Writing the ode as

$$y'' - 2xy' + x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 569: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(x)\, e^{\frac{x^2}{2}}\right) + c_2 \left(\cos(x)\, e^{\frac{x^2}{2}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x)\, e^{\frac{x^2}{2}} + c_2 \sin(x)\, e^{\frac{x^2}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos(x)\, e^{\frac{x^2}{2}} + c_2 \sin(x)\, e^{\frac{x^2}{2}}
$$

Verified OK.

2.299.1 Maple step by step solution

Let's solve

$y'' - 2xy' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  ◦ Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + (6a_3 - 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k k + a_{k-2}\right) x^{k}\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 - 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = 0, a_3 = \frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 2a_k k + a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 2a_{k+2}(k+2) + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} - a_k + 4a_{k+2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x^2}{2}} \cos(x) + c_2 \mathrm{e}^{\frac{x^2}{2}} \sin(x)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 39

```
DSolve[y''[x]-2*x*y'[x]+x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{\frac{1}{2}x(x-2i)} \left(2c_1 - ic_2 e^{2ix}\right)$$

2.300 problem 303

2.300.1 Maple step by step solution 2888

Internal problem ID [7790]
Internal file name [OUTPUT/6723_Sunday_June_05_2022_05_07_28_PM_79880697/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 303.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(-x^2+2)\,y''-(x^2+4x+2)\,((1-x)\,y'+y)=0}$$

Writing the ode as

$$\left(-x^3+2x\right)y''+\left(x^3+3x^2-2x-2\right)y'+\left(-x^2-4x-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+2x \\ B &= x^3+3x^2-2x-2 \\ C &= -x^2-4x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12 \\ t &= 4\left(x^3 - 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 571: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} + \frac{3}{4x^2} + \frac{3}{4\left(x - \sqrt{2}\right)^2} + \frac{3}{4\left(x + \sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x - \sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x + \sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} - \frac{3}{2x^3} + \frac{21}{4x^4} - \frac{43}{4x^5} + \frac{135}{4x^6} - \frac{147}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4x^6-16x^4+16x^2}\\
&= Q + \frac{R}{4x^6-16x^4+16x^2}\\
&= \left(\frac{1}{4}\right) + \left(\frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}\right)\\
&= \frac{1}{4} + \frac{2x^5-x^4-16x^3+20x^2+24x+12}{4x^6-16x^4+16x^2}
\end{aligned}$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0)\\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6+2x^5-5x^4-16x^3+24x^2+24x+12}{4\left(x^3-2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2} \\ &= \frac{x^3 + x^2 - 2x - 6}{2x^3 - 4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{2\left(x-\sqrt{2}\right)^2} + \frac{1}{2\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{3}{2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right) dx} \\
&= \frac{x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}}{\sqrt{x-\sqrt{2}}\,\sqrt{x+\sqrt{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x^2-2\right)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x^2-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x^2-2)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{(x-1)\,\mathrm{e}^{-x}}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x \left(-\frac{(x-1)\,\mathrm{e}^{-x}}{x^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^2 \mathrm{e}^x c_1 + c_2(1-x) \tag{1}
$$

Verification of solutions

$$
y = x^2 \mathrm{e}^x c_1 + c_2(1-x)
$$

Verified OK.

2.300.1 Maple step by step solution

Let's solve

$(-x^3+2x)\,y'' + (x^3+3x^2-2x-2)\,y' + (-x^2-4x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+4x+2)y}{x(x^2-2)} + \frac{(x^3+3x^2-2x-2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^3+3x^2-2x-2)y'}{x(x^2-2)} + \frac{(x^2+4x+2)y}{x(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x^3+3x^2-2x-2}{x(x^2-2)}, P_3(x) = \frac{x^2+4x+2}{x(x^2-2)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x(x^2-2) + (-x^3-3x^2+2x+2)\, y' + (x^2+4x+2)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$-2a_0r(-2+r)\, x^{-1+r} + (-2a_1(1+r)\,(-1+r) + 2a_0(1+r))\, x^r + \left(-2a_2(2+r)\, r + 2a_1(2+r) +\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- The coefficients of each power of x must be 0

  $$\left[-2a_1(1+r)(-1+r)+2a_0(1+r)=0, -2a_2(2+r)r+2a_1(2+r)+a_0(-2+r)^2=0\right]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1=\frac{a_0}{-1+r}, a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)(k+r-1)+(2a_k-a_{k-2})k+(2a_k-a_{k-2})r+2a_k+3a_{k-}$$

- Shift index using $k->k+2$

  $$a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)(k+r+1)+(2a_{k+2}-a_k)(k+2)+(2a_{k+2}-a_k)r+2a_{k+2}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$$

- Recursion relation for $r=0$

  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=-\frac{5a_0}{2}\right]$$

- Recursion relation for $r=2$

  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$$

- Solution for $r=2$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}, a_1=a_0, a_2=\frac{a_0}{2}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 17

```
dsolve(x*(2-x^2)*diff(y(x),x$2)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x))=0,y(x), singsol=all)
```

$$y(x) = c_1(x-1) + c_2 e^x x^2$$

✓ Solution by Mathematica
Time used: 0.14 (sec). Leaf size: 21

```
DSolve[x*(2-x^2)*y''[x]-(x^2+4*x+2)*((1-x)*y'[x]+y[x])==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 e^x x^2 + c_2(x-1)$$

2.301 problem 304

2.301.1 Maple step by step solution 2897

Internal problem ID [7791]
Internal file name [OUTPUT/6724_Sunday_June_05_2022_05_07_31_PM_20045850/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 304.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-(2x+1)\,(xy'-y)=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-2x^2-x\right)y'+(2x+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -2x^2-x \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1 - 4x}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1 - 4x$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-1 - 4x}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 573: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} + \frac{3}{4(1+x)^2} - \frac{1}{4x^2} + \frac{1}{2x+2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-1-4x}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(1+x)} + \frac{1}{2x} + (0) \\
&= -\frac{1}{2\,(1+x)} + \frac{1}{2x} \\
&= \frac{1}{2x\,(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-1-4x}{4(x^2+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)} \, dx} \\ &= z_1 e^{\frac{\ln(x(1+x))}{2}} \\ &= z_1 \left(\sqrt{x(1+x)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}\sqrt{x(1+x)}}{\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(1+x))}}{(y_1)^2}\,dx \\
&= y_1(x + \ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}\,\sqrt{x(1+x)}}{\sqrt{1+x}}\right) + c_2 \left(\frac{\sqrt{x}\,\sqrt{x(1+x)}}{\sqrt{1+x}}(x + \ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}\,\sqrt{x(1+x)}}{\sqrt{1+x}} + \frac{c_2\sqrt{x}\,\sqrt{x(1+x)}\,(x+\ln(x))}{\sqrt{1+x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}\,\sqrt{x(1+x)}}{\sqrt{1+x}} + \frac{c_2\sqrt{x}\,\sqrt{x(1+x)}\,(x+\ln(x))}{\sqrt{1+x}}
$$

Verified OK.

2.301.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y'' + (-2x^2 - x)\,y' + (2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y}{x^2(1+x)} + \frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{x(1+x)} + \frac{(2x+1)y}{x^2(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2x+1}{x(1+x)}, P_3(x) = \frac{2x+1}{x^2(1+x)}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -1$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$
  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$
- Multiply by denominators

  $x^2(1+x) y'' - x(2x+1) y' + (2x+1) y = 0$
- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u) \left(\frac{d^2}{du^2} y(u)\right) + (-2u^2 + 3u - 1) \left(\frac{d}{du} y(u)\right) + (2u - 1) y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, u^{-1+r} + \left(a_1(1+r)(-1+r) - a_0(2r^2-5r+1)\right) u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term must be 0

  $$a_1(1+r)(-1+r) - a_0(2r^2-5r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 5a_k - 5a_{k-1})\, k + (-2a_k + a_{k-1} + a_{k+1})$$

- Shift index using $k- > k+1$

  $$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 5a_{k+1} - 5a_k)\,(k+1) + (-2a_{k+1} +$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 3k a_k + k a_{k+1} - 3r a_k + r a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2kr + r^2 + 2k + 2r}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_k + k a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_k + k a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}$$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-(1+2*x)*(x*diff(y(x),x)-y(x))=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 x(x + \ln(x))$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 132

```
DSolve[x^2*(1+x)*y''[x]-(1+2*x)*(x*y'[x]+y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^{1+\sqrt{2}} \text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2}, -\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2}, 1+2\sqrt{2}, -x\right) + c_1 x^{1-\sqrt{2}} \text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right), \frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right), 1 - 2\sqrt{2}, -x\right)$$

2.302 problem 305

2.302.1 Maple step by step solution 2906

Internal problem ID [7792]
Internal file name [`OUTPUT/6725_Sunday_June_05_2022_05_07_34_PM_47645531/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 305.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(-x+2)\,x^2y''-(-x+4)\,xy'+(3-x)\,y=0}$$

Writing the ode as

$$(-2x^3+4x^2)\,y''+(x^2-4x)\,y'+(3-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+4x^2 \\ B &= x^2-4x \\ C &= 3-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16(x-2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16(x-2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16(x-2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 575: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16\,(x-2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x-8} + (-)\,(0) \\
&= \frac{1}{4x-8} \\
&= \frac{1}{4x-8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x-8}\right)(0) + \left(\left(-\frac{1}{4(x-2)^2}\right) + \left(\frac{1}{4x-8}\right)^2 - \left(-\frac{3}{16(x-2)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{4x-8} dx} \\ &= (x-2)^{\frac{1}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2-4x}{-2x^3+4x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{\ln(x-2)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(x-2)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-4x}{-2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{\ln(x-2)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(2\sqrt{x-2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(2\sqrt{x-2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x} + 2c_2\sqrt{x}\,\sqrt{x-2} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x} + 2c_2\sqrt{x}\,\sqrt{x-2}
$$

Verified OK.

2.302.1 Maple step by step solution

Let's solve

$(-2x^3 + 4x^2)\, y'' + (x^2 - 4x)\, y' + (3 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y'}{2x(x-2)} - \frac{(-3+x)y}{2x^2(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-4)y'}{2x(x-2)} + \frac{(-3+x)y}{2x^2(x-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{x-4}{2x(x-2)}, P_3(x) = \frac{-3+x}{2(x-2)x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{3}{4}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$2y''(x-2)\,x^2 - x(x-4)\,y' + (-3+x)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+2r)(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} (-a_k(2k+2r-1)(2k+2r-3) + a_{k-1}(2k+2r-3)(k-2$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-1+2r)(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-4\left(\left(-\frac{k}{2} - \frac{r}{2} + 1\right) a_{k-1} + a_k\left(k + r - \frac{1}{2}\right)\right)\left(k + r - \frac{3}{2}\right) = 0$$

- Shift index using $k- >k+1$

  $$-4\left(\left(-\frac{k}{2} + \frac{1}{2} - \frac{r}{2}\right) a_k + a_{k+1}\left(k + \frac{1}{2} + r\right)\right)\left(k + r - \frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(k+r-1)a_k}{2k+1+2r}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = \frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 21

```
dsolve(2*(2-x)*x^2*diff(y(x),x$2)-(4-x)*x*diff(y(x),x)+(3-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x^2 - 2x}$$

✓ Solution by Mathematica
Time used: 0.064 (sec). Leaf size: 41

```
DSolve[2*(2-x)*x^2*y''[x]-(4-x)*x*y'[x]+(3-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{\sqrt[4]{x-2}\sqrt{x}\left(2c_2\sqrt{x-2} + c_1\right)}{\sqrt[4]{2-x}}$$

2.303 problem 306

Internal problem ID [7793]
Internal file name [`OUTPUT/6726_Sunday_June_05_2022_05_07_37_PM_73594286/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 306.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= 5x^2-4x \\ C &= 6-9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x+4}{4x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x+4 \\ t &= 4x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x+4}{4x(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 577: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(x-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+4}{4x\left(x-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{1}{2\left(x-1\right)} \\
&= \frac{x-2}{2x\left(x-1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2\,(x-1)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{2\,(x-1)}\right)^2 - \left(\frac{-x+4}{4x\,(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{2(x-1)}\right) dx} \\ &= \frac{x}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2-4x}{-x^3+x^2}\, dx} \\ &= z_1 e^{2\ln(x) + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(x^2\sqrt{x-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)+\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^3) + c_2 \left(x^3 \left(\frac{1}{x} + \ln(x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^2 (x \ln(x) + 1) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^2 (x \ln(x) + 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1-x)*x^2*diff(y(x),x$2)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^3 + c_2x^2(x\ln(x) + 1)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 24

```
DSolve[(1-x)*x^2*y''[x]+(5*x-4)*x*y'[x]+(6-9*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x^2(c_1x - c_2(x\log(x) + 1))$$

2.304 problem 307

2.304.1 Maple step by step solution 2922

Internal problem ID [7794]
Internal file name [`OUTPUT/6727_Sunday_June_05_2022_05_07_40_PM_5446021/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 307.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (4x^2+1)\,y' + 4x(x^2+1)\,y = 0}$$

Writing the ode as

$$xy'' + (4x^2+1)\,y' + (4x^3+4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2+1 \\ C &= 4x^3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 578: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x}\, dx} \\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x^2}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x^2 - \ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(\ln(x))\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \ln(x) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \ln(x)
$$

Verified OK.

2.304.1 Maple step by step solution

Let's solve
$x y'' + (4x^2+1) y' + (4x^3+4x) y = 0$

- Highest derivative means the order of the ODE is 2
y''
- Isolate 2nd derivative
$y'' = (-4x^2-4) y - \frac{(4x^2+1)y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$y'' + \frac{(4x^2+1)y'}{x} + (4x^2+4) y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions
$\left[P_2(x) = \frac{4x^2+1}{x}, P_3(x) = 4x^2+4\right]$
- $x \cdot P_2(x)$ is analytic at $x=0$
$\left.(x \cdot P_2(x))\right|_{x=0} = 1$
- $x^2 \cdot P_3(x)$ is analytic at $x=0$
$\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (4x^2 + 1)\, y' + 4x(x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 1..3$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$

- ○ Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1 + r)^2 x^r + \left(a_2 (2 + r)^2 + 4a_0 (1 + r)\right) x^{1+r} + \left(a_3 (3 + r)^2 + 4a_1 (2 + r)\right) x^{2+r} + \Big($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- The coefficients of each power of x must be 0

  $[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$

- Shift index using $k- > k+3$

  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} \ln(x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 21

```
DSolve[x*y''[x]+(4*x^2+1)*y'[x]+4*x*(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 \log(x) + c_1)$$

2.305 problem 308

Internal problem ID [7795]
Internal file name [`OUTPUT/6728_Sunday_June_05_2022_05_07_42_PM_79691299/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 308.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$\begin{align*} y'' - 2xy' + 8y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1\\ B &= -2x \tag{3}\\ C &= 8 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{x^2-9}{1}\\ &= Q + \frac{R}{1}\\ &= (x^2-9) + (0)\\ &= x^2 - 9\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned}b &= (-9) - (0)\\ &= -9\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= x\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &= -5\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &= 4\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^-\\ &= 4\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\,(x) \\
&= -x \\
&= -x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2(-x)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left((-1) + (-x)^2 - \left(x^2 - 9\right)\right) = 0$$
$$2a_3x^3 + 4(3 + a_2)\,x^2 + 6(a_1 + a_3)\,x + 8a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\
&= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\
&= \frac{\left(4x^4 - 12x^2 + 3\right) e^{-\frac{x^2}{2}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2\left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 1.113 (sec). Leaf size: 63

```
DSolve[y''[x]-2*x*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1\left(x^4-3x^2+\frac{3}{4}\right)-\frac{1}{12}c_2\left(\sqrt{\pi}\left(-4x^4+12x^2-3\right)\text{erfi}(x)+2e^{x^2}x\left(2x^2-5\right)\right)$$

2.306 problem 309

Internal problem ID [7796]
Internal file name [`OUTPUT/6729_Sunday_June_05_2022_05_07_46_PM_64552380/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 309.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 581: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 9) + (0) \\ &= x^2 - 9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &= -5 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(x) \\ &= -x \\ &= -x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} \left(12x^2 + 6x a_3 + 2a_2\right) + 2(-x)\left(4x^3 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left((-1) + (-x)^2 - \left(x^2 - 9\right)\right) &= 0 \\ 2a_3 x^3 + 4(3 + a_2)\,x^2 + 6(a_1 + a_3)\,x + 8a_0 + 2a_2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\ &= \frac{\left(4x^4 - 12x^2 + 3\right) e^{-\frac{x^2}{2}}}{4} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2 \left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2 (4x^4 - 12x^2 + 3) \left(\int \frac{\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 63

```
DSolve[y''[x]-2*x*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) - \frac{1}{12}c_2\left(\sqrt{\pi}\left(-4x^4 + 12x^2 - 3\right)\operatorname{erfi}(x) + 2e^{x^2}x\left(2x^2 - 5\right)\right)$$

2.307 problem 310

2.307.1 Maple step by step solution 2948

Internal problem ID [7797]
Internal file name [OUTPUT/6730_Sunday_June_05_2022_05_07_49_PM_28409189/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 310.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-2xy'+12y=0}$$

Writing the ode as

$$(1-x^2)\,y''-2xy'+12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x^2\\ B&=-2x\\ C&=12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12x^2 - 13 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12x^2 - 13}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 582: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x=1$ of order 2. There is a pole at $x=-1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{25}{4(x-1)} - \frac{25}{4(1+x)}$$

For the pole at $x=1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2-13}{(x^2-1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 4 - (1) \\
&= 3
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\
&= \frac{x}{x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{2x-2} + \frac{}{2x}\right.\right.$$

$$\frac{-6a_2x^2 + (-10a_1}{x^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - \frac{3}{5}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= \frac{(5x^3 - 3x)\sqrt{x-1}\,\sqrt{1+x}}{5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{625x}{180x^2 - 108} + \frac{25}{9x} - \frac{25\ln(1 + x)}{8} + \frac{25\ln(x - 1)}{8}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}}\right) \\
&\quad + c_2 \left(\frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}} \left(\frac{625x}{180x^2 - 108} + \frac{25}{9x} - \frac{25\ln(1 + x)}{8} + \frac{25\ln(x - 1)}{8}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}} \\
&\quad + \frac{5c_2(15\ln(x - 1)\, x^3 - 15\ln(1 + x)\, x^3 - 9\ln(x - 1)\, x + 9\ln(1 + x)\, x + 30x^2 - 8)\sqrt{x^2 - 1}}{24\sqrt{1 + x}\,\sqrt{x - 1}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{1+x}} + \frac{5c_2(15\ln(x-1)\,x^3-15\ln(1+x)\,x^3-9\ln(x-1)\,x+9\ln(1+x)\,x+30x^2-8)\sqrt{x^2-1}}{24\sqrt{1+x}\,\sqrt{x-1}}$$

Verified OK.

2.307.1 Maple step by step solution

Let's solve

$(1-x^2)\,y''-2xy'+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-1}+\frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{12y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{12}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''+2xy'-12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-12y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+4)\,(k+r-3)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2=0$$

- Values of r that satisfy the indicial equation

$$r=0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2+a_k(k+4)\,(k-3)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Recursion relation for $r=0$; series terminates at $k=3$

$$a_{k+1}=\frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Apply recursion relation for $k=0$

$a_1 = -6a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{15a_0}{2}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$

- Express in terms of a_0

  $a_3 = -\frac{5a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{5}{3}x^3 + x\right) + c_2\left(-\frac{5\ln(x+1)\,x^3}{24} + \frac{5\ln(x-1)\,x^3}{24} + \frac{\ln(x+1)\,x}{8} - \frac{\ln(x-1)\,x}{8} + \frac{5x^2}{12} - \frac{1}{9}\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 59

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_1 x(5x^2-3) + c_2\left(-\frac{5x^2}{2} - \frac{1}{4}(5x^2-3)\,x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

2.308 problem 311

2.308.1 Maple step by step solution 2958

Internal problem ID [7798]
Internal file name [OUTPUT/6731_Sunday_June_05_2022_05_07_52_PM_58707844/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 311.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+2)\, y''+2(1+x)\, y'-2y=0}$$

Writing the ode as

$$\left(x^{2}+2x\right) y''+\left(2x+2\right) y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2x \\ B &= 2x+2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 + 4x - 1 \\ t &= (x^2 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4x} - \frac{5}{4(x+2)} - \frac{1}{4(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2+4x-1}{(x^2+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= 2-(1)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\ &= \frac{1}{2x+4} + \frac{1}{2x} \\ &= \frac{1+x}{x\,(x+2)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2\,(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) = 0$$

$$\frac{2-2a_0}{x\,(x+2)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (1+x)\,e^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\ &= (1+x)\,e^{\frac{\ln(x)}{2} + \frac{\ln(x+2)}{2}} \\ &= (1+x)\sqrt{x}\,\sqrt{x+2}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(x+2)}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1 + x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\ln(x)}{2} + \frac{1}{1+x} - \frac{\ln(x+2)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1+x) + c_2\left(1 + x\left(\frac{\ln(x)}{2} + \frac{1}{1+x} - \frac{\ln(x+2)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x) + c_2\left(\frac{(-x-1)\ln(x+2)}{2} + 1 + \frac{\ln(x)(1+x)}{2}\right) \tag{1}
$$

Verification of solutions

$$y = c_1(1+x) + c_2\left(\frac{(-x-1)\ln(x+2)}{2} + 1 + \frac{\ln(x)(1+x)}{2}\right)$$

Verified OK.

2.308.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (2x+2)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x+2)} - \frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x(x+2)} + \frac{2(1+x)y'}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2(1+x)}{x(x+2)}, P_3(x) = -\frac{2}{x(x+2)}\right]$

- ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left.((x+2)\cdot P_2(x))\right|_{x=-2} = 1$

- ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left.((x+2)^2\cdot P_3(x))\right|_{x=-2} = 0$

- ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (2x+2)\,y' - 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$
- Revert the change of variables $u = x+2$

  $[y = a_0(-x-1)]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x*(x+2)*diff(y(x),x$2)+2*(x+1)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2\left(\frac{x\ln(x)}{2} - \frac{\ln(x+2)\,x}{2} + \frac{\ln(x)}{2} - \frac{\ln(x+2)}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 37

```
DSolve[x*(x+2)*y''[x]+2*(x+1)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

2.309 problem 313

2.309.1 Maple step by step solution 2967

Internal problem ID [7799]
Internal file name [`OUTPUT/6732_Sunday_June_05_2022_05_07_54_PM_86235736/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 313.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x(x+2)\,y''+(1+x)\,y'-4y=0}$$

Writing the ode as

$$\left(x^2+2x\right)y''+(1+x)\,y'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2+2x\\B&=1+x\\C&=-4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 + 30x - 3 \\ t &= 4\left(x^2 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{16x} - \frac{33}{16(x+2)} - \frac{3}{16(x+2)^2} - \frac{3}{16x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4(x^2 + 2x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\left(x+2\right)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}\right)\right) = \frac{3-3a_0}{x\left(x+2\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (1+x)\, e^{\frac{3\ln(x)}{4} + \frac{3\ln(x+2)}{4}} \\
&= (1+x)\, x^{\frac{3}{4}} (x+2)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1 \left(\frac{1}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(1+x\right) x^{\frac{3}{4}} \left(x+2\right)^{\frac{3}{4}}}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x^2+2x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{2\left(x^2+2x+\frac{1}{2}\right)}{\sqrt{x+2}\,\sqrt{x}\,\left(1+x\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(1+x\right) x^{\frac{3}{4}} \left(x+2\right)^{\frac{3}{4}}}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}}\right) + c_2 \left(\frac{\left(1+x\right) x^{\frac{3}{4}} \left(x+2\right)^{\frac{3}{4}}}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}}\left(-\frac{2\left(x^2+2x+\frac{1}{2}\right)}{\sqrt{x+2}\,\sqrt{x}\,\left(1+x\right)}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1\left(1+x\right) x^{\frac{3}{4}} \left(x+2\right)^{\frac{3}{4}}}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}} - \frac{2c_2 x^{\frac{1}{4}} \left(x+2\right)^{\frac{1}{4}} \left(x^2+2x+\frac{1}{2}\right)}{\left(x\left(x+2\right)\right)^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+x)\, x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x\,(x+2))^{\frac{1}{4}}} - \frac{2c_2 x^{\frac{1}{4}}(x+2)^{\frac{1}{4}}\left(x^2+2x+\frac{1}{2}\right)}{(x\,(x+2))^{\frac{1}{4}}}$$

Verified OK.

2.309.1 Maple step by step solution

Let's solve

$(x^2+2x)\, y'' + (1+x)\, y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{x(x+2)} - \frac{(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x(x+2)} - \frac{4y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x(x+2)}, P_3(x) = -\frac{4}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left.((x+2)\cdot P_2(x))\right|_{x=-2} = \frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left.((x+2)^2\cdot P_3(x))\right|_{x=-2} = 0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (1+x)\, y' - 4y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

$$\left(u^2-2u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-1+u\right)\left(\frac{d}{du}y(u)\right)-4y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

 ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

 ◦ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

 ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

 ◦ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k+r+2)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+a_k(k+r+2)\,(k+r-2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+2)(k+r-2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$; series terminates at $k=2$

$$a_{k+1}=\frac{a_k(k+2)(k-2)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k=0$

$a_1 = -4a_0$

- Apply recursion relation for $k = 1$

$a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

$a_2 = 2a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot (2u^2 - 4u + 1)$

- Revert the change of variables $u = x + 2$

$[y = a_0(2x^2 + 4x + 1)]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Revert the change of variables $u = x + 2$

$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0(2x^2 + 4x + 1) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. Attempt
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*(x+2)*diff(y(x),x$2)+(x+1)*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(2x^2 + 4x + 1) + c_2(x + 1)\sqrt{x(x + 2)}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 53

```
DSolve[x*(x+2)*y''[x]+(x+1)*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(4\log\left(\sqrt{x + 2} - \sqrt{x}\right)\right) - ic_2 \sinh\left(4\log\left(\sqrt{x + 2} - \sqrt{x}\right)\right)$$

2.310 problem 314

2.310.1 Maple step by step solution 2977

Internal problem ID [7800]
Internal file name [`OUTPUT/6733_Sunday_June_05_2022_05_07_57_PM_60929359/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 314.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0)\\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\ &= y_1\left(-x\,e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-x\,e^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 x$$

Verified OK.

2.310.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$

◦ Shift index using $k->k+1$

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1))\,u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+e^x c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

2.311 problem 315

Internal problem ID [7801]
Internal file name [`OUTPUT/6734_Sunday_June_05_2022_05_08_00_PM_47419187/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 315.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$\begin{align*}(x^2+1)\,y''-2xy'+2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+1\\ B&=-2x \tag{3}\\ C&=2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{\left(x^2+1\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= \left(x^2+1\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1\left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2\left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2 (x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2 (x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 21

```
DSolve[(1+x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

2.312 problem 316

Internal problem ID [7802]
Internal file name [`OUTPUT/6735_Sunday_June_05_2022_05_08_03_PM_31052204/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 316.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2 - 2x + 10)\, y'' + xy' - 4y = 0}$$

Writing the ode as

$$(x^2 - 2x + 10)\, y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 2x + 10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 32x + 180 \\ t &= 4\left(x^2 - 2x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 591: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12} \end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\,(x^2 - 2x + 10)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$1+3i$	2	0	$\frac{3}{4}+\frac{i}{12}$	$\frac{1}{4}-\frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4}-\frac{i}{12}$	$\frac{1}{4}+\frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} \\
&= \frac{3x - 4}{2x^2 - 4x + 20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)(1) + \left(\left(\frac{-\frac{3}{4} - \frac{i}{12}}{(x - 1 - 3i)^2} + \frac{-\frac{3}{4} + \frac{i}{12}}{(x - 1 + 3i)^2}\right) + \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right.\right.
$$

$$
- \frac{3\ldots}{(x \ldots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3 \ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(-\frac{1}{3} + \frac{x}{3}\right)}{6}} \\
&= \frac{(3x - 4)\left(x^2 - 2x + 10\right)^{\frac{3}{4}} e^{-\frac{\arctan\left(-\frac{1}{3} + \frac{x}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\right) \\
&\quad + c_2 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2+\frac{6}{205}x-\frac{9}{82}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2+\frac{6}{205}x-\frac{9}{82}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^2-\frac{4}{3}x+5\right)+c_2(3x-4)\sqrt{x^2-2x+10}\left(\frac{-x+1+3i}{x-1+3i}\right)^{\frac{i}{6}}$$

✓ Solution by Mathematica

Time used: 0.842 (sec). Leaf size: 92

```
DSolve[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2 - 2x + 10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2 \int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4 - 3K[1])^2 \left(K[1]^2 - 2K[1] + 10\right)^{3/2}} dK[1] + c_1\right)$$

2.313 problem 317

Internal problem ID [7803]
Internal file name [`OUTPUT/6736_Sunday_June_05_2022_05_08_06_PM_18566677/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 317.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-2x+10)\,y''+xy'-4y=0}$$

Writing the ode as

$$(x^2-2x+10)\,y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 32x + 180 \\ t &= 4\left(x^2 - 2x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 592: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216 (x - 1 - 3i)} + \frac{149i}{216 (x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12}
\end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4 (x^2 - 2x + 10)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$1+3i$	2	0	$\frac{3}{4}+\frac{i}{12}$	$\frac{1}{4}-\frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4}-\frac{i}{12}$	$\frac{1}{4}+\frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right)\\
&= \frac{5}{2}-\left(\frac{3}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3
$$

$$
-\frac{3}{(x
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{\frac{3}{4}} e^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\right) \\
&\quad + c_2 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2+\frac{6}{205}x-\frac{9}{82}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2+\frac{6}{205}x-\frac{9}{82}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^2-\frac{4}{3}x+5\right)+c_2(3x-4)\sqrt{x^2-2x+10}\left(\frac{-x+1+3i}{x-1+3i}\right)^{\frac{i}{6}}$$

✓ Solution by Mathematica

Time used: 0.571 (sec). Leaf size: 92

```
DSolve[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2 - 2x + 10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2 \int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4 - 3K[1])^2 (K[1]^2 - 2K[1] + 10)^{3/2}} dK[1] + c_1\right)$$

2.314 problem 318

2.314.1 Maple step by step solution 3010

Internal problem ID [7804]
Internal file name [OUTPUT/6737_Sunday_June_05_2022_05_08_09_PM_6841783/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 318.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' + 2y = 0}$$

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 593: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 - 1\right) + c_2 \left(x^2 - 1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.314.1 Maple step by step solution

Let's solve

$y'' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2+3k+2)\, a_{k+2} - a_k(k-2) = 0$

- Recursion relation; series terminates at $k=2$

  $a_{k+2} = \frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k=0$

  $a_2 = -a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x - a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 - 1) + c_2(x^2 - 1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x-1)^2(x+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 54

```
DSolve[y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

2.315 problem 319

2.315.1 Maple step by step solution 3020

Internal problem ID [7805]
Internal file name [`OUTPUT/6738_Sunday_June_05_2022_05_08_14_PM_3659847/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 319.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+2)\,y''+xy'-y=0}$$

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{4 (x+2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{4 (x+2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 595: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{4(x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x+2} - \frac{1}{2} \\
&= -\frac{4+x}{2(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{(x+2)^2}\right) + \left(-\frac{1}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+12}{4(x+2)^2}\right)\right) = 0
$$

$$
\frac{a_0 - 4}{x+2} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, e^{\int \left(-\frac{1}{x+2} - \frac{1}{2}\right) dx} \\
&= (4+x)\, e^{-\frac{x}{2} - \ln(x+2)} \\
&= \frac{(4+x)\, e^{-\frac{x}{2}}}{x+2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (4+x)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x\,\mathrm{e}^{x}}{4+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)\,\mathrm{e}^{-x}\right) + c_2\left((4+x)\,\mathrm{e}^{-x}\left(\frac{x\,\mathrm{e}^{x}}{4+x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(4+x)\,\mathrm{e}^{-x} + c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1(4+x)\,\mathrm{e}^{-x} + c_2 x
$$

Verified OK.

2.315.1 Maple step by step solution

Let's solve

$(x+2)\,y''+xy'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x+2}-\frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x+2}-\frac{y}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\,y''+xy'-y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-1)}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+1} = -\frac{a_k(k-1)}{(k+1)(k-2)}$$

- Apply recursion relation for $k=0$

$$a_1 = -\frac{a_0}{2}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$$

- Revert the change of variables $u = x+2$

$$\left[y = -\frac{a_0 x}{2}\right]$$

- Recursion relation for $r=3$

$a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = -\frac{a_0 x}{2} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+3}\right), b_{k+1} = -\frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x+2)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 e^{-x}(x+4)$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 72

```
DSolve[(x+2)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2\sqrt{\frac{2}{\pi}}e^{-x-2}\sqrt{x+2}(c_1(e^{x+2}x+x+4)-ic_2((e^{x+2}-1)\,x-4))}{\sqrt{-i(x+2)}}$$

2.316 problem 320

Internal problem ID [7806]
Internal file name [`OUTPUT/6739_Sunday_June_05_2022_05_08_17_PM_10638099/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 320.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{(x^2+1)\, y''-6y=0}$$

Writing the ode as

$$(x^2+1)\, y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2 + 1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 + 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2 + 1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 597: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x-i}\right)(2x + a_1) + \left(\left(-\frac{1}{(x-i)^2} \right) + \left(\frac{1}{x-i} \right)^2 - \left(\frac{6}{x^2+1} \right) \right) = 0$$

$$2 + \frac{-4x - 2a_1}{-x+i} + \frac{-6x^2 - 6a_1 x - 6a_0}{x^2+1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + ix$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + ix\right) \mathrm{e}^{\int \frac{1}{x-i} dx} \\
&= \left(x^2 + ix\right) \mathrm{e}^{\frac{\ln\left(x^2+1\right)}{2} + i \arctan(x)} \\
&= x(x+i)(ix+1)
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x(x+i)(ix+1)
\end{aligned}$$

Which simplifies to

$$y_1 = ix^3 + ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= ix^3 + ix \int \frac{1}{\left(ix^3 + ix\right)^2} \, dx \\
&= ix^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2+2} + \frac{3\arctan(x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (i x^3 + i x) + c_2 \left(i x^3 + i x \left(\frac{1}{x} + \frac{x}{2x^2 + 2} + \frac{3 \arctan(x)}{2} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 (i x^3 + i x) + i c_2 \left(1 + \frac{3(x^3 + x) \arctan(x)}{2} + \frac{3x^2}{2} \right) \tag{1}
$$

Verification of solutions

$$
y = c_1 (i x^3 + i x) + i c_2 \left(1 + \frac{3(x^3 + x) \arctan(x)}{2} + \frac{3x^2}{2} \right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((x^2+1)*diff(y(x),x$2)-6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1 (x^3 + x) + c_2 \left(\frac{3 \arctan(x) x^3}{2} + \frac{3 \arctan(x) x}{2} + \frac{3x^2}{2} + 1 \right)
$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 36

```
DSolve[(x^2+1)*y''[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x^3 + x) - \frac{1}{2}c_2(3(x^3 + x)\arctan(x) + 3x^2 + 2)$$

2.317 problem 321

Internal problem ID [7807]
Internal file name [`OUTPUT/6740_Sunday_June_05_2022_05_08_20_PM_57855755/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 321.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+2)\,y''+3xy'-y=0}$$

Writing the ode as

$$(x^2+2)\,y''+3xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2 \\ B &= 3x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 20}{4\left(x^2 + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 20$$
$$t = 4\left(x^2 + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 20}{4\left(x^2 + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 598: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4\left(x^2+2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1, 2, 3\}$
$-i\sqrt{2}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{\left(x - (i\sqrt{2})\right)} + \frac{1}{\left(x - (-i\sqrt{2})\right)}\right) \\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\right) w + \frac{7x^2 + 16}{4\left(\sqrt{2} + ix\right)^2 \left(x + i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{2x^2 + 4}}{2x^2 + 4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{x + 2\sqrt{2x^2+4}}{2x^2+4} dx} \\ &= \left(x^2 + 2\right)^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x}{x^2+2} \, dx} \\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}} \\ &= z_1 \left(\frac{1}{\left(x^2 + 2\right)^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}\right) + c_2 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} + \frac{c_2 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)}{\sqrt{x^2+2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} + \frac{c_2 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)}{\sqrt{x^2+2}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve((x^2+2)*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\left(\sqrt{2}\,x+\sqrt{2}\,\sqrt{x^2+2}\right)^{\sqrt{2}}}{\sqrt{x^2+2}} + \frac{c_2\left(\frac{\sqrt{2}}{2\sqrt{x^2+2}+2x}\right)^{\sqrt{2}}}{\sqrt{x^2+2}}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 92

```
DSolve[(x^2+2)*y''[x]+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2^{3/4}c_1\cos\left(2\sqrt{2}\arcsin\left(\frac{1}{2}\sqrt{2-i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}} + \frac{c_2 Q^{\frac{1}{2}}_{-\frac{1}{2}+\sqrt{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}$$

2.318 problem 322

2.318.1 Maple step by step solution 3044

Internal problem ID [7808]
Internal file name [`OUTPUT/6741_Sunday_June_05_2022_05_08_23_PM_60505126/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 322.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x-1\right)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\left(x-1\right)^2}\right) + \left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4\left(x-1\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} - c_2 x$$

Verified OK.

2.318.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.319 problem 323

Internal problem ID [7809]
Internal file name [`OUTPUT/6742_Sunday_June_05_2022_05_08_26_PM_87785353/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 323.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 - 9\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 601: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2 - 9) + (0) \\ &= x^2 - 9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &= -5 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(12x^2 + 6xa_3 + 2a_2) + 2(-x)\left(4x^3 + 3x^2 a_3 + 2xa_2 + a_1\right) + \left((-1) + (-x)^2 - (x^2 - 9)\right) &= 0 \\ 2a_3 x^3 + 4(3 + a_2)\, x^2 + 6(a_1 + a_3)\, x + 8a_0 + 2a_2 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\ &= \frac{(4x^4 - 12x^2 + 3)\, e^{-\frac{x^2}{2}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2 \left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2 (4x^4 - 12x^2 + 3) \left(\int \frac{\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 49

```
DSolve[y''[x]-2*x*y''[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{4x-2}\left(c_1 \operatorname{BesselI}\left(1, 4\sqrt{x-\frac{1}{2}}\right) - c_2 K_1\left(4\sqrt{x-\frac{1}{2}}\right)\right)$$

2.320 problem 325

2.320.1 Maple step by step solution 3063

Internal problem ID [7810]
Internal file name [OUTPUT/6743_Sunday_June_05_2022_05_08_29_PM_67284273/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 325.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0}$$

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 + 30x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 602: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{6x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{6}\right) - (0) \\
&= \frac{5}{6}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 30x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= \frac{5}{6} - \left(-\frac{1}{6}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{6x} + \frac{1}{2} \\
&= -\frac{1}{6x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x} + \frac{1}{2}\right)(1) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} + \frac{1}{2}\right)^2 - \left(\frac{9x^2 + 30x + 7}{36x^2}\right)\right) = 0$$

$$\frac{-1 - 3a_0}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{1}{6x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\frac{x}{2} - \frac{\ln(x)}{6}} \\
&= \frac{(3x - 1)\, \mathrm{e}^{\frac{x}{2}}}{3x^{\frac{1}{6}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{5}{3}x + x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{3x - 1}{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{5}{3}x + x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x - \frac{5\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{9x^{\frac{1}{3}} \mathrm{e}^{-x}}{(3x - 1)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{3x-1}{3x}\right) + c_2 \left(\frac{3x-1}{3x}\left(\int \frac{9x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(3x-1)}{3x} + \frac{c_2(9x-3)\left(\int \frac{x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(3x-1)}{3x} + \frac{c_2(9x-3)\left(\int \frac{x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)}{x}
$$

Verified OK.

2.320.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(\frac{5}{3}x + x^2\right) y' - \frac{y}{3} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{3x^2} - \frac{(5+3x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(5+3x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5+3x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(5+3x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r-\frac{1}{3}\right)(k+r+1)\,a_k + 3a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

  $3\left(k+\frac{2}{3}+r\right)(k+2+r)\,a_{k+1} + 3a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{3a_k(k+r)}{(3k+2+3r)(k+2+r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{3a_k(k-1)}{(3k-1)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (-3x + 1)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (-3x + 1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k\left(k+\frac{1}{3}\right)}{(3k+3)\left(k+\frac{7}{3}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(x^2*diff(y(x),x$2)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x-1)}{x} + \frac{c_2(3x-1)\left(\int \frac{x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.355 (sec). Leaf size: 47

```
DSolve[x^2*y''[x]+(5/3*x+x^2)*y'[x]-1/3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-3c_1x + 3c_2e^{-x}\sqrt[3]{x} + c_2(1-3x)\Gamma\left(\frac{1}{3}, x\right) + c_1}{3x}$$

2.321 problem 326

2.321.1 Maple step by step solution 3073

Internal problem ID [7811]
Internal file name [`OUTPUT/6744_Sunday_June_05_2022_05_08_32_PM_32798533/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 326.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 604: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)\\ &= -\frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4\,(1+4x)\,x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\ &= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1 \left(x^{\frac{1}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
&\quad + \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
&\quad + \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned}
$$

Verified OK.

2.321.1 Maple step by step solution

Let's solve

$2xy'' - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$
- Multiply by denominators
  $2xy'' - y' + 2y = 0$
- Assume series solution for y
  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion
    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$
    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion
    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$
    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$
    Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(2i\sqrt{x}-1\right)}{1+2i\sqrt{x}}} + c_2 e^{-2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(1+2i\sqrt{x}\right)}{2i\sqrt{x}-1}}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 59

```
DSolve[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right) + \frac{1}{8} c_2 e^{-2i\sqrt{x}}\left(1+2i\sqrt{x}\right)$$

2.322 problem 327

2.322.1 Maple step by step solution 3084

Internal problem ID [7812]
Internal file name [OUTPUT/6745_Sunday_June_05_2022_05_08_35_PM_37550172/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 327.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{2xy'' - (3+2x)\,y' + y = 0}$$

Writing the ode as

$$\begin{align*} 2xy'' + (-2x-3)\,y' + y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 2x \\ B &= -2x-3 \tag{3}\\ C &= 1 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 4x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 606: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 4x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\
&= -\frac{3}{4x} + \frac{1}{2} \\
&= -\frac{3}{4x} + \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+21}{16x^2}\right)\right) = 0
$$

$$
\frac{-3-2a_0}{2x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{3}{2}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{3}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) e^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) e^{\frac{x}{2} - \frac{3\ln(x)}{4}} \\
&= \frac{(2x-3)\, e^{\frac{x}{2}}}{2x^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-3}{2x}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{3}{4}} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(2x-3)\,\mathrm{e}^{x}}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-3}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{4x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(2x-3)\,\mathrm{e}^{x}}{2}\right) + c_2 \left(\frac{(2x-3)\,\mathrm{e}^{x}}{2}\left(\int \frac{4x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1(2x-3)\,\mathrm{e}^{x}}{2} + c_2(4x-6)\,\mathrm{e}^{x}\left(\int \frac{x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(2x-3)\,\mathrm{e}^x}{2} + c_2(4x-6)\,\mathrm{e}^x \left(\int \frac{x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right)$$

Verified OK.

2.322.1 Maple step by step solution

Let's solve

$2xy'' + (-2x-3)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} + \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3+2x)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{3+2x}{2x}, P_3(x) = \frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-2x-3)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$
  - Shift index using $k- > k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - Shift index using $k- > k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-5+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_k(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{5}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{3}{2}+r\right)a_{k+1} - 2\left(k-\frac{1}{2}+r\right)a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(2k+2r-1)a_k}{(k+1+r)(2k-3+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(2k-1)a_k}{(k+1)(2k-3)}$
- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(2k-1)a_k}{(k+1)(2k-3)}\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+1} = \frac{(2k+4)a_k}{\left(k+\frac{7}{2}\right)(2k+2)}$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{(2k+4) a_k}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{(2k-1) a_k}{(k+1)(2k-3)}, b_{k+1} = \frac{(2k+4) b_k}{\left(k+\frac{7}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(2*x*diff(y(x),x$2)-(3+2*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{x}(-3+2x)}{2} + c_2 \mathrm{e}^{x}(-3+2x)\left(\int \frac{x^{\frac{3}{2}} \mathrm{e}^{-x}}{(-3+2x)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.575 (sec). Leaf size: 54

```
DSolve[2*x*y''[x]-(3+2*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{\pi} c_2 e^x (2x-3) \operatorname{erf}\left(\sqrt{x}\right) + 2c_1 e^x (2x-3) - 6c_2 \sqrt{x}\right)$$

2.323 problem 328

2.323.1 Maple step by step solution 3093

Internal problem ID [7813]
Internal file name [OUTPUT/6746_Sunday_June_05_2022_05_08_39_PM_37316014/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 328.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 3xy' + (2x - 1)\,y = 0}$$

Writing the ode as

$$2x^2y'' + 3xy' + (2x - 1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 608: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$$
$$= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)$$
$$= -\frac{1}{2x}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\phi = \theta + \frac{p'}{p}$$
$$= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4(1+4x)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2\\
&= c_1\left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\right)\\
&\quad + c_2\left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\\
&+ \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}\,x}
\end{aligned}
\tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y = {} & \frac{c_1\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\\
&+ \frac{c_2\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}\,x}
\end{aligned}
$$

Verified OK.

2.323.1 Maple step by step solution

Let's solve

$2x^2y'' + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2x} - \frac{(2x-1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $2x^2 y'' + 3xy' + (2x-1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, \frac{1}{2}\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+r+1)a_k+2a_{k-1}=0$

- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+2+r)a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k}{(2k+1+2r)(k+2+r)}$

- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}, b_{k+1}=-\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{2i\sqrt{x}} \sqrt{\frac{(1+4x)(2i\sqrt{x}-1)}{1+2i\sqrt{x}}}}{x} + \frac{c_2 \mathrm{e}^{-2i\sqrt{x}} \sqrt{\frac{(1+4x)(1+2i\sqrt{x})}{2i\sqrt{x}-1}}}{x}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 64

```
DSolve[2*x^2*y''[x]+3*x*y'[x]+(2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-2i\sqrt{x}} \left(8c_1 e^{4i\sqrt{x}} (2\sqrt{x} + i) + c_2 (1 + 2i\sqrt{x})\right)}{8x}$$

2.324 problem 329

2.324.1 Maple step by step solution 3100

Internal problem ID [7814]
Internal file name [OUTPUT/6747_Sunday_June_05_2022_05_08_42_PM_85193613/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 329.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' - yx = 0}$$

Writing the ode as

$$xy'' + 2y' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 610: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-x}}{x}\right) + c_2 \left(\frac{e^{-x}}{x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x}$$

Verified OK.

2.324.1 Maple step by step solution

Let's solve

$xy'' + 2y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' - yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- > k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- > k + 1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$

  - Shift index using $k- > k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, (k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1 + r)\, x^{-1+r} + a_1 (1 + r)\, (2 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + r + 1)\, (k + 2 + r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)-a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x)}{x} + \frac{c_2 \cosh(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 28

```
DSolve[x*y''[x]+2*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^{x}}{2x}$$

2.325 problem 330

2.325.1 Maple step by step solution 3107

Internal problem ID [7815]
Internal file name [`OUTPUT/6748_Sunday_June_05_2022_05_08_44_PM_55111855/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 330.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 612: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.325.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^{2}y'' + 4xy' + (4x^{2} - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.326 problem 331

2.326.1 Maple step by step solution 3118

Internal problem ID [7816]
Internal file name [OUTPUT/6749_Sunday_June_05_2022_05_08_46_PM_72099395/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 331.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (x-6)\,y' - 3y = 0}$$

Writing the ode as

$$xy'' + (x-6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x-6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 48 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 614: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 48}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\
&= \frac{1}{4} + \frac{12}{x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 48}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-3) \\
&= 3
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{3}{x} - \frac{1}{2} \\ &= -\frac{x+6}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)(3x^2 + 2a_2 x + a_1) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+48}{4x^2}\right)\right) = 0$$

$$\frac{(a_2 - 12)\, x^2 + 2(a_1 - 5a_2)\, x + 3a_0 - 6a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 120, a_1 = 60, a_2 = 12\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 12x^2 + 60x + 120$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 + 12x^2 + 60x + 120\right) e^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\ &= \left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2} - 3\ln(x)} \\ &= \frac{\left(x^3 + 12x^2 + 60x + 120\right) e^{-\frac{x}{2}}}{x^3}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-6}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+3\ln(x)} \\
&= z_1 \left(x^3 \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-x+6\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{x}\left(x^3 - 12x^2 + 60x - 120\right)}{x^3 + 12x^2 + 60x + 120}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-x}\right) \\
&\quad + c_2\left(\left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{x}\left(x^3 - 12x^2 + 60x - 120\right)}{x^3 + 12x^2 + 60x + 120}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-x} + c_2\left(x^3 - 12x^2 + 60x - 120\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(x^3 + 12x^2 + 60x + 120\right) \mathrm{e}^{-x} + c_2\left(x^3 - 12x^2 + 60x - 120\right)
$$

Verified OK.

2.326.1 Maple step by step solution

Let's solve

$xy'' + (x-6)\,y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x} - \frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-6)y'}{x} - \frac{3y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-6}{x}, P_3(x) = -\frac{3}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (x-6)\,y' - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-7+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-7+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 7\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 3$

  $$a_{k+1} = -\frac{a_k(k-3)}{(k+1)(k-6)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{a_0}{2}$$

- Apply recursion relation for $k = 1$

  $$a_2 = -\frac{a_1}{5}$$

- Express in terms of a_0

  $$a_2 = \frac{a_0}{10}$$

- Apply recursion relation for $k = 2$

  $$a_3 = -\frac{a_2}{12}$$

- Express in terms of a_0

  $$a_3 = -\frac{a_0}{120}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$

- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$

- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x*diff(y(x),x$2)+(x-6)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^3 - 12x^2 + 60x - 120) + c_2 e^{-x}(x^3 + 12x^2 + 60x + 120)$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 98

```
DSolve[x*y''[x]+(x-6)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left((c_1x^3 + 12ic_2x^2 + 60c_1x + 120ic_2)\cosh\left(\frac{x}{2}\right) - (12c_1(x^2+10) + ic_2x(x^2+60))\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

2.327 problem 332

Internal problem ID [7817]
Internal file name [`OUTPUT/6750_Sunday_June_05_2022_05_08_49_PM_4811039/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 332.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x^4y'' + \lambda y = 0}$$

Writing the ode as

$$x^4y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 616: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{\lambda}{x^4}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)(0) \\ &= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\ &= \frac{-i\sqrt{\lambda} + x}{x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x\,\mathrm{e}^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}} \tag{1}$$

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x \,\mathrm{e}^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(x^4*diff(y(x),x$2)+lambda*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 x \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 52

```
DSolve[x^4*y''[x]+\[Lambda]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

2.328 problem 333

2.328.1 Maple step by step solution 3136

Internal problem ID [7818]
Internal file name [`OUTPUT/6751_Sunday_June_05_2022_05_08_52_PM_78754354/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 333.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0}$$

Writing the ode as

$$4x^2y'' + 4xy' + \left(4x^2 - 25\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 617: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 - 3ix - 3)\, e^{\int (-\frac{2}{x} - i) dx} \\
&= (x^2 - 3ix - 3)\, e^{-ix - 2\ln(x)} \\
&= \frac{(x^2 - 3ix - 3)\, e^{-ix}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}$$

Verified OK.

2.328.1 Maple step by step solution

Let's solve

$4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{25}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\,x^r + a_1(7+2r)\,(-3+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5) + 4a_{\cdot}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(5+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(7+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r+5)\,(2k+2r-5) + 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

  $$a_{k+2}(2k+9+2r)\,(2k-1+2r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$$

- Recursion relation for $r = -\frac{5}{2}$

  $$a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$$

- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-25)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{ix}(x^2 + 3ix - 3)}{x^{\frac{5}{2}}} + \frac{c_2 e^{-ix}(x^2 - 3ix - 3)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 59

```
DSolve[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-25)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2+3c_1x+3c_2)\cos(x)+(c_1(x^2-3)+3c_2x)\sin(x))}{x^{5/2}}$$

2.329 problem 334

2.329.1 Maple step by step solution 3143

Internal problem ID [7819]
Internal file name [OUTPUT/6752_Sunday_June_05_2022_05_08_56_PM_9251467/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 334.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -36$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 619: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(6x)}{6}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(6x)}{\sqrt{x}}\left(\frac{\tan(6x)}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(6x)}{\sqrt{x}} + \frac{c_2 \sin(6x)}{6\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(6x)}{\sqrt{x}} + \frac{c_2 \sin(6x)}{6\sqrt{x}}
$$

Verified OK.

2.329.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(36x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(144x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_{0} = 0$ is a regular singular point

  $x_{0} = 0$

- Multiply by denominators

  $4x^{2}y'' + 4xy' + \left(144x^{2} - 1\right) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k} x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^{m} \cdot y$ to series expansion for $m = 0..2$

    $x^{m} \cdot y = \sum_{k=0}^{\infty} a_{k} x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_{k}(k + r)\, x^{k+r}$

  - Convert $x^{2} \cdot y''$ to series expansion

    $x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1 + 2r)\,(-1 + 2r)\, x^{r} + a_{1}(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(2k + 2r + 1)\,(2k + 2r - 1) + 144a\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_{1}(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 144a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(36*x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(6x)}{\sqrt{x}} + \frac{c_2 \cos(6x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(36*x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2 e^{12ix})}{12\sqrt{x}}$$

2.330 problem 335

2.330.1 Maple step by step solution 3154

Internal problem ID [7820]
Internal file name [`OUTPUT/6753_Sunday_June_05_2022_05_08_58_PM_19814557/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 335.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y(x^2 - 2) = 0}$$

Writing the ode as

$$x^2y'' + y(x^2 - 2) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 621: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) = 0$$
$$\frac{2ia_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x - i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\ &= (x - i)\, e^{-ix - \ln(x)} \\ &= \frac{(x - i)\, e^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(x - i)\, e^{-ix}}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\
&= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\right) + c_2 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x} \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x-i)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x-i)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x}$$

Verified OK.

2.330.1 Maple step by step solution

Let's solve

$x^2 y'' + y(x^2 - 2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y(x^2-2)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y(x^2-2)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + y(x^2 - 2) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\left(k+r-1\right)x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\left(-2+r\right)x^{r}+a_{1}(2+r)\left(-1+r\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r+1)\left(k+r-2\right)+a_{k-2}\right)x^{k+r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\left(-2+r\right)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,2\}$
- Each term must be 0

  $a_{1}(2+r)\left(-1+r\right)=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)\left(k+r-2\right)+a_{k-2}=0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)\left(k+r\right)+a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}}{(k+3+r)(k+r)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_{k}}{(k+2)(k-1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-1},a_{k+2}=-\frac{a_{k}}{(k+2)(k-1)},a_{1}=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_{k}}{(k+5)(k+2)}$
- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(x^2*diff(y(x),x$2)+(x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(-\sin(x)+\cos(x)x)}{x}+\frac{c_2(\cos(x)+x\sin(x))}{x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+(x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x(c_1 j_1(x)-c_2 y_1(x))$$

2.331 problem 336

2.331.1 Maple step by step solution 3163

Internal problem ID [7821]
Internal file name [OUTPUT/6754_Sunday_June_05_2022_05_09_03_PM_29877184/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 336.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 3y' + yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + yx^3 = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^4 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 623: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= ix \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-4x^4 + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-x^2) + \left(\frac{3}{4x^2}\right) \\
&= -x^2 + \frac{3}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(ix) \\
&= -\frac{1}{2x} - ix \\
&= -\frac{1}{2x} - ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - ix\right) dx} \\ &= \frac{e^{-\frac{ix^2}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{x} \, dx} \\ &= z_1 e^{-\frac{3 \ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{ix^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2} dx \\ &= y_1 \left(-\frac{i e^{i x^2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2}\right) + c_2 \left(\frac{e^{-\frac{i x^2}{2}}}{x^2} \left(-\frac{i e^{i x^2}}{2}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 e^{-\frac{i x^2}{2}}}{x^2} - \frac{i c_2 e^{\frac{i x^2}{2}}}{2x^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 e^{-\frac{i x^2}{2}}}{x^2} - \frac{i c_2 e^{\frac{i x^2}{2}}}{2x^2}$$

Verified OK.

2.331.1 Maple step by step solution

Let's solve

$x y'' + 3y' + y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 3y' + yx^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- > k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

◦ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r) x^{-1+r} + a_1(1+r)(3+r) x^r + a_2(2+r)(4+r) x^{1+r} + a_3(3+r)(5+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3) + a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)(k+6+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin\left(\frac{x^2}{2}\right)}{x^2}+\frac{c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 43

```
DSolve[x*y''[x]+3*y'[x]+x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-\frac{ix^2}{2}}\left(2c_1-ic_2e^{ix^2}\right)}{2x^2}$$

2.332 problem 337

2.332.1 Maple step by step solution 3170

Internal problem ID [7822]
Internal file name [OUTPUT/6755_Sunday_June_05_2022_05_09_06_PM_63765216/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 337.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + (x^2 + 2)\, y = 0}$$

Writing the ode as

$$x^2y'' + 4xy' + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 625: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2} (\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2}
$$

Verified OK.

2.332.1 Maple step by step solution

Let's solve

$x^2 y'' + 4xy' + (x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 4xy' + \left(x^2 + 2\right) y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2 + r)\,(1 + r)\, x^r + a_1(3 + r)\,(2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 2)\,(k + r + 1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3 + r)\,(2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k + r + 2)(k + r + 1) + a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}(k + 4 + r)(k + 3 + r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x^2} + \frac{c_2 \cos(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.333 problem 338

2.333.1 Maple step by step solution 3180

Internal problem ID [7823]
Internal file name [OUTPUT/6756_Sunday_June_05_2022_05_09_08_PM_9607050/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 338.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 32xy' + (x^4 - 12)\,y = 0}$$

Writing the ode as

$$16x^2y'' + 32xy' + (x^4 - 12)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4 + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 627: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{ix}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^4 + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\ &= -\frac{x^2}{16} + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{ix}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\ &= -\frac{1}{2x} - \frac{ix}{4} \\ &= -\frac{1}{2x} - \frac{ix}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4 + 12}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\ &= \frac{e^{-\frac{ix^2}{8}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{32x}{16x^2} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-2i\,\mathrm{e}^{\frac{i x^2}{4}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} \left(-2i\,\mathrm{e}^{\frac{i x^2}{4}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} - \frac{2 i c_2 \mathrm{e}^{\frac{i x^2}{8}}}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{8}}}{x^{\frac{3}{2}}} - \frac{2 i c_2 \mathrm{e}^{\frac{i x^2}{8}}}{x^{\frac{3}{2}}}
$$

Verified OK.

2.333.1 Maple step by step solution

Let's solve

$16x^2 y'' + 32 x y' + (x^4 - 12)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{(x^4-12)\, y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ○ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$
  - ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$
  - ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{3}{4}$
  - ○ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$
- • Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$
  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

    Rewrite ODE with series expansions

$$4a_0(3+2r)(-1+2r)x^r+4a_1(5+2r)(1+2r)x^{1+r}+4a_2(7+2r)(3+2r)x^{2+r}+4a_3(9+2r)($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4(3+2r)(-1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{-\frac{3}{2},\frac{1}{2}\right\}$$

- The coefficients of each power of x must be 0

  $$[4a_1(5+2r)(1+2r)=0,4a_2(7+2r)(3+2r)=0,4a_3(9+2r)(5+2r)=0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1=0,a_2=0,a_3=0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$16\left(k+r+\frac{3}{2}\right)\left(k-\frac{1}{2}+r\right)a_k+a_{k-4}=0$$

- Shift index using $k->k+4$

  $$16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right)a_{k+4}+a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4}=-\frac{a_k}{4(2k+11+2r)(2k+7+2r)}$$

- Recursion relation for $r=-\frac{3}{2}$

  $$a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)}$$

- Solution for $r=-\frac{3}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}},a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)},a_1=0,a_2=0,a_3=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+4}=-\frac{a_k}{4(2k+12)(2k+8)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+4}=-\frac{a_k}{4(2k+12)(2k+8)},a_1=0,a_2=0,a_3=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)},a_1=0,a_2=0,a_3=0,b_{k+4}=-\frac{}{4(2k}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(16*x^2*diff(y(x),x$2)+32*x*diff(y(x),x)+(x^4-12)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x^2}{8}\right)}{x^{\frac{3}{2}}} + \frac{c_2 \cos\left(\frac{x^2}{8}\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 42

```
DSolve[16*x^2*y''[x]+32*x*y'[x]+(x^4-12)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{\frac{ix^2}{4}}\right)}{x^{3/2}}$$

2.334 problem 339

2.334.1 Maple step by step solution 3190

Internal problem ID [7824]
Internal file name [`OUTPUT/6757_Sunday_June_05_2022_05_09_11_PM_49338061/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 339.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2 y' + yx = 0}$$

Writing the ode as

$$y'' - x^2 y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x(x^3 - 8)}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x(x^3 - 8) \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 629: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3 - 8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\ &= \frac{1}{4}x^4 - 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3 - 8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-) \left(\frac{x^2}{2} \right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3 - 8)}{4}\right)\right) = 0$$

$$x a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x^2}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^3}{6}} \\
&= x\, \mathrm{e}^{-\frac{x^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-3(-x^3)^{\frac{2}{3}}\, \mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}\, x} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}\,x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}
$$

Verified OK.

2.334.1 Maple step by step solution

Let's solve

$y'' - x^2 y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+1}$$

- Shift index using $k- >k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2) \, (k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2) \, (k+1) - a_{k-1}(k-2)\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2) \, a_{k+2} - a_{k-1}(k-2) = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k(k-1) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x - \frac{c_2 3^{\frac{1}{3}} \left(6(-x^3)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) 3^{\frac{2}{3}} - 6(-x^3)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) 3^{\frac{2}{3}} + 18 \,\mathrm{e}^{\frac{x^3}{3}}\right)}{3 \left(1 + \sqrt{-3}\right)}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 41

```
DSolve[y''[x]-x^2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3} \Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3 \sqrt[3]{3}}$$

2.335 problem 340

2.335.1 Maple step by step solution 3199

Internal problem ID [7825]
Internal file name [OUTPUT/6758_Sunday_June_05_2022_05_09_15_PM_42801058/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 340.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+2)\,y' + 2y = 0}$$

Writing the ode as

$$xy'' + (-x-2)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 631: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}}-0\right) &&= -1\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}}-0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2-4x+8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\
&= -1 - (-1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x-2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \ln(x)} \\
&= z_1 \left(x \, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(-e^{-x}\left(x^2+2x+2\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(-e^{-x}\left(x^2+2x+2\right)\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^x + c_2\left(-x^2-2x-2\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2\left(-x^2-2x-2\right)$$

Verified OK.

2.335.1 Maple step by step solution

Let's solve

$x y'' + (-x-2)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (-x-2)\, y' + 2y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- ○ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k}{k+4}$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)-(x+2)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 + 2x + 2) + e^x c_2$$

✓ Solution by Mathematica
Time used: 0.033 (sec). Leaf size: 24

```
DSolve[x*y''[x]-(x+2)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2(x^2 + 2x + 2)$$

2.336 problem 341

2.336.1 Maple step by step solution 3209

Internal problem ID [7826]
Internal file name [OUTPUT/6759_Sunday_June_05_2022_05_09_18_PM_65584959/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 341.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 633: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2}\right) \\
&= -\frac{x}{2} \\
&= -\frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\
a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\ &= x\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\, \mathrm{e}^{\frac{x^2}{2}}}{2x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.336.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^{k}$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k}(k+2)\right) x^{k} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\,(k a_{k+2} + a_{k} + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+2} = -\frac{a_{k}}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_{1} x\,\mathrm{e}^{-\frac{x^{2}}{2}} - \frac{c_{2} \mathrm{e}^{-\frac{x^{2}}{2}} \left(i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x + 2\,\mathrm{e}^{\frac{x^{2}}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.337 problem 342

2.337.1 Maple step by step solution 3218

Internal problem ID [7827]
Internal file name [`OUTPUT/6760_Sunday_June_05_2022_05_09_21_PM_32069475/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 342.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(1-x^2)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-x^2\\B&=-2x\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 635: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{5}{4(x-1)} - \frac{5}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2 - 3}{(x^2 - 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right.
$$
$$
- \frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x)\, e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= x\sqrt{x-1}\,\sqrt{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2\sqrt{x^2-1}\,(\ln(x-1)\,x-\ln(1+x)\,x+2)}{2\sqrt{x-1}\,\sqrt{1+x}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2 \sqrt{x^2-1}\,(\ln(x-1)\,x - \ln(1+x)\,x + 2)}{2\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

2.337.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{2}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = 1$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2xy' - 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ○ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ○ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = 1 + x$

  $[y = -a_0 x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(-\frac{\ln(x+1)\, x}{2} + \frac{\ln(x-1)\, x}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.338 problem 343

2.338.1 Maple step by step solution 3224

Internal problem ID [7828]
Internal file name [OUTPUT/6761_Sunday_June_05_2022_05_09_24_PM_15900690/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 343.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 637: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{x^2}\right) + c_2 \left(\mathrm{e}^{x^2} (x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\, \mathrm{e}^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\, \mathrm{e}^{x^2}
$$

Verified OK.

2.338.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x \, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.339 problem 344

2.339.1 Maple step by step solution 3233

Internal problem ID [7829]
Internal file name [OUTPUT/6762_Sunday_June_05_2022_05_09_26_PM_25266187/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 344.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\, y''-2xy'+30y=0}$$

Writing the ode as

$$(1-x^2)\, y''-2xy'+30y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2\\ B &= -2x \\ C &= 30 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{30x^2 - 31}{\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 30x^2 - 31 \\ t &= \left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{30x^2 - 31}{\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 639: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{61}{4(x-1)} - \frac{61}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2 - 31}{(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 30$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 6 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 6$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 6 - (1) \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\ &= \frac{x}{x^2 - 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2(x-\right.\right.$$

$$-10a_4x^4 + (-18a_3 - 20$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \mathrm{e}^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) \mathrm{e}^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x-1}\,\sqrt{1+x}}{63}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3087x(1449x^2-935)}{1600\,(63x^4-70x^2+15)} + \frac{441}{25x} - \frac{3969\ln(1+x)}{128} + \frac{3969\ln(x-1)}{128}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}}\right) \\
&\quad + c_2 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} \left(\frac{3087x(1449x^2-935)}{1600\,(63x^4-70x^2+15)} + \frac{441}{25x} \right.\right. \\
&\qquad \left.\left. - \frac{3969\ln(1+x)}{128} + \frac{3969\ln(x-1)}{128}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} \\
&\quad + \frac{3969c_2\sqrt{x^2-1}\left(\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)\ln(x-1) + \left(-x^5 + \frac{10}{9}x^3 - \frac{5}{21}x\right)\ln(1+x) + 2x^4 - \frac{14x^2}{9} + \frac{128}{945}\right)}{128\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} + \frac{3969c_2\sqrt{x^2-1}\left(\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)\ln(x-1) + \left(-x^5 + \frac{10}{9}x^3 - \frac{5}{21}x\right)\ln(1+x) + 2x^4 - \frac{14x^2}{9} + \frac{128}{945}\right)}{128\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

2.339.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 2xy' + 30y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{30y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{30y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2xy' - 30y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u - 2)\left(\frac{d}{du}y(u)\right) - 30y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$$

  - Shift index using $k- > k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, u^{k+r-2+m}$$

  - Shift index using $k- > k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k + 1 + r)^2 + a_k(k + r + 6)\,(k + r - 5)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k + 1)^2 + a_k(k + 6)\,(k - 5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 5$

$$a_{k+1} = \frac{a_k(k+6)(k-5)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

  $a_1 = -15a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{7a_1}{2}$
- Express in terms of a_0

  $a_2 = \frac{105a_0}{2}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{3}$
- Express in terms of a_0

  $a_3 = -70a_0$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{9a_3}{16}$
- Express in terms of a_0

  $a_4 = \frac{315a_0}{8}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 83

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+30*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{21}{5}x^5 - \frac{14}{3}x^3 + x\right) + c_2\left(-\frac{21\ln(x+1)x^5}{640} + \frac{21\ln(x-1)x^5}{640} + \frac{7\ln(x+1)x^3}{192} - \frac{7\ln(x-1)x^3}{192} + \frac{21x^4}{320} - \frac{\ln(x+1)x}{128} + \frac{\ln(x-1)x}{128} - \frac{49x^2}{960} + \frac{1}{225}\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 76

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+30*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}c_1x(63x^4 - 70x^2 + 15) + c_2\left(-\frac{63x^4}{8} + \frac{49x^2}{8} - \frac{1}{16}(63x^4 - 70x^2 + 15)x(\log(1-x) - \log(x+1)) - \frac{8}{15}\right)$$

2.340 problem 345

2.340.1 Maple step by step solution 3240

Internal problem ID [7830]
Internal file name [OUTPUT/6763_Sunday_June_05_2022_05_09_29_PM_91058357/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 345.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 641: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.340.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\,(k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1 + r)\, x^{-1+r} + a_1 (1 + r)\,(2 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + r + 1)\,(k + 2 + r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- > k+1$

$a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

$a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.341 problem 346

2.341.1 Maple step by step solution 3249

Internal problem ID [7831]
Internal file name [`OUTPUT/6764_Sunday_June_05_2022_05_09_31_PM_42180961/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 346.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (2x+1)\,y' + (1+x)\,y = 0}$$

Writing the ode as

$$xy'' + (2x+1)\,y' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x+1 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 643: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x+1}{x} \, dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\ln (x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \ln (x) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \ln (x)
$$

Verified OK.

2.341.1 Maple step by step solution

Let's solve

$x y'' + (2x+1) y' + (1+x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y'}{x} - \frac{(1+x)y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} + \frac{(1+x)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{1+x}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (2x+1)\,y' + (1+x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\,x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 + a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 2a_k k + a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+(2*x+1)*diff(y(x),x)+(x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{-x}\ln(x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2*x+1)*y'[x]+(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

2.342 problem 347

2.342.1 Maple step by step solution 3259

Internal problem ID [7832]
Internal file name [`OUTPUT/6765_Sunday_June_05_2022_05_09_34_PM_17943667/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 347.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{2x(x-1)\,y''-(1+x)\,y'+y=0}$$

Writing the ode as

$$(2x^2-2x)\,y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-2x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 18x - 3}{16 (x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 18x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 18x - 3}{16 (x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 645: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 2 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x} + \frac{3}{4(x-1)^2} - \frac{3}{16x^2} - \frac{3}{4(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{1}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{1}{2\left(x-1\right)} \\
&= \frac{-3+x}{4x\left(x-1\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4x} - \frac{1}{2\left(x-1\right)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2\left(x-1\right)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2\left(x-1\right)}\right)^2 - \left(\frac{-3x^2+18x-3}{16\left(x^2-x\right)^2}\right)\right) = 0
$$
$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right)dx} \\
&= \frac{x^{\frac{3}{4}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{2x^2-2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\frac{\sqrt{x-1}}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2x+2}{\sqrt{x}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\right) + c_2 \left(\sqrt{x}\left(\frac{2x+2}{\sqrt{x}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} + c_2(2x+2) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + c_2(2x+2)$$

Verified OK.

2.342.1 Maple step by step solution

Let's solve

$(2x^2-2x)\,y''+(-x-1)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(1+x)y'}{2x(x-1)}-\frac{y}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(1+x)y'}{2x(x-1)}+\frac{y}{2x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1+x}{2x(x-1)},P_3(x)=\frac{1}{2x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-x-1)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)\,(k+r-1)\right) x^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(2k+2r-1)(k+r-1)a_k}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $$a_{k+1} = \frac{(2k-1)(k-1)a_k}{(2k+1)(k+1)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = a_0$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $$y = a_0 \cdot (1+x)$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = \frac{2k\left(k-\frac{1}{2}\right)a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2k\left(k-\frac{1}{2}\right)a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1+x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2k\left(k-\frac{1}{2}\right) b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*x*(x-1)*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 21

```
DSolve[2*x*(x-1)*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt{x} - 2c_2(x+1)$$

2.343 problem 348

2.343.1 Maple step by step solution 3265

Internal problem ID [7833]
Internal file name [OUTPUT/6766_Sunday_June_05_2022_05_09_37_PM_23347882/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 348.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' + 4yx = 0}$$

Writing the ode as

$$xy'' + 2y' + 4yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 647: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(2x)}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x}\right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x}
$$

Verified OK.

2.343.1 Maple step by step solution

Let's solve

$xy'' + 2y' + 4yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + 4y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + 4yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- > k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)+4a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)+4a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{4a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{4a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{4a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{4b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(2x)}{x} + \frac{c_2 \cos(2x)}{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+4*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x}$$

2.344 problem 349

2.344.1 Maple step by step solution 3272

Internal problem ID [7834]
Internal file name [OUTPUT/6767_Sunday_June_05_2022_05_09_39_PM_73299547/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 349.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-2x + 2)\, y' + (x - 2)\, y = 0}$$

Writing the ode as

$$xy'' + (-2x + 2)\, y' + (x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x + 2 \\ C &= x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 649: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x+2}{x}\,dx} \\ &= z_1 e^{x-\ln(x)} \\ &= z_1\left(\frac{e^x}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{x} + c_2 e^x \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{x} + c_2 e^x$$

Verified OK.

2.344.1 Maple step by step solution

Let's solve

$x y'' + (-2x + 2) y' + (x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$xy'' + (-2x+2)\,y' + (x-2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\,x^{-1+r} + \left(a_1(1+r)\,(2+r) - 2a_0(1+r)\right)x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) - 2a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) - 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) - 2a_{k+1}(k+1) - 2ra_{k+1} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^x}{x} + \mathrm{e}^x c_2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{x}$$

2.345 problem 350

2.345.1 Maple step by step solution 3279

Internal problem ID [7835]
Internal file name [`OUTPUT/6768_Sunday_June_05_2022_05_09_41_PM_73393708/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 350.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 6xy' + (4x^2 + 6)\, y = 0}$$

Writing the ode as

$$x^2y'' + 6xy' + (4x^2 + 6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 651: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\,dx} \\
&= z_1 e^{-3\ln(x)} \\
&= z_1\left(\frac{1}{x^3}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(2x)}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x^3}\right) + c_2 \left(\frac{\cos(2x)}{x^3} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(2x)}{x^3} + \frac{c_2 \sin(2x)}{2x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(2x)}{x^3} + \frac{c_2 \sin(2x)}{2x^3}
$$

Verified OK.

2.345.1 Maple step by step solution

Let's solve

$x^2 y'' + 6xy' + (4x^2 + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(2x^2+3)y}{x^2} - \frac{6y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6y'}{x} + \frac{2(2x^2+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 6xy' + (4x^2 + 6)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3 + r)\,(2 + r)\, x^r + a_1(4 + r)\,(3 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 3)\,(k + r + 2) + 4a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3 + r)\,(2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4 + r)\,(3 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r+2) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+5+r)(k+4+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+6*x*diff(y(x),x)+(4*x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(2x)}{x^3} + \frac{c_2 \cos(2x)}{x^3}$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+6*x*y'[x]+(4*x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

2.346 problem 351

2.346.1 Maple step by step solution 3288

Internal problem ID [7836]
Internal file name [`OUTPUT/6769_Sunday_June_05_2022_05_09_43_PM_47352609/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 351.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\,y' + (x-1)\,y = 0}$$

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 653: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x - \ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \ln(x)$$

Verified OK.

2.346.1 Maple step by step solution

Let's solve

$xy'' + (1-2x) y' + (x-1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- ○ Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- ○ Shift index using $k- >k + 1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1 (1 + r)^2 - a_0 (1 + 2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + 1 + r)^2 - a_k (2k + 2r + 1) + a_{k-1}\right) x^k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $$a_1(1+r)^2 - a_0(1+2r) = 0$$
- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$$
- Shift index using $k- > k+1$

  $$a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$
- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$$
- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x \ln(x)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

2.347 problem 352

2.347.1 Maple step by step solution 3298

Internal problem ID [7837]
Internal file name [OUTPUT/6770_Sunday_June_05_2022_05_09_46_PM_70493527/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 352.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\,y''+\left(2x+\frac{1}{2}\right)y'-2y=0}$$

Writing the ode as

$$\left(-x^2+x\right)y''+\left(2x+\frac{1}{2}\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^2+x\\ B&=2x+\frac{1}{2}\\ C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x - 3}{16(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x - 3}{16(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 655: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 1 \\
&= 3
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{8x} + \frac{45}{16(x-1)^2} - \frac{3}{16x^2} - \frac{21}{8(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}
$$

Since the order of r at ∞ is $3 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{5}{4\left(x - 1\right)} + (0) \\ &= \frac{1}{4x} - \frac{5}{4\left(x - 1\right)} \\ &= -\frac{1 + 4x}{4x\left(x - 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{5}{4\,(x-1)}\right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{5}{4\,(x-1)^2}\right) + \left(\frac{1}{4x} - \frac{5}{4\,(x-1)}\right)^2 - \left(\frac{48x-3}{16\,(x^2-x)^2}\right)\right) = 0$$

$$\frac{-1+4a_0}{2x\,(x-1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = \frac{1}{4} + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \left(\frac{1}{4} + x\right) \mathrm{e}^{\int \left(\frac{1}{4x} - \frac{5}{4(x-1)}\right)dx} \\
&= \left(\frac{1}{4} + x\right) \mathrm{e}^{\frac{\ln(x)}{4} - \frac{5\ln(x-1)}{4}} \\
&= \frac{\left(\frac{1}{4} + x\right) x^{\frac{1}{4}}}{(x-1)^{\frac{5}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+\frac{1}{2}}{-x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{5\ln(x-1)}{4}} \\
&= z_1 \left(\frac{(x-1)^{\frac{5}{4}}}{x^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{4} + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+\frac{1}{2}}{-x^2+x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\frac{5\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{x}\,\sqrt{x-1}\,\left(12x\ln(2) - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 4\sqrt{x\,(x-1)}\,x + 3\ln(2) - 3\ln\left(2x - \right.\right.}{\sqrt{x\,(x-1)}\,(1+4x)} \right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{1}{4}+x\right) + c_2\left(\frac{1}{4}\right. \\
&\quad +x\left(\frac{\sqrt{x}\,\sqrt{x-1}\,\left(12x\ln(2) - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 4\sqrt{x\,(x-1)}\,x + 3\ln(2) - 3\ln\left(2x - \right.\right.}{\sqrt{x\,(x-1)}\,(1+4x)}\right.
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned}
y = c_1\left(\frac{1}{4}+x\right) \\
-\frac{c_2\sqrt{x}\,\sqrt{x-1}\,\left(12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) - 12x\ln(2) - 4\sqrt{x\,(x-1)}\,x + 3\ln\left(2x-1+2\sqrt{x\,(x\right.\right.}{4\sqrt{x\,(x-1)}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\left(\frac{1}{4}+x\right) - \frac{c_2\sqrt{x}\,\sqrt{x-1}\left(12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)-12x\ln\left(2\right)-4\sqrt{x\,(x-1)}\,x+3\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)\right.}{4\sqrt{x\,(x-1)}}$$

Verified OK.

2.347.1 Maple step by step solution

Let's solve

$(-x^2+x)\,y''+\left(2x+\frac{1}{2}\right)y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x-1)}+\frac{(1+4x)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(1+4x)y'}{2x(x-1)}+\frac{2y}{x(x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{1+4x}{2x(x-1)},P_3(x)=\frac{2}{x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-1-4x)\,y'+4y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + 2a_k(k+r-1)\,(k+r-2)\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r-1)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{2a_k(k-1)(k-2)}{(2k+1)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 4a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot (1 + 4x)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1 + 4x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve(x*(1-x)*diff(y(x),x$2)+(1/2+2*x)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(1 + 4x) + c_2\left(4\sqrt{x(x-1)}\,x - 12\ln\left(x - \frac{1}{2} + \sqrt{x(x-1)}\right)x + 26\sqrt{x(x-1)} - 3\ln\left(x - \frac{1}{2} + \sqrt{x(x-1)}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.416 (sec). Leaf size: 64

```
DSolve[x*(1-x)*y''[x]+(1/2+2*x)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_2\left(\sqrt{-((x-1)x)}(2x+13) - 6(4x+1)\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right)\right) + c_1\left(x+\frac{1}{4}\right)$$

2.348 problem 353

2.348.1 Maple step by step solution 3308

Internal problem ID [7838]
Internal file name [`OUTPUT/6771_Sunday_June_05_2022_05_09_59_PM_82981702/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 353.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4(t^2-3t+2)\,y''-2y'+y=0}$$

Writing the ode as

$$y''(4t^2-12t+8)+y-2y'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2-12t+8 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4t^2 + 20t - 19$$
$$t = 16\left(t^2 - 3t + 2\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 657: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{8(t-1)} + \frac{5}{16(t-2)^2} - \frac{3}{8(t-2)} - \frac{3}{16(t-1)^2}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16(t^2 - 3t + 2)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(t-2\right)} + \frac{3}{4\left(t-1\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(t-2\right)} + \frac{3}{4\left(t-1\right)} \\
&= \frac{2t-5}{4\left(t-1\right)\left(t-2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(t-2\right)} + \frac{3}{4\left(t-1\right)}\right)(0) + \left(\left(\frac{1}{4\left(t-2\right)^2} - \frac{3}{4\left(t-1\right)^2}\right) + \left(-\frac{1}{4\left(t-2\right)} + \frac{3}{4\left(t-1\right)}\right)^2 - \left(\frac{-4}{16(}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\
&= \frac{\left(t-1\right)^{\frac{3}{4}}}{\left(t-2\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{4t^2-12t+8}\, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{4} + \frac{\ln(t-2)}{4}} \\
&= z_1 \left(\frac{\left(t-2\right)^{\frac{1}{4}}}{\left(t-1\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{\ln(t-1)}{2}+\frac{\ln(t-2)}{2}}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4}{\sqrt{t-1}\,\sqrt{t-2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{t-1}\right) \\
&\quad + c_2\left(\sqrt{t-1}\left(\frac{\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4}{\sqrt{t-1}\,\sqrt{t-2}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{t-1} + \frac{c_2\left(\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4\right)}{\sqrt{t-2}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{t-1} + \frac{c_2\left(\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4\right)}{\sqrt{t-2}}$$

Verified OK.

2.348.1 Maple step by step solution

Let's solve

$y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4(t^2-3t+2)} + \frac{y'}{2(t^2-3t+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2(t^2-3t+2)} + \frac{y}{4(t^2-3t+2)} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{1}{2(t^2-3t+2)}, P_3(t) = \frac{1}{4(t^2-3t+2)}\right]$

  ◦ $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Big|_{t=1} = \frac{1}{2}$

  ◦ $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Big|_{t=1} = 0$

  ◦ $t = 1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $(4u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$$

- Shift index using $k->k+1$

  $$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-1+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)^2\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r-1)^2 - 4\left(k+\frac{1}{2}+r\right)a_{k+1}(k+1+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(2k+2r-1)^2}{2(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}\right]$$

- Revert the change of variables $u = t-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k k^2}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Revert the change of variables $u = t - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (t-1)^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k (2k-1)^2}{2(2k+1)(k+1)}, b_{k+1} = \frac{2b_k k^2}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(4*(t^2-3*t+2)*diff(y(t),t$2)-2*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\sqrt{t-1} + \frac{c_2\sqrt{t-2}\,(t-1)\left(\ln\left(t-\frac{3}{2}+\sqrt{t^2-3t+2}\right)\sqrt{t^2-3t+2}-2t+4\right)}{t^2-3t+2}$$

✓ Solution by Mathematica

Time used: 0.257 (sec). Leaf size: 53

```
DSolve[4*(t^2-3*t+2)*y''[t]-2*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{1-t}\left(-2c_2\text{arctanh}\left(\frac{1}{\sqrt{\frac{t-1}{t-2}}}\right)+\frac{2c_2}{\sqrt{\frac{t-1}{t-2}}}+c_1\right)$$

2.349 problem 354

2.349.1 Maple step by step solution 3318

Internal problem ID [7839]
Internal file name [OUTPUT/6772_Sunday_June_05_2022_05_10_02_PM_97281057/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 354.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(t^2-5t+6)\,y''+(2t-3)\,y'-8y=0}$$

Writing the ode as

$$y''(2t^2-10t+12)+(2t-3)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2-10t+12 \\ B &= 2t-3 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 60t^2 - 308t + 381$$
$$t = 16\left(t^2 - 5t + 6\right)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 659: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{29}{8(t-3)} - \frac{3}{16(t-3)^2} + \frac{5}{16(t-2)^2} - \frac{29}{8(t-2)}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16(t^2 - 5t + 6)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{60t^2-308t+381}{16\left(t^2-5t+6\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\ &= \frac{5}{2}-\left(\frac{3}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} \\
&= \frac{6t-17}{4(t-2)(t-3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - \cdots}{16(t^2 - \cdots}\right.\right. \quad \frac{-6\cdots}{2t^2 - \cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{17}{6}\right\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t - \frac{17}{6}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\
&= \left(t - \frac{17}{6}\right) e^{\frac{\ln(t-3)}{4} + \frac{5\ln(t-2)}{4}} \\
&= \left(t - \frac{17}{6}\right) (t-3)^{\frac{1}{4}} (t-2)^{\frac{5}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2t-3}{2t^2-10t+12}\,dt} \\
&= z_1 e^{\frac{\ln(t-2)}{4}-\frac{3\ln(t-3)}{4}} \\
&= z_1 \left(\frac{(t-2)^{\frac{1}{4}}}{(t-3)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{\ln(t-2)}{2}-\frac{3\ln(t-3)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{(576t^2-2496t+2664)\sqrt{t-3}}{(t-2)^{\frac{3}{2}}\,(30t-85)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}}\right) + c_2 \left(\frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}} \left(\frac{(576t^2-2496t+2664)\sqrt{t-3}}{(t-2)^{\frac{3}{2}}\,(30t-85)}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}} + c_2\left(\frac{96}{5}t^2 - \frac{416}{5}t + \frac{444}{5}\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(6t-17)(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}} + c_2\left(\frac{96}{5}t^2 - \frac{416}{5}t + \frac{444}{5}\right)$$

Verified OK.

2.349.1 Maple step by step solution

Let's solve

$y''(2t^2-10t+12)+(2t-3)y'-8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{t^2-5t+6} - \frac{(2t-3)y'}{2(t^2-5t+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2t-3)y'}{2(t^2-5t+6)} - \frac{4y}{t^2-5t+6} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = \frac{2t-3}{2(t^2-5t+6)}, P_3(t) = -\frac{4}{t^2-5t+6}\right]$

  ◦ $(t-2)\cdot P_2(t)$ is analytic at $t=2$

    $\left.\left((t-2)\cdot P_2(t)\right)\right|_{t=2} = -\frac{1}{2}$

  ◦ $(t-2)^2\cdot P_3(t)$ is analytic at $t=2$

    $\left.\left((t-2)^2\cdot P_3(t)\right)\right|_{t=2} = 0$

  ◦ $t=2$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 2$

- Multiply by denominators

  $y''(2t^2-10t+12)+(2t-3)y'-8y=0$

- Change variables using $t=u+2$ so that the regular singular point is at $u=0$

  $(2u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u+1)\left(\frac{d}{du}y(u)\right)-8y(u)=0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(2k-1+2r)(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k+2)(k-2)}{(2k-1)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 8a_0$

- Apply recursion relation for $k=1$

  $a_2=-3a_1$
- Express in terms of a_0

  $a_2=-24a_0$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(-24u^2+8u+1\right)$
- Revert the change of variables $u=t-2$

  $[y=a_0(-24t^2+104t-111)]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}$
- Solution for $r=\frac{3}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$
- Revert the change of variables $u=t-2$

  $\left[y=\sum_{k=0}^{\infty}a_k(t-2)^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0(-24t^2+104t-111)+\left(\sum_{k=0}^{\infty}b_k(t-2)^{k+\frac{3}{2}}\right), b_{k+1}=\frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 32

```
dsolve(2*(t^2-5*t+6)*diff(y(t),t$2)+(2*t-3)*diff(y(t),t)-8*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\left(t^2 - \frac{13}{3}t + \frac{37}{8}\right) + \frac{c_2(6t-17)\,(t-2)^{\frac{3}{2}}}{\sqrt{t-3}}$$

✓ Solution by Mathematica
Time used: 0.267 (sec). Leaf size: 84

```
DSolve[2*(t^2-5*t+6)*y''[t]+(2*t-3)*y'[t]-8*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt[4]{2-t}\left(5c_1\sqrt[4]{t-3}\sqrt{t-2}(6t^2-29t+34)+24c_2(t-3)^{3/4}\,(24t^2-104t+111)\right)}{30(3-t)^{3/4}\sqrt[4]{t-2}}$$

2.350 problem 355

2.350.1 Maple step by step solution 3328

Internal problem ID [7840]
Internal file name [`OUTPUT/6773_Sunday_June_05_2022_05_10_05_PM_5622771/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 355.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3t(t+1)\, y''+t y'-y=0}$$

Writing the ode as

$$\left(3t^{2}+3t\right) y''+t y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^{2}+3t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7t + 12}{36 (t + 1)^2 t} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7t + 12$$
$$t = 36(t + 1)^2 t$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{7t + 12}{36t (t + 1)^2} \right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 661: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(t+1)^2 t$. There is a pole at $t = -1$ of order 2. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{3t} - \frac{1}{3(t+1)} - \frac{5}{36(t+1)^2}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7t+12}{36t(t+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{6} \\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{7t+12}{36t\,(t+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{7}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right) \\ &= \frac{7}{6}-\left(\frac{7}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{t} + \frac{1}{6t+6} + (0) \\
&= \frac{1}{t} + \frac{1}{6t+6} \\
&= \frac{1}{t} + \frac{1}{6t+6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{t} + \frac{1}{6t+6}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6(t+1)^2}\right) + \left(\frac{1}{t} + \frac{1}{6t+6}\right)^2 - \left(\frac{7t+12}{36t(t+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{t} + \frac{1}{6t+6}\right) dt} \\
&= t(t+1)^{\frac{1}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{3t^2+3t} \, dt} \\
&= z_1 e^{-\frac{\ln(t+1)}{6}} \\
&= z_1 \left(\frac{1}{(t+1)^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{\ln(t+1)}{3}}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{-2\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t - 6(t+1)^{\frac{2}{3}} - 2\ln\left((t+1)^{\frac{1}{3}} - 1\right) t + \ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right.}{6\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) \\
&\quad + c_2 \left(t \left(\frac{-2\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t - 6(t+1)^{\frac{2}{3}} - 2\ln\left((t+1)^{\frac{1}{3}} - 1\right) t + \ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}}\right.}{6\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \right. \right.
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 t - \frac{c_2\left(\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t + \ln\left((t+1)^{\frac{1}{3}} - 1\right) t - \frac{\ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right) t}{2} + 3(t+1)^{\frac{2}{3}}\right) t}{3\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \tag{1}$$

Verification of solutions

$$y = c_1 t - \frac{c_2 \left(\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t + \ln\left((t+1)^{\frac{1}{3}}-1\right) t - \frac{\ln\left((t+1)^{\frac{2}{3}}+(t+1)^{\frac{1}{3}}+1\right) t}{2} + 3(t+1)^{\frac{2}{3}}\right) t}{3\left((t+1)^{\frac{2}{3}}+(t+1)^{\frac{1}{3}}+1\right)\left((t+1)^{\frac{1}{3}}-1\right)}$$

Verified OK.

2.350.1 Maple step by step solution

Let's solve

$(3t^2+3t)\, y''+ty'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3t(t+1)}-\frac{y'}{3(t+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{3(t+1)}-\frac{y}{3t(t+1)}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t)=\frac{1}{3(t+1)}, P_3(t)=-\frac{1}{3t(t+1)}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left((t+1)\cdot P_2(t)\right)\bigg|_{t=-1}=\frac{1}{3}$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left((t+1)^2\cdot P_3(t)\right)\bigg|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0=-1$

- Multiply by denominators

  $3t(t+1)\, y''+ty'-y=0$

- Change variables using $t = u - 1$ so that the regular singular point is at $u = 0$

$$(3u^2 - 3u)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-2+3r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(3k+3r+1) + a_k(3k+3r+1)(k+r-1)\right) u^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k+r+\frac{1}{3}\right)\left((-k-r-1)\, a_{k+1} + a_k(k+r-1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k-1)}{k+1}$$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$

- Revert the change of variables $u = t + 1$

  $[y = -a_0 t]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$

- Solution for $r = \frac{2}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Revert the change of variables $u = t + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (t+1)^{k+\frac{2}{3}}, a_{k+1} = \frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Combine solutions and rename parameters

  $\left[y = -a_0 t + \left(\sum_{k=0}^{\infty} b_k (t+1)^{k+\frac{2}{3}}\right), b_{k+1} = \frac{b_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(3*t*(1+t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t \left(\int \frac{1}{(t+1)^{\frac{1}{3}} t^2} dt\right)$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 93

```
DSolve[3*t*(1+t)*y''[t]+t*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{6c_1t - c_2\left(2\sqrt{3}t\arctan\left(\frac{2\sqrt[3]{t+1}+1}{\sqrt{3}}\right) + 6(t+1)^{2/3} + 2t\log\left(\sqrt[3]{t+1}-1\right) - t\log\left((t+1)^{2/3} + \sqrt[3]{t+1}\right.\right.}{6\sqrt[6]{3}}$$

2.351 problem 356

2.351.1 Maple step by step solution 3338

Internal problem ID [7841]
Internal file name [OUTPUT/6774_Sunday_June_05_2022_05_10_10_PM_18578703/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 356.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + \frac{\left(\frac{3}{4} + x\right) y}{4} = 0}$$

Writing the ode as

$$x^2y'' + \left(\frac{3}{16} + \frac{x}{4}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{3}{16} + \frac{x}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 4x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3 - 4x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 - 4x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 663: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1 - (1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d = 0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{-x}+1}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{-x}+1}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{2\sqrt{-x}}} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \left(\frac{\sqrt{-x}\left(-1 + \mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}\right) + c_2\left(x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}\left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{x^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{x^{\frac{1}{4}}}
$$

Verified OK.

2.351.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(\frac{3}{16} + \frac{x}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3+4x)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+4x)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{3+4x}{16x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + (3+4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)\,(-3+4r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)\,(4k+4r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)\,(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) a_k + 4a_{k-1} = 0$

- Shift index using $k- > k + 1$

  $16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right) a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$

- Solution for $r = \frac{3}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+1/4*(x+3/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\sqrt{x}\right) x^{\frac{1}{4}} + c_2 x^{\frac{1}{4}} \cos\left(\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 43

```
DSolve[x^2*y''[x]+1/4*(x+3/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-i\sqrt{x}} \sqrt[4]{x} \left(c_1 e^{2i\sqrt{x}} + i c_2\right)$$

2.352 problem 357

2.352.1 Maple step by step solution 3345

Internal problem ID [7842]
Internal file name [OUTPUT/6775_Sunday_June_05_2022_05_10_13_PM_61468890/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 357.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 665: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(2\tan\left(\frac{x}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}} \left(2\tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{2c_2 \sin\left(\frac{x}{2}\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{2c_2 \sin\left(\frac{x}{2}\right)}{\sqrt{x}}
$$

Verified OK.

2.352.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3+2r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

$a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$a_k(4k^2+8kr+4r^2-1) + a_{k-2} = 0$

- Shift index using $k- > k+2$

$a_{k+2}\left(4(k+2)^2+8(k+2)\,r+4r^2-1\right) + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{b_k}{4k^2+20k+24}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+1/4*(x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 36

```
DSolve[x^2*y''[x]+x*y'[x]+1/4*(x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix}{2}}(c_1 - ic_2 e^{ix})}{\sqrt{x}}$$

2.353 problem 358

2.353.1 Maple step by step solution 3354

Internal problem ID [7843]
Internal file name [`OUTPUT/6776_Sunday_June_05_2022_05_10_15_PM_81846759/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 358.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\, y' + (x-1)\, y = 0}$$

Writing the ode as

$$xy'' + (1-2x)\, y' + (x-1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 667: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x - \ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x \ln(x)
$$

Verified OK.

2.353.1 Maple step by step solution

Let's solve

$xy'' + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (1-2x)\, y' + (x-1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x \ln(x)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

2.354 problem 359

2.354.1 Maple step by step solution 3364

Internal problem ID [7844]
Internal file name [`OUTPUT/6777_Sunday_June_05_2022_05_10_17_PM_65916189/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 359.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (1+x)\,y' + y = 0}$$

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 669: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2} dx \\ &= y_1 \left(-(1+x) e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-(1+x) e^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2(-x-1) \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2(-x-1)$$

Verified OK.

2.354.1 Maple step by step solution

Let's solve

$xy'' + (-x-1) y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x} + \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Bigg|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $xy'' + (-x-1)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$
  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$
  - Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x + 1) + e^x c_2$$

✓ Solution by Mathematica
Time used: 0.027 (sec). Leaf size: 19

```
DSolve[x*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2(x + 1)$$

2.355 problem 360

2.355.1 Maple step by step solution 3374

Internal problem ID [7845]
Internal file name [OUTPUT/6778_Sunday_June_05_2022_05_10_20_PM_5746542/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 360.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + 3y' + 4yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 671: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(-4x^2\right) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\,\mathrm{e}^{2i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-i x^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-i x^2}}{x^2}\left(-\frac{i\,\mathrm{e}^{2i x^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2}
$$

Verified OK.

2.355.1 Maple step by step solution

Let's solve

$x y'' + 3y' + 4y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + 4x^2y = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $xy'' + 3y' + 4yx^3 = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - Shift index using $k- >k-3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r) x^{-1+r} + a_1(1+r)(3+r) x^r + a_2(2+r)(4+r) x^{1+r} + a_3(3+r)(5+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+3) + 4a_{k-3} = 0$$

- Shift index using $k->k+3$

$$a_{k+4}(k+4+r)(k+6+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$$a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin(x^2)}{x^2}+\frac{c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 41

```
DSolve[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.356 problem 361

Internal problem ID [7846]
Internal file name [OUTPUT/6779_Sunday_June_05_2022_05_10_23_PM_29865828/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 361.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x^2)\,y''+2x(1-x^2)\,y'-2y=0}$$

Writing the ode as

$$\left(-x^4+x^2\right)y''+\left(-2x^3+2x\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+x^2 \\ B &= -2x^3+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{x^2 (x^2 - 1)} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= x^2 (x^2 - 1) \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{x^2 (x^2 - 1)}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 673: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2 - 1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{1}{x-1} + \frac{1}{1+x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2(x^2-1)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 1 - (0) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x-1} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{x-1} - \frac{1}{x} \\
&= \frac{1}{x^2-x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2(x^2-1)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (1+x)\, e^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\ &= (1+x)\, e^{\ln(x-1) - \ln(x)} \\ &= \frac{x^2 - 1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3 + 2x}{-x^4 + x^2} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^2-1}{x^2}\right) + c_2 \left(\frac{x^2-1}{x^2}\left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(\ln(x-1)\, x^2 - \ln(1+x)\, x^2 - \ln(x-1) + \ln(1+x) - 2x)}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(\ln(x-1)\, x^2 - \ln(1+x)\, x^2 - \ln(x-1) + \ln(1+x) - 2x)}{4x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
dsolve(x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-x^2)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2-1)}{x^2} + \frac{c_2\left(-\frac{\ln(x+1)x^2}{4} + \frac{\ln(x-1)x^2}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} - \frac{x}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 56

```
DSolve[x^2*(1-x^2)*y''[x]+2*x*(1-x^2)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{-4c_1x^2 - c_2(x^2-1)\log(1-x) + c_2(x^2-1)\log(x+1) + 2c_2x + 4c_1}{4x^2}$$

2.357 problem 362

2.357.1 Maple step by step solution 3391

Internal problem ID [7847]
Internal file name [OUTPUT/6780_Sunday_June_05_2022_05_10_27_PM_1821066/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 362.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (x-2)\,y' - y = 0}$$

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 674: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{1}{2x} - \frac{1}{4} \\
&= -\frac{x+2}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 + 4x + 12}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x-2}{2x} \, dx} \\
&= z_1 e^{-\frac{x}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(2(x-2)\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(2(x-2)\,\mathrm{e}^{\frac{x}{2}}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2(2x-4) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2(2x-4)$$

Verified OK.

2.357.1 Maple step by step solution

Let's solve

$2xy'' + (x-2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{y}{2x} - \frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-2)y'}{2x} - \frac{y}{2x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (x-2)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-2+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(a_{k+1}(k+1+r) + \frac{a_k}{2}\right)(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{2(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{a_k}{2(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)}\right]$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{a_k}{2(k+3)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = -\frac{a_k}{2(k+1)}, b_{k+1} = -\frac{b_k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(2*x*diff(y(x),x$2)+(x-2)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-2) + c_2 e^{-\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 23

```
DSolve[2*x*y''[x]+(x-2)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} + 2c_2(x-2)$$

2.358 problem 363

2.358.1 Maple step by step solution 3398

Internal problem ID [7848]
Internal file name [`OUTPUT/6781_Sunday_June_05_2022_05_10_29_PM_41777111/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 363.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 676: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.358.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r (1+r)\, x^{-1+r} + a_1 (1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.359 problem 364

2.359.1 Maple step by step solution 3405

Internal problem ID [7849]
Internal file name [OUTPUT/6782_Sunday_June_05_2022_05_10_31_PM_13741856/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 364.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2x^2y' + (x^4 + 2x - 1)\, y = 0}$$

Writing the ode as

$$y'' + 2x^2y' + (x^4 + 2x - 1)\, y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 678: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\,dx}\\ &= z_1 e^{-\frac{x^3}{3}}\\ &= z_1\left(\mathrm{e}^{-\frac{x^3}{3}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x(x^2+3)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x(x^2+3)}{3}} \right) + c_2 \left(\mathrm{e}^{-\frac{x(x^2+3)}{3}} \left(\frac{\mathrm{e}^{2x}}{2} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x(x^2+3)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x(x^2-3)}{3}}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x(x^2+3)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x(x^2-3)}{3}}}{2}
$$

Verified OK.

2.359.1 Maple step by step solution

Let's solve

$y'' + 2x^2 y' + (x^4 + 2x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\, x + (12a_4 - a_2 + 4a_1)\, x^2 + (20a_5 - a_3 + 6a_2)\, x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$$

- Shift index using $k- >k+4$

$$\left((k+4)^2 + 3k + 14\right) a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x^4+2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{1}{3}x^3-x} + c_2 \mathrm{e}^{-\frac{1}{3}x^3+x}$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 34

```
DSolve[y''[x]+2*x^2*y'[x]+(x^4+2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

2.360 problem 365

2.360.1 Maple step by step solution 3413

Internal problem ID [7850]
Internal file name [`OUTPUT/6783_Sunday_June_05_2022_05_10_34_PM_34954768/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 365.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{u'' + \frac{u}{x^2} = 0}$$

The ode can be written as

$$u''x^2 + u = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$u''x^2 + u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 680: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in u is found from

$$u_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} u_1 &= z_1 \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

Which simplifies to

$$u_1 = x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{u_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
u_2 &= u_1 \int \frac{1}{u_1^2}\, dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\, dx \\
&= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} \tag{1}
$$

Verification of solutions

$$
u = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}
$$

Verified OK.

2.360.1 Maple step by step solution

Let's solve

$u''x^2 + u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Isolate 2nd derivative

  $u'' = -\frac{u}{x^2}$

- Group terms with u on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$u'' + \frac{u}{x^2} = 0$

- Multiply by denominators of the ODE

$u'' x^2 + u = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of u with respect to x , using the chain rule

  $u' = \left(\frac{d}{dt}u(t)\right) t'(x)$

  - Compute derivative

  $u' = \frac{\frac{d}{dt}u(t)}{x}$

  - Calculate the 2nd derivative of u with respect to x , using the chain rule

  $u'' = \left(\frac{d^2}{dt^2}u(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}u(t)\right)$

  - Compute derivative

  $u'' = \frac{\frac{d^2}{dt^2}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{dt^2}u(t)}{x^2} - \frac{\frac{d}{dt}u(t)}{x^2}\right) x^2 + u(t) = 0$

- Simplify

$\frac{d^2}{dt^2}u(t) - \frac{d}{dt}u(t) + u(t) = 0$

- Characteristic polynomial of ODE

$r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

$r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

$r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

$u_1(t) = e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

$$u_2(t) = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$$

- General solution of the ODE

$$u(t) = c_1 u_1(t) + c_2 u_2(t)$$

- Substitute in solutions

$$u(t) = c_1 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$$

- Change variables back using $t = \ln(x)$

$$u = c_1 \sqrt{x}\, \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) + c_2 \sqrt{x}\, \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)$$

- Simplify

$$u = \sqrt{x}\left(c_1 \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(u(x),x$2)+1/x^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = c_1 \sqrt{x}\, x^{\frac{\sqrt{-3}}{2}} + c_2 \sqrt{x}\, x^{-\frac{\sqrt{-3}}{2}}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 42

```
DSolve[u''[x]+1/x^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

2.361 problem 366

2.361.1 Maple step by step solution 3419

Internal problem ID [7851]
Internal file name [OUTPUT/6784_Sunday_June_05_2022_05_10_36_PM_80849899/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 366.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' - (2x+1)\,u' + \left(x^2+x-1\right)u = 0}$$

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2+x-1\right)u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x-1 \\ C &= x^2+x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 682: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\,dx} \\ &= z_1 e^{\frac{1}{2}x^2+\frac{1}{2}x} \\ &= z_1\left(\mathrm{e}^{\frac{x(1+x)}{2}}\right)\end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\ &= u_1(\mathrm{e}^x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x^2}{2}}(\mathrm{e}^{x})\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$u = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{x(x+2)}{2}} \tag{1}$$

Verification of solutions

$$u = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{x(x+2)}{2}}$$

Verified OK.

2.361.1 Maple step by step solution

Let's solve

$u'' + (-2x-1)\,u' + (x^2+x-1)\,u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot u$ to series expansion for $m = 0..2$

    $x^m \cdot u = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^m \cdot u = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x^m \cdot u'$ to series expansion for $m = 0..1$

    $x^m \cdot u' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot u' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert u'' to series expansion

$$u'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$u'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k->k+2$

$$(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} + k a_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2+7k+12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x)=0,u(x), singsol=all)
```

$$u(x) = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{1}{2}x^2+x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 24

```
DSolve[u''[x]-(2*x+1)*u'[x]+(x^2+x-1)*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to e^{\frac{x^2}{2}}(c_2 e^x + c_1)$$

2.362 problem 367

2.362.1 Maple step by step solution 3427

Internal problem ID [7852]
Internal file name [OUTPUT/6785_Sunday_June_05_2022_05_10_38_PM_31214161/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 367.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0}$$

Writing the ode as

$$y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 + \frac{2}{(3x+1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{(3x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = (3x+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{(3x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 684: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (3x+1)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x+\frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(3x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(3x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3x+1} + (-)\,(0) \\ &= \frac{1}{3x+1} \\ &= \frac{1}{3x+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x+1}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{3x+1}\right)^2 - \left(-\frac{2}{(3x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{3x+1} dx} \\ &= (3x+1)^{\frac{1}{3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1} \, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (3x+1)^{\frac{1}{3}} \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left((3x+1)^{\frac{1}{3}} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} \right) + c_2 \left((3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} \left((3x+1)^{\frac{1}{3}} \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 (3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} + c_2 (3x+1)^{\frac{2}{3}} \mathrm{e}^{-x} \tag{1}
$$

Verification of solutions

$$
y = c_1 (3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} + c_2 (3x+1)^{\frac{2}{3}} \mathrm{e}^{-x}
$$

Verified OK.

2.362.1 Maple step by step solution

Let's solve

$y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3(3x^2+2x+1)y}{(3x+1)^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{3(3x^2+2x+1)y}{(3x+1)^2} = 0$

□ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

- Define functions

  $\left[P_2(x)=2, P_3(x)=\frac{3(3x^2+2x+1)}{(3x+1)^2}\right]$

- $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

- $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$

- $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $y''(3x+1)^2+2y'(3x+1)^2+(9x^2+6x+3)\,y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $9u^2\left(\frac{d^2}{du^2}y(u)\right)+18u^2\left(\frac{d}{du}y(u)\right)+(9u^2+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$

- Shift index using $k->k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$

- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k-1+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)(-2+3r)\,u^r + (a_1(2+3r)(1+3r)+18a_0r)\,u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(3k+3r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-2+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0

  $a_1(2+3r)(1+3r) + 18a_0r = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{18a_0r}{9r^2+9r+2}$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right)a_k + 18a_{k-1}k + 18a_{k-1}r + 9a_{k-2} - 18a_{k-1} = 0$

- Shift index using $k->k+2$

  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right)a_{k+2} + 18a_{k+1}(k+2) + 18a_{k+1}r + 9a_k - 18a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$

- Solution for $r = \frac{1}{3}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$

- Revert the change of variables $u = x + \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2k a_{k+1}+a_k+\frac{10}{3} a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2k a_{k+1}+a_k+\frac{10}{3} a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9\left(2k a_{k+1}+a_k+\frac{8}{3} a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0, b_{k+2} =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+(1+2/(1+3*x)^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} + c_2(3x+1)^{\frac{2}{3}} \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 35

```
DSolve[y''[x]+2*y'[x]+(1+2/(1+3*x)^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1}+c_1\right)$$

2.363 problem 368

2.363.1 Maple step by step solution 3435

Internal problem ID [7853]
Internal file name [`OUTPUT/6786_Sunday_June_05_2022_05_10_41_PM_54967396/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 368.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2+2)\,y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 686: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x\cos(x) + c_2 x\sin(x)$$

Verified OK.

2.363.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2+2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.364 problem 369

2.364.1 Maple step by step solution 3444

Internal problem ID [7854]
Internal file name [`OUTPUT/6787_Sunday_June_05_2022_05_10_43_PM_58115425/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 369.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0}$$

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = (1+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 688: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{(1+x)^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{(1+x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + (-)\,(0) \\
&= -\frac{1}{1+x} \\
&= -\frac{1}{1+x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{1+x} dx} \\ &= \frac{1}{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 + x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(1+x)^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2+x}\right) + c_2 \left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2+x} + \frac{c_2(1+x)^2}{3x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2+x} + \frac{c_2(1+x)^2}{3x}$$

Verified OK.

2.364.1 Maple step by step solution

Let's solve

$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(1+x)^2}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x=-1$

  $((1+x) \cdot P_2(x))\Big|_{x=-1} = 0$
- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $((1+x)^2 \cdot P_3(x))\Big|_{x=-1} = -2$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 y'' + 2y'(1+x)^2 - 2yx = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3 - u^2)\left(\frac{d^2}{du^2}y(u)\right) + 2u^2\left(\frac{d}{du}y(u)\right) + (-2u+2)\, y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^2 \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r+1}$
- Shift index using $k->k-1$

  $u^2 \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

  Rewrite ODE with series expansions

  $$-a_{0} \left(1+r \right) \left(-2+r \right) u^{r}+\left(\sum_{k =1}^{\infty}\left(-a_{k} \left(k +r +1\right) \left(k +r -2\right)+a_{k -1} \left(k +r +1\right) \left(k +r -2\right)\right) u^{k +r}\right)$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-\left(1+r \right) \left(-2+r \right)=0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, 2\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-\left(k +r +1\right) \left(k +r -2\right) \left(a_{k}-a_{k -1}\right)=0$$

- Shift index using $k->k+1$

  $$-\left(k +r +2\right) \left(k -1+r \right) \left(a_{k +1}-a_{k}\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k +1}=a_{k}$$

- Recursion relation for $r=-1$

  $$a_{k +1}=a_{k}$$

- Solution for $r=-1$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k -1}, a_{k +1}=a_{k}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k =0}^{\infty}a_{k} \left(1+x \right)^{k -1}, a_{k +1}=a_{k}\right]$$

- Recursion relation for $r=2$

  $$a_{k +1}=a_{k}$$

- Solution for $r=2$

  $$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +2}, a_{k +1}=a_{k}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k =0}^{\infty}a_{k} \left(1+x \right)^{k +2}, a_{k +1}=a_{k}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k+2}\right), a_{k+1}=a_k, b_{k+1}=b_k\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+2/x*diff(y(x),x)-2/(1+x)^2*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x\,(x+1)}+\frac{c_2(x^3+3x^2+3x)}{x\,(x+1)}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 34

```
DSolve[y''[x]+2/x*y'[x]-2/(1+x)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{c_2x(x^2+3x+3)+3c_1}{3x(x+1)}$$

2.365 problem 370

Internal problem ID [7855]
Internal file name [`OUTPUT/6788_Sunday_June_05_2022_05_10_47_PM_76451549/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 370.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{y'' + \frac{y}{2x^4} = 0}$$

Writing the ode as

$$y'' + \frac{y}{2x^4} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{1}{2x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{2x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 2x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{2x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 690: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 2x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{2x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{2}}{2x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{2}}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{i\sqrt{2}}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{2}}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{2}}{2}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{2x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{2}}{2x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i\sqrt{2}}{2x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{2} + 2x}{2x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{i\sqrt{2}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{2}}{2x^2} + \frac{1}{x}\right)^2 - \left(-\frac{1}{2x^4}\right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{i\sqrt{2}}{2x^2}+\frac{1}{x}\right)dx} \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{i\sqrt{2}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}}\right) + c_2 \left(x\,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} \left(-\frac{i\sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{x}}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} - \frac{i c_2 x \sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{2x}}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{\frac{i\sqrt{2}}{2x}} - \frac{i c_2 x \sqrt{2}\,\mathrm{e}^{-\frac{i\sqrt{2}}{2x}}}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+1/(2*x^4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin\left(\frac{\sqrt{2}}{2x}\right) + c_2 x \cos\left(\frac{\sqrt{2}}{2x}\right)$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 50

```
DSolve[y''[x]+1/(2*x^4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{i}{\sqrt{2}x}} x - \frac{i c_2 e^{-\frac{i}{\sqrt{2}x}} x}{\sqrt{2}}$$

2.366 problem 371

2.366.1 Maple step by step solution 3462

Internal problem ID [7856]
Internal file name [`OUTPUT/6789_Sunday_June_05_2022_05_10_50_PM_12945432/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 371.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 691: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.366.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.181 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.367 problem 372

2.367.1 Maple step by step solution 3471

Internal problem ID [7857]
Internal file name [`OUTPUT/6790_Sunday_June_05_2022_05_10_53_PM_5389391/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 372.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 693: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+2)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.367.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.368 problem 373

2.368.1 Maple step by step solution 3480

Internal problem ID [7858]
Internal file name [OUTPUT/6791_Sunday_June_05_2022_05_10_56_PM_38688456/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 373.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 695: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+2)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.368.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2+3k+5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}x^2+2x} + 2\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.369 problem 374

2.369.1 Maple step by step solution 3489

Internal problem ID [7859]
Internal file name [`OUTPUT/6792_Sunday_June_05_2022_05_10_58_PM_61523399/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 374.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 697: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$y = c_1 y_1 + c_2 y_2$

$$= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.369.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ◦ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\, e^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{1}{2}x^2+2x} + 2\pi\, e^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.370 problem 375

2.370.1 Maple step by step solution 3498

Internal problem ID [7860]
Internal file name [OUTPUT/6793_Sunday_June_05_2022_05_11_01_PM_84972958/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 375.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 699: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.370.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\, e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) x - i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{1}{2}x^2+2x} + 2\pi\, e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right) e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.371 problem 376

2.371.1 Maple step by step solution 3507

Internal problem ID [7861]
Internal file name [OUTPUT/6794_Sunday_June_05_2022_05_11_04_PM_39779047/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 376.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 701: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (x+2)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.371.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}x^2+2x} + 2\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.372 problem 377

2.372.1 Maple step by step solution 3516

Internal problem ID [7862]
Internal file name [`OUTPUT/6795_Sunday_June_05_2022_05_11_07_PM_40831016/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 377.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 703: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\
&= (x+2)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.372.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.373 problem 378

2.373.1 Maple step by step solution 3525

Internal problem ID [7863]
Internal file name [OUTPUT/6796_Sunday_June_05_2022_05_11_09_PM_15044721/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 378.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 705: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int (-1-\frac{x}{2}) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.373.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.374 problem 379

2.374.1 Maple step by step solution 3534

Internal problem ID [7864]
Internal file name [`OUTPUT/6797_Sunday_June_05_2022_05_11_12_PM_23793907/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 379.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 707: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\; \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((x+2)\,e^{-x}\right) + c_2\left((x+2)\,e^{-x}\left(\frac{-i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,e^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.374.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2+3k+5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\text{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.375 problem 380

2.375.1 Maple step by step solution 3543

Internal problem ID [7865]
Internal file name [OUTPUT/6798_Sunday_June_05_2022_05_11_15_PM_99117579/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 380.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 709: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \,dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \,dx}}{y_1^2} \,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \,dx}}{(y_1)^2} \,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \,dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.375.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) x - i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}x^2+2x} + 2\pi e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right) e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.376 problem 381

2.376.1 Maple step by step solution 3552

Internal problem ID [7866]
Internal file name [OUTPUT/6799_Sunday_June_05_2022_05_11_18_PM_40173007/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 381.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 711: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha^{\pm}_{\infty}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha^{-}_{\infty} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha^{-}_{\infty} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_{\infty} \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+2)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.376.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.377 problem 382

2.377.1 Maple step by step solution 3558

Internal problem ID [7867]
Internal file name [OUTPUT/6800_Sunday_June_05_2022_05_11_21_PM_23387836/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 382.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 713: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\cos(x)}{x} + \frac{c_2\sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\cos(x)}{x} + \frac{c_2\sin(x)}{x}
$$

Verified OK.

2.377.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.378 problem 383

2.378.1 Maple step by step solution 3568

Internal problem ID [7868]
Internal file name [`OUTPUT/6801_Sunday_June_05_2022_05_11_23_PM_40209735/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 383.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{2x^2y'' + 3xy' - yx = 0}$$

The ODE is

$$2x^2y'' + 3xy' - yx = 0$$

Or

$$x(2xy'' - y + 3y') = 0$$

For $x \neq 0$ the above simplifies to

$$2xy'' - y + 3y' = 0$$

Writing the ode as

$$2x^2y'' + 3xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 715: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1 - (1)) \\
&= 0
\end{aligned}$$

We now form the following rational function

$$\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\
&= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\
&= \frac{1}{2x}
\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-8x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{\frac{1}{4}}\mathrm{e}^{\sqrt{2}\,\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} \left(-\frac{\mathrm{e}^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} - \frac{c_2 \sqrt{2}\,\mathrm{e}^{-\sqrt{2}\,\sqrt{x}}}{2\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} - \frac{c_2 \sqrt{2}\,\mathrm{e}^{-\sqrt{2}\,\sqrt{x}}}{2\sqrt{x}}$$

Verified OK. `{x <> 0}`

2.378.1 Maple step by step solution

Let's solve

$2x^2y'' + 3xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2x} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} - \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2xy'' - y + 3y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k}{(2k+3)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sinh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}+\frac{c_2 \cosh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 56

```
DSolve[2*x^2*y''[x]+3*x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1 e^{2\sqrt{2}\sqrt{x}}-\sqrt{2}c_2\right)}{2\sqrt{x}}$$

2.379 problem 384

2.379.1 Maple step by step solution 3578

Internal problem ID [7869]
Internal file name [OUTPUT/6802_Sunday_June_05_2022_05_11_25_PM_23361458/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 384.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2 y'' + (3x^2 + 2x)\, y' - 2y = 0}$$

Writing the ode as

$$x^2 y'' + (3x^2 + 2x)\, y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2 + 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 717: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) &&= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 12x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\
&= -\frac{1}{x} - \frac{3}{2} \\
&= -\frac{1}{x} - \frac{3}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{3x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2 + 2x}{x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-3x}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{3x}(9x^2-6x+2)}{27}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-3x}}{x^2}\right) + c_2 \left(\frac{e^{-3x}}{x^2}\left(\frac{e^{3x}(9x^2-6x+2)}{27}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-3x}}{x^2} + \frac{c_2(9x^2-6x+2)}{27x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-3x}}{x^2} + \frac{c_2(9x^2-6x+2)}{27x^2}$$

Verified OK.

2.379.1 Maple step by step solution

Let's solve

$x^2y'' + (3x^2 + 2x)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2} - \frac{(3x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x+2)y'}{x} - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(3x + 2)\,y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r+2) + 3a_{k-1}) = 0$$

- Shift index using $k- >k+1$

  $$(k+r)\,(a_{k+1}(k+3+r) + 3a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{3a_k}{k+3+r}$$

- Recursion relation for $r = -2$

  $$a_{k+1} = -\frac{3a_k}{k+1}$$

- Solution for $r = -2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{3a_k}{k+4}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x), x, x) + (2*x+3*x^2)*diff(y(x),x)-2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(9x^2 - 6x + 2)}{x^2} + \frac{c_2 e^{-3x}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]+(2*x+3*x^2)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1(9x^2 - 6x + 2) + 27c_2 e^{-3x}}{27x^2}$$

2.380 problem 385

2.380.1 Maple step by step solution 3587

Internal problem ID [7870]
Internal file name [OUTPUT/6803_Sunday_June_05_2022_05_11_28_PM_855979/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 385.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0}$$

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^4 + 18x^3 + 27x^2 - 2x - 3$$
$$t = 16\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 719: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2+3x+1}{4x\left(x^2+x+1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right) dx} \\
&= 2\sqrt{2}\, x^{\frac{1}{4}} \left(x^2+x+1\right)^{\frac{3}{4}} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{\frac{9}{4}} \left(x^2+x+1\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{2}\, \sqrt{x^2+x+1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2} - \frac{\ln\left(x^2+x+1\right)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\, \left(x^2+x+1\right)^{\frac{3}{2}}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}\right) \\
&\quad + c_2 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\,\left(x^2+x+1\right)^{\frac{3}{2}}} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \\
&\quad + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx\right)}{4x^2}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \\
&\quad + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx\right)}{4x^2}
\end{aligned}
$$

Verified OK.

2.380.1 Maple step by step solution

Let's solve

$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(7x^2+10x+6) y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9) y'}{2x(x^2+x+1)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(11x^2+11x+9) y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6) y}{2x^2(x^2+x+1)} = 0$$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = \frac{9}{2}$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 3$$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2(x^2+x+1) y'' + x(11x^2+11x+9) y' + (7x^2+10x+6) y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(3+2r)\, x^r + (a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-2, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(2+r)a_0}{3+r}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{3}{2}+r\right)\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k->k+2$

$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{k b_k + k b_{k+1} - \frac{1}{2} b_k +}{k + \frac{5}{2}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 141

```
dsolve(2*x^2*(1+x+x^2)*diff(y(x), x$2) + x*(9+11*x+11*x^2)*diff(y(x), x) + (6+10*x+7*x^2)*y(
```

$$y(x) = \frac{c_1\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}}{x^2} + \frac{c_2\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}\left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{6}}}{(x^2+x+1)^{\frac{3}{2}}\sqrt{x}}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.708 (sec). Leaf size: 93

```
DSolve[2*x^2*(1+x+x^2)*y''[x] + x*(9+11*x+11*x^2)*y'[x] + (6+10*x+7*x^2)*y[x] == 0,y[x],x,In
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

2.381 problem 388

2.381.1 Maple step by step solution 3599

Internal problem ID [7871]
Internal file name [OUTPUT/6804_Sunday_June_05_2022_05_11_36_PM_73434126/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 388.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1+x)\,y' + 2y = 0}$$

Writing the ode as

$$xy'' + (1+x)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1+x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 721: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 6x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 1)\, \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (x - 1)\, \mathrm{e}^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (x - 1)\sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-1)\,e^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - e^{x}}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x-1)\,e^{-x}\big) + c_2\left((x-1)\,e^{-x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - e^{x}}{x-1}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1(x-1)\,e^{-x} + c_2\big(-1-(x-1)\,e^{-x}\operatorname{expIntegral}_1(-x)\big) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1(x-1)\,e^{-x} + c_2\big(-1-(x-1)\,e^{-x}\operatorname{expIntegral}_1(-x)\big)
$$

Verified OK.

2.381.1 Maple step by step solution

Let's solve

$xy'' + (1+x)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+x)y'}{x} - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (1+x)\, y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 + a_k(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(x*diff(y(x), x$2) +(1+x)*diff(y(x),x)+2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x-1) + c_2(\operatorname{expIntegral}_1(-x)\, x - \operatorname{expIntegral}_1(-x) + e^x)\, e^{-x}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 33

```
DSolve[x*y''[x] +(1+x)*y'[x]+2*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2(x-1)\operatorname{ExpIntegralEi}(x) + c_1(x-1) - c_2 e^x)$$

2.382 problem 389

2.382.1 Maple step by step solution 3608

Internal problem ID [7872]
Internal file name [OUTPUT/6805_Sunday_June_05_2022_05_11_39_PM_45908116/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 389.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(4+x)\,y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 10x - 1 \\ t &= 4x^2(x - 1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 723: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2\,(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-2}{2} + 2\right) &&= \frac{1}{2} \\ \alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-2}{2} + 2\right) &&= \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= 0 \\ \alpha_\infty^{-} &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2\,(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\ &= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\ &= \frac{2x^2+x+1}{2x(x-1)^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\ &= z_1 \left(\frac{x^{\frac{3}{2}}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2} dx \\
&= y_1 \left(\mathrm{e}^{-4} \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}\, x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}\, x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} \left(\mathrm{e}^{-4} \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}\, x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}} \mathrm{e}^{-\frac{4x}{x-1}} \sqrt{x(x-1)}\, \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}} \sqrt{x(x-1)}\, x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}} \mathrm{e}^{-\frac{4x}{x-1}} \sqrt{x(x-1)}\, \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}}
$$

Verified OK.

2.382.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (-x^2-3x)\, y' + (4+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4+x)y}{x^2(x-1)^2} + \frac{(x+3)y'}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x(x-1)^2} + \frac{(4+x)y}{x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x(x-1)^2}, P_3(x) = \frac{4+x}{x^2(x-1)^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1)^2 - x(x+3)\,y' + (4+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term must be 0

  $$a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = \frac{a_0(1+2r)}{-1+r}$$

- Each term in the series must be 0, giving the recursion relation

  $$\left((a_k + a_{k-2} - 2a_{k-1})\, k + (a_k + a_{k-2} - 2a_{k-1})\, r - 2a_k - 3a_{k-2} + a_{k-1}\right)(k+r-2) = 0$$

- Shift index using $k->k+2$

  $$\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})\, r - 2a_{k+2} - 3a_k + a_{k+1}\right)(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k a_k - 2k a_{k+1} + r a_k - 2r a_{k+1} - a_k - 3a_{k+1}}{k+r}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
dsolve(x^2*(1-2*x+x^2)*diff(y(x), x$2) -x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,y(x), singsol=al
```

$$y(x) = \frac{c_1 x^2 e^{-\frac{4}{x-1}}}{x-1} + \frac{c_2 x^2 \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right) e^{-\frac{4x}{x-1}}}{x-1}$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 54

```
DSolve[x^2*(1-2*x+x^2)*y''[x] -x*(3+x)*y'[x]+(4+x)*y[x] == 0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}} \sqrt{1-x} x^2 \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right) + e^4 c_1\right)}{(x-1)^{3/2}}$$

2.383 problem 390

2.383.1 Maple step by step solution 3617

Internal problem ID [7873]
Internal file name [OUTPUT/6806_Sunday_June_05_2022_05_11_41_PM_94327905/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 390.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+2)\,y''+5x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\ B&=5x^2\\ C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 - 24x - 16 \\ t &= 16\left(x^2 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 725: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} + \frac{1}{8x + 16} + \frac{5}{16 (x + 2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16 (x^2 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4\left(x+2\right)} + \frac{1}{2x} + (-)\left(0\right) \\ &= -\frac{1}{4\left(x+2\right)} + \frac{1}{2x} \\ &= \frac{4+x}{4x\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x+2\right)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4\left(x+2\right)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4\left(x+2\right)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16\left(x^2+2x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{\left(x+2\right)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{5x^2}{2x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{\left(x+2\right)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(x+2\right)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{x+2} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\left(2\sqrt{x+2} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} - \frac{2c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) - \sqrt{x+2}\right)}{(x+2)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} - \frac{2c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) - \sqrt{x+2}\right)}{(x+2)^{\frac{3}{2}}}
$$

Verified OK.

2.383.1 Maple step by step solution

Let's solve

$(2x^3 + 4x^2)\,y'' + 5x^2 y' + (1 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(1+x)y}{2x^2(x+2)} - \frac{5y'}{2(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{5y'}{2(x+2)} + \frac{(1+x)y}{2x^2(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{5}{2(x+2)}, P_3(x) = \frac{1+x}{2x^2(x+2)}\right]$

- ○ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

$\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = \frac{5}{2}$

- ○ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

$\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

- ○ $x = -2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -2$

- Multiply by denominators

$2x^2(x+2)\, y'' + 5x^2y' + (1+x)\, y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\, y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- >k - m$

$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\,u^{-1+r}+\left(4a_1(1+r)\,(5+2r)-a_0(8r^2+12r+1)\right)u^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+2r)-a_0(8r^2+12r+1)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k+a_{k-1}+4a_{k+1})\,k^2+\left(4(-4a_k+a_{k-1}+4a_{k+1})\,r-12a_k-a_{k-1}+28a_{k+1}\right)k+2(-4a_k+$$

- Shift index using $k->k+1$

$$2(-4a_{k+1}+a_k+4a_{k+2})\,(k+1)^2+\left(4(-4a_{k+1}+a_k+4a_{k+2})\,r-12a_{k+1}-a_k+28a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+4kra_k-16kra_{k+1}+2r^2a_k-8r^2a_{k+1}+3ka_k-28ka_{k+1}+3ra_k-28ra_{k+1}+a_k-21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+3ka_k-28ka_{k+1}+a_k-21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+3ka_k-28ka_{k+1}+a_k-21a_{k+1}}{4(2k^2+11k+14)},20a_1-a_0=0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(2*x^2*(2+x)*diff(y(x), x$2) +5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{c_2\sqrt{2}\left(2\sqrt{2}\,\sqrt{x+2} - 4\ \mathrm{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\sqrt{x}}{2\,(x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 55

```
DSolve[2*x^2*(2+x)*y''[x] +5*x^2*y'[x]+(1+x)*y[x] == 0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\mathrm{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

2.384 problem 391

2.384.1 Maple step by step solution 3625

Internal problem ID [7874]
Internal file name [OUTPUT/6807_Sunday_June_05_2022_05_11_44_PM_25427070/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 391.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + (x^2 + 2)\, y = 0}$$

Writing the ode as

$$x^2y'' + 4xy' + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 727: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x^2}\right) + c_2 \left(\frac{\cos(x)}{x^2} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2}
$$

Verified OK.

2.384.1 Maple step by step solution

Let's solve

$x^2 y'' + 4xy' + (x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 4xy' + (x^2+2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\, x^r + a_1(3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3+r)\,(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x), x, x) + 4*x*diff(y(x), x) + (x^2+2)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x^2} + \frac{c_2 \cos(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.385 problem 392

2.385.1 Maple step by step solution 3632

Internal problem ID [7875]
Internal file name [OUTPUT/6808_Sunday_June_05_2022_05_11_46_PM_68505208/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 392.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 729: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.385.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k + 2)^2 + 8(k + 2) r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.386 problem 394

2.386.1 Maple step by step solution 3643

Internal problem ID [7876]
Internal file name [`OUTPUT/6809_Sunday_June_05_2022_05_11_49_PM_75100555/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 394.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 731: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{1} - 0\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{1} - 0\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(1) \\
&= -1 - \frac{1}{x} \\
&= -\frac{1 + x}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1 - \frac{1}{x}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-1 - \frac{1}{x}\right)^2 - \left(\frac{x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{-2 + 2a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(-1 - \frac{1}{x}\right) dx} \\
&= (1 + x)\, e^{-x - \ln(x)} \\
&= \frac{(1 + x)\, e^{-x}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(x-1)\,\mathrm{e}^{2x}}{2x+2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\right) + c_2\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\left(\frac{(x-1)\,\mathrm{e}^{2x}}{2x+2}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}} + \frac{c_2(x-1)\,\mathrm{e}^{x}}{2\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}} + \frac{c_2(x-1)\,\mathrm{e}^{x}}{2\sqrt{x}}$$

Verified OK.

2.386.1 Maple step by step solution

Let's solve

$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+5)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(4x^2+5)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{4x^2+5}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{5}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-4x^2 - 5)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-5+2r)\, x^r + a_1(3+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-5) - 4a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-5+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(-3+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(-\frac{5}{2}+k+r\right) a_k - 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $4\left(k+\frac{5}{2}+r\right)\left(-\frac{1}{2}+k+r\right) a_{k+2} - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{(2k+5+2r)(-1+2k+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{(2k+4)(-2+2k)}$
- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{(2k+4)(-2+2k)}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = \frac{4a_k}{(2k+10)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = \frac{4a_k}{(2k+4)(-2+2k)}, a_1 = 0, b_{k+2} = \frac{4b_k}{(2k+10)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-(x^2+5/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{x}(x-1)}{\sqrt{x}} + \frac{c_2 e^{-x}(x+1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 53

```
DSolve[x^2*y''[x]-x*y'[x]-(x^2+5/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((ic_2x + c_1)\sinh(x) - (c_1x + ic_2)\cosh(x))}{\sqrt{-ix}}$$

2.387 problem 395

2.387.1 Maple step by step solution 3650

Internal problem ID [7877]
Internal file name [OUTPUT/6810_Sunday_June_05_2022_05_11_52_PM_91203988/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 395.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 733: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.387.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.388 problem 396

2.388.1 Maple step by step solution 3661

Internal problem ID [7878]
Internal file name [`OUTPUT/6811_Sunday_June_05_2022_05_11_54_PM_85581829/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 396.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 3xy' + 4y x^4 = 0}$$

The ODE is

$$x^2y'' + 3xy' + 4y x^4 = 0$$

Or

$$x(xy'' + 3y' + 4y x^3) = 0$$

For $x \neq 0$ the above simplifies to

$$xy'' + 3y' + 4y x^3 = 0$$

Writing the ode as

$$x^2y'' + 3xy' + 4y x^4 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 735: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -4x^2 + \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4+3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4+3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(2ix\right) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-ix^2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\mathrm{e}^{-ix^2}}{x^2} - \frac{ic_2\mathrm{e}^{ix^2}}{4x^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4 x^2}$$

Verified OK. `{x <> 0}`

2.388.1 Maple step by step solution

Let's solve

$x^2 y'' + 3 x y' + 4 y x^4 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + 4x^2 y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x y'' + 3 y' + 4 y x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)\,(k+6+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r=-2$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k-2}, a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$$
- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k}, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$
- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k}\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+4*x^4*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x^2)}{x^2} + \frac{c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]+3*x*y'[x]+4*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.389 problem 398

2.389.1 Maple step by step solution 3671

Internal problem ID [7879]
Internal file name [`OUTPUT/6812_Sunday_June_05_2022_05_11_57_PM_61097509/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 398.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - (x^2+3)\, y = 0}$$

Writing the ode as

$$y'' + (-x^2-3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 737: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+3}{1} \\
&= Q + \frac{R}{1} \\
&= (x^2+3) + (0) \\
&= x^2+3
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned}
b &= (3) - (0) \\
&= 3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned}
d &= \alpha_\infty^{+} \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\ -2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int x dx} \\ &= (x)\, e^{\frac{x^2}{2}} \\ &= x\, e^{\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= x\,\mathrm{e}^{\frac{x^2}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\,dx\\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^2}{2}}\left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right)\right)\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right)$$

Verified OK.

2.389.1 Maple step by step solution

Let's solve

$y'' + (-x^2 - 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 3a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k- >k + 2$

$$\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2=\frac{3a_0}{2}, a_3=\frac{a_1}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)=(x^2+3)*y(x),y(x), singsol=all)
```

$$y(x)=c_1\mathrm{e}^{\frac{x^2}{2}}x+c_2\mathrm{e}^{\frac{x^2}{2}}\left(\sqrt{\pi}\,\mathrm{erf}\,(x)\,x+\mathrm{e}^{-x^2}\right)$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 46

```
DSolve[y''[x]==(x^2+3)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\mathrm{erf}(x)+c_1e^{x^2}x-c_2\right)$$

2.390 problem 399

2.390.1 Maple step by step solution 3676

Internal problem ID [7880]
Internal file name [`OUTPUT/6813_Sunday_June_05_2022_05_12_00_PM_89496352/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 399.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 2xy' + (x^2+1)\, y = 0}$$

Writing the ode as

$$y'' + 2xy' + (x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 739: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x^2}{2}} + c_2 x\, \mathrm{e}^{-\frac{x^2}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x^2}{2}} + c_2 x\, \mathrm{e}^{-\frac{x^2}{2}}
$$

Verified OK.

2.390.1 Maple step by step solution

Let's solve

$y'' + 2xy' + (x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + 3a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k+2) + a_{k+2} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2} + a_k + 5a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{x^2}{2}} + c_2 e^{-\frac{x^2}{2}} x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[y''[x]+2*x*y'[x]+(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{2}}(c_2 x + c_1)$$

2.391 problem 400

Internal problem ID [7881]
Internal file name [`OUTPUT/6814_Sunday_June_05_2022_05_12_02_PM_93986180/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 400.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y'' + y' - \frac{y}{x} = 0}$$

Writing the ode as

$$x^3y'' + y' - \frac{y}{x} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3 \\ B &= 1 \\ C &= -\frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-2x^2 + 1}{4x^6} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -2x^2 + 1 \\
t &= 4x^6
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-2x^2 + 1}{4x^6}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 741: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{1}{2x^4}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{1}{2x} - \frac{x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{1}{2}$. Therefore

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^3} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &&= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} + 3\right) &&= 2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-2x^2 + 1}{4x^6}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	6	$\frac{1}{2x^3}$	1	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x^3} + \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} + \frac{1}{x} \\
&= \frac{1}{2x^3} + \frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x^3} + \frac{1}{x}\right)(0) + \left(\left(-\frac{3}{2x^4} - \frac{1}{x^2} \right) + \left(\frac{1}{2x^3} + \frac{1}{x} \right)^2 - \left(\frac{-2x^2 + 1}{4x^6} \right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x^3}+\frac{1}{x}\right)dx} \\
&= x \, \mathrm{e}^{-\frac{1}{4x^2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{x^3} \, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{4x^2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x^3} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{i c_2 x \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{i c_2 x \sqrt{\pi}\, \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}}{2x}\right)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^3*diff(y(x),x$2)+ diff(y(x),x)-1/x*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 x \operatorname{erf}\left(\frac{i\sqrt{2}}{2x}\right)$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 34

```
DSolve[x^3*y''[x]+ y'[x]-1/x*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \sqrt{\frac{\pi}{2}} c_2 x \text{erfi}\left(\frac{1}{\sqrt{2}x}\right)$$

2.392 problem 401

2.392.1 Maple step by step solution 3690

Internal problem ID [7882]
Internal file name [`OUTPUT/6815_Sunday_June_05_2022_05_12_05_PM_17824705/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 401.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 742: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.392.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.393 problem 402

2.393.1 Maple step by step solution 3697

Internal problem ID [7883]
Internal file name [`OUTPUT/6816_Sunday_June_05_2022_05_12_07_PM_10505507/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 402.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 744: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\ &= z_1 e^{x-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1\left(\frac{e^x}{\sqrt{x}}\right) + c_2\left(\frac{e^x}{\sqrt{x}}(x)\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2\sqrt{x}\, e^x \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2\sqrt{x}\, e^x$$

Verified OK.

2.393.1 Maple step by step solution

Let's solve

$4x^2 y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4x(2x-1)\,y' + (4x^2 - 4x - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + (a_1(3+2r)\,(1+2r) - 4a_0(1+2r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r)-4a_0(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=\frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+(-8k-8r+4)\,a_{k-1}+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)\,r+4r^2-1\right)+(-8k-12-8r)\,a_{k+1}+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1=a_0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0, b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x), sin
```

$$y(x) = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(-8*x^2+4*x)*y'[x]+(4*x^2-4*x-1)*y[x] == 0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

2.394 problem 404

2.394.1 Maple step by step solution 3705

Internal problem ID [7884]
Internal file name [OUTPUT/6817_Sunday_June_05_2022_05_12_10_PM_45043938/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 404.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' + y = 0}$$

Writing the ode as

$$y'' - y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 746: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2 c_2 \sqrt{3}\,\mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \tag{1}
$$

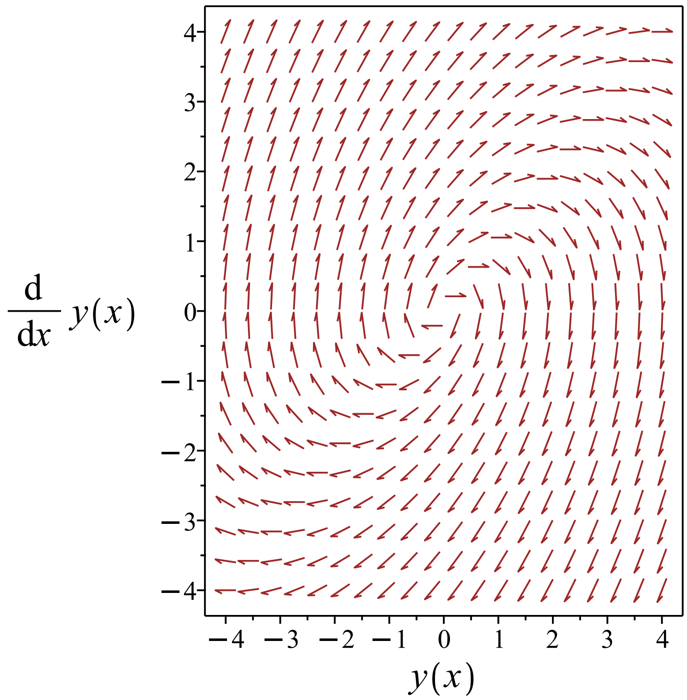

Figure 4: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\sqrt{3}\,e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}$$

Verified OK.

2.394.1 Maple step by step solution

Let's solve

$y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 42

```
DSolve[y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2}\left(c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.395 problem 405

2.395.1 Maple step by step solution 3713

Internal problem ID [7885]
Internal file name [`OUTPUT/6818_Sunday_June_05_2022_05_12_12_PM_13175369/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 405.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2-1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = \left(x^2 - 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 748: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 0\\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2} + \frac{3}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{3}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(1+x)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-1)} + \frac{3}{2\,(1+x)} \\
&= \frac{x-2}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(1+x)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{3}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\ &= z_1\left(\sqrt{x-1}\,\sqrt{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(1+x)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^2\right) + c_2\left((1+x)^2 \left(-\frac{x}{(1+x)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)^2 - c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^2 - c_2 x
$$

Verified OK.

2.395.1 Maple step by step solution

Let's solve

$(x^2-1)\,y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$
- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -1$
- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' - 2xy' + 2y = 0$
- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$$-2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-1)+a_k(k+r-1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$
- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1}+a_k(k+r-2))\,(k+r-1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r=0$; series terminates at $k=2$

  $a_{k+1}=\frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k=0$

  $a_1=-a_0$
- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{4}$
- Express in terms of a_0

  $a_2=\frac{a_0}{4}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-u+\frac{1}{4}u^2\right)$
- Revert the change of variables $u=1+x$

  $\left[y=\frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_kk}{2(k+3)}$
- Solution for $r=2$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2},a_{k+1}=\frac{a_kk}{2(k+3)}\right]$
- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\frac{a_0 (x-1)^2}{4}+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), b_{k+1}=\frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2-1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1 x+c_2 (x^2+1)$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 39

```
DSolve[(x^2-1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\sqrt{x^2-1}(c_1 (x-1)^2+c_2 x)}{\sqrt{1-x^2}}$$

2.396 problem 406

2.396.1 Maple step by step solution 3720

Internal problem ID [7886]
Internal file name [`OUTPUT/6819_Sunday_June_05_2022_05_12_14_PM_57641045/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 406.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+2)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 - 2x\right)y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 2x \\ C &= x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 750: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-2x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\ln(x)} \\
&= z_1\left(x\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x + \mathrm{e}^x c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x + \mathrm{e}^x c_2 x$$

Verified OK.

2.396.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 2x)\, y' + (x+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x^2} + \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+2)y'}{x} + \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2 y'' - x(x+2)\, y' + (x+2)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+2}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=\frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x$2)-x*(x+2)*diff(y(x),x)+(x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + e^x c_2x$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-x*(x+2)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2e^x + c_1)$$

2.397 problem 407

2.397.1 Maple step by step solution 3730

Internal problem ID [7887]
Internal file name [OUTPUT/6820_Sunday_June_05_2022_05_12_17_PM_36225893/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 407.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+x)\,y''-(x+2)\,y'+y=0}$$

Writing the ode as

$$(1+x)\,y''+(-x-2)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1+x \\ B &= -x-2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2}{4(1 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 2 \\ t &= 4(1 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2}{4(1 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 752: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(1+x)^2} - \frac{1}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{3}{4x^4} - \frac{3}{4x^5} + \frac{1}{x^6} - \frac{1}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+2}{4x^2+8x+4} \\ &= Q + \frac{R}{4x^2+8x+4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{1-2x}{4x^2+8x+4}\right) \\ &= \frac{1}{4} + \frac{1-2x}{4x^2+8x+4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0)\\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2}{4\left(1 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right)\\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(1+x)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\,(1+x)} + \frac{1}{2} \\
&= \frac{x}{2x+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\,(1+x)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\,(1+x)^2}\right) + \left(-\frac{1}{2\,(1+x)} + \frac{1}{2}\right)^2 - \left(\frac{x^2+2}{4\,(1+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{1+x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{1+x}\,dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(1+x)}{2}} \\
&= z_1 \left(\sqrt{1+x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{1+x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1\left(-(x+2)\, e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(-(x+2)\, e^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2(-x-2) \tag{1}$$

Verification of solutions

$$y = c_1 e^x + c_2(-x-2)$$

Verified OK.

2.397.1 Maple step by step solution

Let's solve

$(1+x)\, y'' + (-x-2)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{1+x} + \frac{(x+2)y'}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(x+2)y'}{1+x} + \frac{y}{1+x} = 0$

☐ Check to see if $x_0 = -1$ is a regular singular point

- ◦ Define functions

$\left[P_2(x) = -\frac{x+2}{1+x}, P_3(x) = \frac{1}{1+x}\right]$

- ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

$\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -1$

- ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- ◦ $x = -1$is a regular singular point

Check to see if $x_0 = -1$ is a regular singular point

$x_0 = -1$

- Multiply by denominators

$(1+x)\, y'' + (-x-2)\, y' + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ◦ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- ◦ Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x+1)*diff(y(x),x$2)-(x+2)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+2) + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 29

```
DSolve[(x+1)*y''[x]-(x+2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^{x+1} - 2c_2(x+2)}{\sqrt{2e}}$$

2.398 problem 408

2.398.1 Maple step by step solution 3739

Internal problem ID [7888]
Internal file name [`OUTPUT/6821_Sunday_June_05_2022_05_12_20_PM_89868378/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 408.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\, y'' + 2xy' - 2y = 0}$$

Writing the ode as

$$(1-x^2)\, y'' + 2xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 754: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(1+x)^2} + \frac{3}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{3}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^{2}-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(1+x\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-1\right)} + \frac{3}{2\left(1+x\right)} \\ &= \frac{x-2}{x^{2}-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(1+x)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{3}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{3}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x}{1-x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\ &= z_1 \left(\sqrt{x-1}\,\sqrt{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(1+x)^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((1+x)^2\right) + c_2 \left((1+x)^2 \left(-\frac{x}{(1+x)^2}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 (1+x)^2 - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 (1+x)^2 - c_2 x$$

Verified OK.

2.398.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' + 2xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x)=-\frac{2x}{x^2-1}, P_3(x)=\frac{2}{x^2-1}\right]$
- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-1$
- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''-2xy'+2y=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-2u+2)\left(\frac{d}{du}y(u)\right)+2y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

Rewrite ODE with series expansions

$-2a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(k+r-1) + a_k(k+r-1)\,(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\, a_{k+1} + a_k(k+r-2))\,(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = 1 + x$

  $\left[y = \frac{a_0(x-1)^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$
- Revert the change of variables $u = 1 + x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1}=\frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\frac{a_0 (x-1)^2}{4}+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), b_{k+1}=\frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve((1-x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1 x+c_2 (x^2+1)$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 39

```
DSolve[(1-x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\sqrt{x^2-1}(c_1 (x-1)^2+c_2 x)}{\sqrt{1-x^2}}$$

2.399 problem 409

2.399.1 Maple step by step solution 3749

Internal problem ID [7889]
Internal file name [OUTPUT/6822_Sunday_June_05_2022_05_12_24_PM_93898388/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 409.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(1-x^2)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 756: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{5}{4(x-1)} - \frac{5}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2-3}{(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2-3}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right.
$$

$$
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x)\, e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= x\sqrt{x-1}\,\sqrt{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2\sqrt{x^2-1}\,(\ln(x-1)\,x-\ln(1+x)\,x+2)}{2\sqrt{x-1}\,\sqrt{1+x}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2 \sqrt{x^2-1}\,(\ln(x-1)\,x - \ln(1+x)\,x + 2)}{2\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

2.399.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{2}{x^2-1}\right]$

  ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2xy' - 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+2)\,(k+r-1)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(k+2)\,(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = 1 + x$

  $[y = -a_0 x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(-\frac{\ln (x+1)\, x}{2} + \frac{\ln (x-1)\, x}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.400 problem 410

2.400.1 Maple step by step solution 3755

Internal problem ID [7890]
Internal file name [OUTPUT/6823_Sunday_June_05_2022_05_12_26_PM_66672557/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 410.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 758: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}$$

Verified OK.

2.400.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

  - ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2) r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.02 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.401 problem 411

2.401.1 Maple step by step solution 3764

Internal problem ID [7891]
Internal file name [OUTPUT/6824_Sunday_June_05_2022_05_12_29_PM_47826417/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 411.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(x^2-1\right)y''-6xy'+12y=0}$$

Writing the ode as

$$\left(x^2-1\right)y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 760: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(1+x)^2} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)} + \frac{15}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)} + (-)\,(0) \\
&= -\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)} \\
&= \frac{x-4}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right)(0) + \left(\left(\frac{3}{2(x-1)^2} - \frac{5}{2(1+x)^2}\right) + \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right)^2 - \Big(\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right) dx} \\ &= \frac{(1+x)^{\frac{5}{2}}}{(x-1)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2-1}\, dx} \\ &= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(1+x)}{2}} \\ &= z_1\left((x-1)^{\frac{3}{2}}(1+x)^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3\ln(x-1)+3\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\frac{x(x^2+1)}{(1+x)^4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)^4\right) + c_2\left((1+x)^4\left(-\frac{x(x^2+1)}{(1+x)^4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)^4 - c_2 x(x^2+1) \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^4 - c_2 x(x^2+1)
$$

Verified OK.

2.401.1 Maple step by step solution

Let's solve

$(x^2-1)\, y'' - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -3$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''-6xy'+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-3)+a_k(k+r-3)\,(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-3)\left((-2k-2r-2)\,a_{k+1}+a_k(k+r-4)\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=4$

$a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$

- Apply recursion relation for $k=0$

$a_1=-2a_0$

- Apply recursion relation for $k=1$

$a_2=-\frac{3a_1}{4}$

- Express in terms of a_0

$a_2=\frac{3a_0}{2}$

- Apply recursion relation for $k=2$

$a_3=-\frac{a_2}{3}$

- Express in terms of a_0

$a_3=-\frac{a_0}{2}$

- Apply recursion relation for $k=3$

$a_4=-\frac{a_3}{8}$

- Express in terms of a_0

$a_4=\frac{a_0}{16}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$

- Revert the change of variables $u=1+x$

$\left[y=\frac{a_0(x-1)^4}{16}\right]$

- Recursion relation for $r=4$

$a_{k+1}=\frac{a_k k}{2(k+5)}$

- Solution for $r=4$

$\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\frac{a_0(x-1)^4}{16}+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+4}\right), b_{k+1}=\frac{b_k k}{2(k+5)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1(x^3+x)+c_2(x^4+6x^2+1)$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 45

```
DSolve[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{x^2-1}(c_2x(x^2+1)+c_1(x-1)^4)}{\sqrt{1-x^2}}$$

2.402 problem 412

Internal problem ID [7892]
Internal file name [`OUTPUT/6825_Sunday_June_05_2022_05_12_31_PM_55778252/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 412.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+3)\,y''-7xy'+16y=0}$$

Writing the ode as

$$(x^2+3)\,y''-7xy'+16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+3 \\ B &= -7x \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 234}{4\left(x^2 + 3\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 234$$
$$t = 4\left(x^2 + 3\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 234}{4\left(x^2 + 3\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 762: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+3)^2$. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{77}{16\left(x - i\sqrt{3}\right)^2} + \frac{77}{16\left(x + i\sqrt{3}\right)^2} + \frac{79i\sqrt{3}}{48\left(x - i\sqrt{3}\right)} - \frac{79i\sqrt{3}}{48\left(x + i\sqrt{3}\right)}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 234}{4\left(x^2+3\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-234}{4\left(x^2+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$
$-i\sqrt{3}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{7}{2}\right)\\
&= 4
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} + (-)\,(0) \\
&= -\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)} \\
&= -\frac{7x}{2x^2+6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(-\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{7}{4\left(x-i\sqrt{3}\right)^2}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{8}, a_1 = 0, a_2 = -9, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 9x^2 + \frac{27}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) e^{\int \left(-\frac{7}{4\left(x-i\sqrt{3}\right)} - \frac{7}{4\left(x+i\sqrt{3}\right)}\right) dx} \\
&= \left(x^4 - 9x^2 + \frac{27}{8}\right) \frac{1}{\left(x^2+3\right)^{\frac{7}{4}}} \\
&= \frac{8x^4 - 72x^2 + 27}{8\left(x^2+3\right)^{\frac{7}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-7x}{x^2+3}\,dx} \\
&= z_1 e^{\frac{7\ln\left(x^2+3\right)}{4}} \\
&= z_1 \left(\left(x^2+3\right)^{\frac{7}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 9x^2 + \frac{27}{8}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7x}{x^2+3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{7\ln\left(x^2+3\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-256\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\left(x^4-9x^2+\frac{27}{8}\right)\ln\left(\sqrt{x^2+3}-x\right)+\left(-3840x^5-720\right.}{256\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\left(x^4-\right.}\right.
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4-9x^2+\frac{27}{8}\right) + c_2\left(x^4-9x^2\right. \\
&\quad + \frac{27}{8}\left(\frac{-256\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\left(x^4-9x^2+\frac{27}{8}\right)\ln\left(\sqrt{x^2+3}-x\right)+\left(-3840x^5-7\right.}{256\left(\left(-x^3-\frac{3}{2}x\right)\sqrt{x^2+3}+x^4+3x^2+\frac{9}{8}\right)\left(x^4\right.}\right.
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + \frac{c_2\left(-256\left(\left(-x^3 - \frac{3}{2}x\right)\sqrt{x^2+3} + x^4 + 3x^2 + \frac{9}{8}\right)\left(x^4 - 9x^2 + \frac{27}{8}\right)\ln\left(\sqrt{x^2+3} - x\right) + \left(-3840x^5 - 720x\right.\right.}{\left(-256x^3 - 384x\right)\sqrt{x^2+3} + 256x^4 + 768x^2 +} \tag{1}$$

Verification of solutions

$$y = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + \frac{c_2\left(-256\left(\left(-x^3 - \frac{3}{2}x\right)\sqrt{x^2+3} + x^4 + 3x^2 + \frac{9}{8}\right)\left(x^4 - 9x^2 + \frac{27}{8}\right)\ln\left(\sqrt{x^2+3} - x\right) + \left(-3840x^5 - 720x\right.\right.}{\left(-256x^3 - 384x\right)\sqrt{x^2+3} + 256x^4 + 768x^2 +}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
dsolve((x^2+3)*diff(y(x),x$2)-7*x*diff(y(x),x)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^4 - 9x^2 + \frac{27}{8}\right) + c_2\left(\frac{\ln\left(\sqrt{x^2+3} - x\right)x^4}{64} + \frac{25\sqrt{x^2+3}\,x^3}{768} + \frac{25x^4}{768} - \frac{9\ln\left(\sqrt{x^2+3} - x\right)x^2}{64} - \frac{55\sqrt{x^2+3}\,x}{512} - \frac{75x^2}{256} + \frac{27\ln\left(\sqrt{x^2+3} - x\right)}{512} + \frac{225}{2048}\right)$$

✓ Solution by Mathematica

Time used: 0.23 (sec). Leaf size: 492

```
DSolve[(x^2+3)*y''[x]-7*x*y'[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24}c_2\Big(12960x^2\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$+5248800x^2\text{RootSum}\Big[21088072057248000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(27353083060732502808000000\#1^3-2723852861741002572000\#1^2-410617$$
$$-4860\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$-1968300\text{RootSum}\Big[21088072057248000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(27353083060732502808000000\#1^3-2723852861741002572000\#1^2-410617$$
$$-1440x^4\text{RootSum}\Big[7838208000\#1^4-188281584000\#1^2-241544908800\#1$$
$$+18453344881\&,\#1\log\Big(-411757211968704000\#1^3-166063274606980800\#1^2+10138703825167113$$
$$-583200x^4\text{RootSum}\Big[21088072057248000000\#1^4-30882886815600000\#1^2$$
$$+97825688064000\#1$$
$$+18453344881\&,\#1\log\Big(27353083060732502808000000\#1^3-2723852861741002572000\#1^2-410617$$
$$+165\sqrt{x^2+3}x+216x^2\log\left(\sqrt{x^2+3}-x\right)-81\log\left(\sqrt{x^2+3}-x\right)$$
$$-24x^4\log\left(\sqrt{x^2+3}-x\right)-50\sqrt{x^2+3}x^3\Big)+c_1\left(x^4-9x^2+\frac{27}{8}\right)$$

2.403 problem 413

2.403.1 Maple step by step solution 3781

Internal problem ID [7893]
Internal file name [`OUTPUT/6826_Sunday_June_05_2022_05_12_37_PM_15702039/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 413.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''+8xy'+12y=0}$$

Writing the ode as

$$(x^2-1)\,y''+8xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= 8x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{\left(x^2-1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8$$
$$t = \left(x^2-1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 763: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x - 1} + \frac{2}{1 + x} + (-)(0) \\
&= -\frac{1}{x - 1} + \frac{2}{1 + x} \\
&= \frac{-3 + x}{x^2 - 1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-1} + \frac{2}{1+x}\right)(0) + \left(\left(\frac{1}{(x-1)^2} - \frac{2}{(1+x)^2}\right) + \left(-\frac{1}{x-1} + \frac{2}{1+x}\right)^2 - \left(\frac{8}{(x^2-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x-1} + \frac{2}{1+x}\right) dx} \\ &= \frac{(1+x)^2}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{8x}{x^2-1} \, dx} \\ &= z_1 e^{-2\ln(x-1) - 2\ln(1+x)} \\ &= z_1 \left(\frac{1}{(x-1)^2 (1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{x^2-1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-3x^2-1}{3\left(1+x\right)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\left(x-1\right)^3}\right) + c_2 \left(\frac{1}{\left(x-1\right)^3}\left(\frac{-3x^2-1}{3\left(1+x\right)^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\left(x-1\right)^3} + \frac{c_2(-3x^2-1)}{3\left(x-1\right)^3\left(1+x\right)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{\left(x-1\right)^3} + \frac{c_2(-3x^2-1)}{3\left(x-1\right)^3\left(1+x\right)^3}
$$

Verified OK.

2.403.1 Maple step by step solution

Let's solve

$(x^2-1)\,y'' + 8xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x=-1$

  $$\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 4$$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x=-1$

  $$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 8xy' + 12y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $$(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (8u-8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)(k+1-m+r)u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_{0}r(3+r)u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(k+r+4)+a_{k}(k+r+4)(k+r+3)\right)u^{k+r}\right)=$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-3,0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+4)\left((-2k-2r-2)a_{k+1}+a_{k}(k+r+3)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r=-3$

  $a_{k+1}=\frac{a_{k}k}{2(k-2)}$

- Series not valid for $r=-3$, division by 0 in the recursion relation at $k=2$

  $a_{k+1}=\frac{a_{k}k}{2(k-2)}$

- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_{k}(k+3)}{2(k+1)}$

- Solution for $r=0$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+1}=\frac{a_{k}(k+3)}{2(k+1)}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(1+x)^{k},a_{k+1}=\frac{a_{k}(k+3)}{2(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve((x^2-1)*diff(y(x),x$2)+8*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x^2+1)}{(x-1)^3 (x+1)^3} + \frac{c_2(x^3+3x)}{(x-1)^3 (x+1)^3}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 37

```
DSolve[(x^2-1)*y''[x]+8*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3(x^2-1)^3}$$

2.404 problem 414

Internal problem ID [7894]
Internal file name [`OUTPUT/6827_Sunday_June_05_2022_05_12_40_PM_62124213/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 414.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3y'' + xy' - 4y = 0}$$

Writing the ode as

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 54}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 54 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{36} + \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 765: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} + \frac{9}{2x} - \frac{243}{4x^3} + \frac{6561}{4x^5} - \frac{885735}{16x^7} + \frac{33480783}{16x^9} - \frac{2711943423}{32x^{11}} + \frac{115063885233}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{6} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+54}{36} \\
&= Q + \frac{R}{36} \\
&= \left(\frac{x^2}{36} + \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{36} + \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= 4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) &&= -5
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{36} + \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{6}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{6}\right) \\ &= \frac{x}{6} \\ &= \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{6}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{6}\right) + \left(\frac{x}{6}\right)^2 - \left(\frac{x^2}{36} + \frac{3}{2}\right)\right) = 0$$

$$-\frac{a_3x^3}{3} + \frac{2(18-a_2)\,x^2}{3} + (-a_1 + 6a_3)\,x - \frac{4a_0}{3} + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 27, a_1 = 0, a_2 = 18, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 18x^2 + 27$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\int \frac{x}{6} dx} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}} \\
&= \left(x^4 + 18x^2 + 27\right) \mathrm{e}^{\frac{x^2}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{3} \, dx} \\
&= z_1 e^{-\frac{x^2}{12}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{12}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 18x^2 + 27
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 18x^2 + 27\right) + c_2 \left(x^4 + 18x^2 + 27 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2} dx \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{e^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(3*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 18x^2 + 27) + c_2(x^4 + 18x^2 + 27)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{6}}}{(x^4 + 18x^2 + 27)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 43

```
DSolve[3*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{6}} \operatorname{HermiteH}\left(-5, \frac{x}{\sqrt{6}}\right) + \frac{1}{27}c_2(x^4 + 18x^2 + 27)$$

2.405 problem 415

2.405.1 Maple step by step solution 3800

Internal problem ID [7895]
Internal file name [`OUTPUT/6828_Sunday_June_05_2022_05_12_44_PM_47383325/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 415.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{5y'' - 2xy' + 10y = 0}$$

Writing the ode as

$$5y'' - 2xy' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= -2x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 55}{25} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 55$$
$$t = 25$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{25} - \frac{11}{5}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 766: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{5} - \frac{11}{2x} - \frac{605}{8x^3} - \frac{33275}{16x^5} - \frac{9150625}{128x^7} - \frac{704598125}{256x^9} - \frac{116258690625}{1024x^{11}} - \frac{10048072546875}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{5}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{5} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{25}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 55}{25} \\ &= Q + \frac{R}{25} \\ &= \left(\frac{x^2}{25} - \frac{11}{5}\right) + (0) \\ &= \frac{x^2}{25} - \frac{11}{5} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{5}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{5}\right) - (0) \\ &= -\frac{11}{5} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{5} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= -6 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) &&= 5 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{25} - \frac{11}{5}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{5}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{5}\right) \\ &= -\frac{x}{5} \\ &= -\frac{x}{5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2) + 2\left(-\frac{x}{5}\right)(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1) + \left(\left(-\frac{1}{5}\right) + \left(-\frac{x}{5}\right)^2 - \left(\frac{x^{2}}{25}\right.\right.$$

$$\frac{2a_4 x^4}{5} + \frac{4(25 + a_3)\, x^3}{5} + \frac{6(a_2 + 10a_4)\, x^2}{5} + \frac{2(4a_1 + 15a_3)\, x}{5} + 2a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{375}{4}, a_2 = 0, a_3 = -25, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 25x^3 + \frac{375}{4}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{\int -\frac{x}{5} dx} \\
&= \left(x^5 - 25x^3 + \frac{375}{4}x\right) e^{-\frac{x^2}{10}} \\
&= \frac{(4x^5 - 100x^3 + 375x) e^{-\frac{x^2}{10}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x}{5} \, dx} \\
&= z_1 e^{\frac{x^2}{10}} \\
&= z_1 \left(e^{\frac{x^2}{10}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - 25x^3 + \frac{375}{4}x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{5}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^{\frac{x^2}{5}}}{x^2\left(4x^4-100x^2+375\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - 25x^3 + \frac{375}{4}x\right) + c_2\left(x^5 - 25x^3 + \frac{375}{4}x\left(\int \frac{16\,\mathrm{e}^{\frac{x^2}{5}}}{x^2\left(4x^4-100x^2+375\right)^2} dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = c_1&\left(x^5 - 25x^3 + \frac{375}{4}x\right) \\
&+ 16c_2 x\left(x^4 - 25x^2 + \frac{375}{4}\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{5}}}{x^2\left(4x^4-100x^2+375\right)^2} dx\right)
\end{aligned} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\left(x^5 - 25x^3 + \frac{375}{4}x\right) + 16c_2 x\left(x^4 - 25x^2 + \frac{375}{4}\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{5}}}{x^2\left(4x^4-100x^2+375\right)^2} dx\right)
$$

Verified OK.

2.405.1 Maple step by step solution

Let's solve

$5y'' - 2xy' + 10y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{2xy'}{5} - 2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2xy'}{5} + 2y = 0$

- Multiply by denominators

$5y'' - 2xy' + 10y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (5a_{k+2}(k+2)\,(k+1) - 2a_k(k-5))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$5(k^2+3k+2)\, a_{k+2} - 2a_k(k-5) = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2a_k(k-5)}{5(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(5*diff(y(x),x$2)-2*x*diff(y(x),x)+10*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^5 - 25x^3 + \frac{375}{4}x\right) + c_2\left(x^5 - 25x^3 + \frac{375}{4}x\right)\left(\int \frac{e^{\frac{x^2}{5}}}{(4x^4 - 100x^2 + 375)^2 x^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.168 (sec). Leaf size: 138

```
DSolve[5*y''[x]-2*x*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{200}\sqrt{\frac{\pi}{5}}c_2\sqrt{x^2}(4x^4-100x^2+375)\operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{5}}\right)+\frac{32c_1x^5}{25\sqrt{5}} - \frac{32c_1x^3}{\sqrt{5}}-\frac{9}{20}c_2e^{\frac{x^2}{5}}x^2+c_2e^{\frac{x^2}{5}}+\frac{1}{50}c_2e^{\frac{x^2}{5}}x^4+24\sqrt{5}c_1x$$

2.406 problem 416

2.406.1 Maple step by step solution 3809

Internal problem ID [7896]
Internal file name [`OUTPUT/6829_Sunday_June_05_2022_05_12_47_PM_20415841/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 416.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - x^2 y' - 3yx = 0}$$

Writing the ode as

$$y'' - x^2 y' - 3yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3+8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3+8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3+8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 768: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= \frac{x^2}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2} \right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) &= 0 \\ -xa_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int \frac{x^2}{2} dx} \\ &= (x)\, e^{\frac{x^3}{6}} \\ &= x\, e^{\frac{x^3}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\ &= z_1 e^{\frac{x^3}{6}} \\ &= z_1 \left(e^{\frac{x^3}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, e^{\frac{x^3}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{e^{-\frac{x^3}{3}}}{x^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{\frac{x^3}{3}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)
$$

Verified OK.

2.406.1 Maple step by step solution

Let's solve

$y'' - x^2 y' - 3yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\,(k a_{k+2} - a_{k-1} + a_{k+2}) = 0$$

- Shift index using $k- >k+1$

$$(k+3)\,((k+1)\, a_{k+3} - a_k + a_{k+3}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 76

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x^3}{3}} x - \frac{9c_2 \mathrm{e}^{\frac{x^3}{3}} 3^{\frac{2}{3}} \mathrm{e}^{-\frac{x^3}{6}} \left(x^6 \,\mathrm{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) + 5 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right) x^3 + 10 \,\mathrm{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right)\right)}{10x^3 \left(x^3\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 51

```
DSolve[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3}\Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.407 problem 417

Internal problem ID [7897]
Internal file name [`OUTPUT/6830_Sunday_June_05_2022_05_12_51_PM_38701365/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 417.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2xy'-2y=0}$$

Writing the ode as

$$\begin{align*}(x^2+1)\,y''+2xy'-2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+1\\ B&=2x \tag{3}\\ C&=-2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 770: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{2x^2+3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\ &= 2-(1)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2(x - i)^2} - \frac{1}{2(x + i)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right.\right.
$$

$$
-\frac{2(x^2 + 1)\, a_0}{(-x + i)^2 (x + i)^2}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x)\sqrt{x^2 + 1} \\
&= \sqrt{x^2 + 1}\, x
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{1}{x} - \arctan\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{1}{x} - \arctan\left(x\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2(-\arctan\left(x\right)x - 1) \tag{1}
$$

Verification of solutions

$$y = c_1x + c_2(-\arctan(x)\,x - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(\arctan(x)\,x + 1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 48

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}i(2c_1x - c_2x\log(1 - ix) + c_2x\log(1 + ix) + 2ic_2)$$

2.408 problem 418

2.408.1 Maple step by step solution 3825

Internal problem ID [7898]
Internal file name [OUTPUT/6831_Sunday_June_05_2022_05_12_53_PM_61634565/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 418.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' - 2y = 0}$$

Writing the ode as

$$y'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 771: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} + \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{2}\right) - (0) \\
&= \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned}d &= \alpha_\infty^+ \\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) = 0$$
$$-a_1 x - 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 + 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2+1\right) + c_2 \left(x^2 + 1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)$$

Verified OK.

2.408.1 Maple step by step solution

Let's solve

$y'' + xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right) a_{k+2}+a_k(k-2)=0$$

- Recursion relation; series terminates at $k=2$

$$a_{k+2}=-\frac{a_k(k-2)}{k^2+3k+2}$$

- Apply recursion relation for $k=0$

$$a_2=a_0$$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

$$y=A_2x^2+A_1x+a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2+1) + c_2(x^2+1)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 35

```
DSolve[y''[x]+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2(x^2+1)$$

2.409 problem 419

Internal problem ID [7899]
Internal file name [`OUTPUT/6832_Sunday_June_05_2022_05_12_56_PM_67565026/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 419.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-6x+10)\,y''-4(-3+x)\,y'+6y=0}$$

Writing the ode as

$$(x^2-6x+10)\,y''+(-4x+12)\,y'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-6x+10 \\ B &= -4x+12 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{\left(x^2 - 6x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= \left(x^2 - 6x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{\left(x^2 - 6x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 773: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 6x + 10)^2$. There is a pole at $x = 3 + i$ of order 2. There is a pole at $x = 3 - i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-3-i)^2} + \frac{2}{(x-3+i)^2} + \frac{2i}{x-3-i} - \frac{2i}{x-3+i}$$

For the pole at $x = 3 + i$ let b be the coefficient of $\frac{1}{(x-3-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = 3 - i$ let b be the coefficient of $\frac{1}{(x-3+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2 - 6x + 10)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3+i$	2	0	2	-1
$3-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-3-i} + \frac{2}{x-3+i} + (-)\,(0) \\ &= -\frac{1}{x-3-i} + \frac{2}{x-3+i} \\ &= \frac{-3-3i+x}{x^2-6x+10} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)(0) + \left(\left(\frac{1}{(x-3-i)^2} - \frac{2}{(x-3+i)^2}\right) + \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right)^2\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-3-i} + \frac{2}{x-3+i}\right) dx} \\
&= \frac{\left(x^2 - 6x + 10\right)^2}{\left(ix - 3i + 1\right)^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x+12}{x^2-6x+10} \, dx} \\
&= z_1 e^{\ln\left(x^2-6x+10\right)} \\
&= z_1 \left(x^2 - 6x + 10\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2 - 6x + 10\right)^3}{\left(ix - 3i + 1\right)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x+12}{x^2-6x+10} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{2\ln\left(x^2-6x+10\right)}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{x^2 - 6x + \frac{26}{3}}{\left(x - 3 + i\right)^3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3}\right) + c_2 \left(\frac{(x^2-6x+10)^3}{(ix-3i+1)^3} \left(\frac{x^2-6x+\frac{26}{3}}{(x-3+i)^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2-6x+10)^3}{(ix-3i+1)^3} + \frac{c_2 (x^2-6x+10)^3 \left(x^2-6x+\frac{26}{3}\right)}{(ix-3i+1)^3 (x-3+i)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2-6x+10)^3}{(ix-3i+1)^3} + \frac{c_2 (x^2-6x+10)^3 \left(x^2-6x+\frac{26}{3}\right)}{(ix-3i+1)^3 (x-3+i)^3}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve((x^2-6*x+10)*diff(y(x),x$2)-4*(x-3)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1 \left(\frac{26}{3} + x^2 - 6x\right) + c_2 (x^3 - 30x + 60)
$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 36

```
DSolve[(x^2-6*x+10)*y''[x]-4*(x-3)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3} i \left(c_2 \left(3x^2 - 18x + 26\right) + 3c_1 (x - (3 + i))^3\right)$$

2.410 problem 420

2.410.1 Maple step by step solution 3841

Internal problem ID [7900]
Internal file name [`OUTPUT/6833_Sunday_June_05_2022_05_12_59_PM_49199144/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 420.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 + 6x)\, y'' + (3x + 9)\, y' - 3y = 0}$$

Writing the ode as

$$(x^2 + 6x)\, y'' + (3x + 9)\, y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 6x \\ B &= 3x + 9 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 + 90x - 27 \\ t &= 4\left(x^2 + 6x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 90x - 27}{4\left(x^2 + 6x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 774: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 6x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -6$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{11}{16x} - \frac{3}{16x^2} - \frac{3}{16(x+6)^2} - \frac{11}{16(x+6)}$$

For the pole at $x = -6$ let b be the coefficient of $\frac{1}{(x+6)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 90x - 27}{4(x^2 + 6x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2+90x-27}{4\left(x^2+6x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-6	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\,(x+6)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2} + \frac{9}{2}}{x\,(x+6)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\,(x+6)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\,(x+6)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2 + 90x - 27}{4\,(x^2+6x)^2}\right)\right) = \frac{9 - 3a_0}{x\,(x+6)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+3)\, e^{\int \left(\frac{3}{4(x+6)} + \frac{3}{4x}\right) dx} \\
&= (x+3)\, e^{\frac{3\ln(x)}{4} + \frac{3\ln(x+6)}{4}} \\
&= (x+3)\, x^{\frac{3}{4}} (x+6)^{\frac{3}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x+9}{x^2+6x}\,dx} \\
&= z_1 e^{-\frac{3\ln(x(x+6))}{4}} \\
&= z_1\left(\frac{1}{(x\,(x+6))^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+9}{x^2+6x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x(x+6))}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-2x^2-12x-9}{81\,(x+3)\,\sqrt{x}\,\sqrt{x+6}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}\right) + c_2\left(\frac{(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}}\left(\frac{-2x^2-12x-9}{81\,(x+3)\,\sqrt{x}\,\sqrt{x+6}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x+3)\,x^{\frac{3}{4}}(x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}} - \frac{2c_2(x+6)^{\frac{1}{4}}\,x^{\frac{1}{4}}\left(x^2+6x+\frac{9}{2}\right)}{81\,(x\,(x+6))^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x+3)\, x^{\frac{3}{4}} (x+6)^{\frac{3}{4}}}{(x\,(x+6))^{\frac{3}{4}}} - \frac{2c_2(x+6)^{\frac{1}{4}}\, x^{\frac{1}{4}} \left(x^2+6x+\frac{9}{2}\right)}{81\,(x\,(x+6))^{\frac{3}{4}}}$$

Verified OK.

2.410.1 Maple step by step solution

Let's solve

$(x^2+6x)\, y'' + (3x+9)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x(x+6)} - \frac{3(x+3)y'}{x(x+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x+3)y'}{x(x+6)} - \frac{3y}{x(x+6)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3(x+3)}{x(x+6)}, P_3(x) = -\frac{3}{x(x+6)}\right]$

  - $(x+6)\cdot P_2(x)$ is analytic at $x=-6$

    $\left((x+6)\cdot P_2(x)\right)\bigg|_{x=-6} = \frac{3}{2}$

  - $(x+6)^2\cdot P_3(x)$ is analytic at $x=-6$

    $\left((x+6)^2\cdot P_3(x)\right)\bigg|_{x=-6} = 0$

  - $x=-6$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -6$

- Multiply by denominators

  $y''x(x+6) + (3x+9)\, y' - 3y = 0$

- Change variables using $x = u-6$ so that the regular singular point is at $u=0$

$$\left(u^2-6u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(3u-9\right)\left(\frac{d}{du}y(u)\right)-3y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0r(1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-3a_{k+1}(k+1+r)\,(2k+3+2r)+a_k(k+r+3)\,(k+r-1)\right)u^{k+r}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-6\left(k+\frac{3}{2}+r\right)(k+1+r)\,a_{k+1}+a_k(k+r+3)\,(k+r-1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+3)(k+r-1)}{3(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+1}=\frac{a_k(k+3)(k-1)}{3(2k+3)(k+1)}$$

- Apply recursion relation for $k=0$

$a_1 = -\frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{u}{3}\right)$

- Revert the change of variables $u = x + 6$

$\left[y = a_0\left(-1 - \frac{x}{3}\right)\right]$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = -\frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Revert the change of variables $u = x + 6$

$\left[y = \sum_{k=0}^{\infty} a_k (x+6)^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0\left(-1 - \frac{x}{3}\right) + \left(\sum_{k=0}^{\infty} b_k (x+6)^{k-\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{3(2k+2)\left(k+\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve((x^2+6*x)*diff(y(x),x$2)+(3*x+9)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+3) + \frac{c_2(2x^2+12x+9)}{\sqrt{x^2+6x}}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 82

```
DSolve[(x^2+6*x)*y''[x]+(3*x+9)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{9\sqrt{\pi}c_2\sqrt[4]{-x(x+6)}Q_{\frac{3}{2}}^{\frac{1}{2}}\left(\frac{x}{3}+1\right) + \sqrt{6}c_1(2x^2+12x+9)}{9\sqrt{\pi}\sqrt[4]{-x^2}\sqrt{x+6}}$$

2.411 problem 421

2.411.1 Maple step by step solution 3852

Internal problem ID [7901]
Internal file name [`OUTPUT/6834_Sunday_June_05_2022_05_13_02_PM_37474984/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 421.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + (t^2 - 1)\, y' + t^3 y = 0}$$

Writing the ode as

$$ty'' + (t^2 - 1)\, y' + t^3 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t^2 - 1 \\ C &= t^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3t^4 + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3t^4 + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3t^4 + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 776: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3t^2}{4} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{1} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{i\sqrt{3}\,t}{2} - \frac{i\sqrt{3}}{4t^3} - \frac{i\sqrt{3}}{16t^7} - \frac{i\sqrt{3}}{32t^{11}} - \frac{5i\sqrt{3}}{256t^{15}} - \frac{7i\sqrt{3}}{512t^{19}} - \frac{21i\sqrt{3}}{2048t^{23}} - \frac{33i\sqrt{3}}{4096t^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i\sqrt{3}}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{i\sqrt{3}\,t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{3t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-3t^4 + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(-\frac{3t^2}{4}\right) + \left(\frac{3}{4t^2}\right) \\ &= -\frac{3t^2}{4} + \frac{3}{4t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{i\sqrt{3}\,t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i\sqrt{3}}{2}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-3t^4 + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{i\sqrt{3}\,t}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + (-)\left(\frac{i\sqrt{3}\, t}{2}\right) \\
&= -\frac{1}{2t} - \frac{i\sqrt{3}\, t}{2} \\
&= \frac{-i\sqrt{3}\, t^2 - 1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2t} - \frac{i\sqrt{3}\, t}{2}\right)(0) + \left(\left(\frac{1}{2t^2} - \frac{i\sqrt{3}}{2}\right) + \left(-\frac{1}{2t} - \frac{i\sqrt{3}\, t}{2}\right)^2 - \left(\frac{-3t^4 + 3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2t} - \frac{i\sqrt{3}\, t}{2}\right) dt} \\
&= \frac{\mathrm{e}^{-\frac{i\sqrt{3}\, t^2}{4}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2 - 1}{t} \, dt} \\
&= z_1 e^{\frac{\ln(t)}{2} - \frac{t^2}{4}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{-\frac{t^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{t^2 \left(1+i\sqrt{3}\right)}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\frac{t^2}{2}+\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{i\sqrt{3}\, \mathrm{e}^{\frac{i\sqrt{3}\, t^2}{2}}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{t^2 \left(1+i\sqrt{3}\right)}{4}}\right) + c_2 \left(\mathrm{e}^{-\frac{t^2 \left(1+i\sqrt{3}\right)}{4}} \left(-\frac{i\sqrt{3}\, \mathrm{e}^{\frac{i\sqrt{3}\, t^2}{2}}}{3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{t^2 \left(1+i\sqrt{3}\right)}{4}} - \frac{i c_2 \sqrt{3}\, \mathrm{e}^{\frac{t^2 \left(i\sqrt{3}-1\right)}{4}}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{t^2 \left(1+i\sqrt{3}\right)}{4}} - \frac{i c_2 \sqrt{3}\, \mathrm{e}^{\frac{t^2 \left(i\sqrt{3}-1\right)}{4}}}{3}$$

Verified OK.

2.411.1 Maple step by step solution

Let's solve

$ty'' + (t^2-1)\,y' + t^3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t^2-1)y'}{t} - t^2y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t^2-1)y'}{t} + t^2y = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = \frac{t^2-1}{t}, P_3(t) = t^2\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -1$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $ty'' + (t^2-1)\,y' + t^3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $t^3 \cdot y$ to series expansion

    $t^3 \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+3}$

- Shift index using $k->k-3$

  $t^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} t^{k+r}$

- Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t \cdot y''$ to series expansion

  $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

  $t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)\,(-1+r)\, t^r + (a_2(2+r)\, r + a_0 r)\, t^{1+r} + (a_3(3+r)\,(1+r) + a_1(1+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- The coefficients of each power of t must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r + a_0 r = 0, a_3(3+r)\,(1+r) + a_1(1+r) = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{a_0}{2+r}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + a_{k-1}(k+r-1) + a_{k-3} = 0$

- Shift index using $k->k+3$

  $a_{k+4}(k+4+r)\,(k+2+r) + a_{k+2}(k+2+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{k a_{k+2} + r a_{k+2} + a_k + 2 a_{k+2}}{(k+4+r)(k+2+r)}$

- Recursion relation for $r=0$

  $a_{k+4} = -\frac{k a_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = -\frac{k a_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}, a_1 = 0, a_2 = -\frac{a_0}{2}, a_3 = 0\right]$

- Recursion relation for $r=2$

  $a_{k+4} = -\frac{k a_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}$

- Solution for $r=2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+4} = -\frac{k a_{k+2}+a_k+4a_{k+2}}{(k+6)(k+4)}, a_1 = 0, a_2 = -\frac{a_0}{4}, a_3 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+4} = -\frac{k a_{k+2}+a_k+2a_{k+2}}{(k+4)(k+2)}, a_1 = 0, a_2 = -\frac{a_0}{2}, a_3 = 0, b_{k+4} = -\frac{k b_k\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(t*diff(y(t),t$2)+ (t^2-1)*diff(y(t),t)+t^3*y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 \mathrm{e}^{-\frac{t^2}{4}} \cos\left(\frac{t^2\sqrt{3}}{4}\right) + c_2 \mathrm{e}^{-\frac{t^2}{4}} \sin\left(\frac{t^2\sqrt{3}}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 48

```
DSolve[t*y''[t]+(t^2-1)*y'[t]+t^3*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-\frac{t^2}{4}} \left(c_2 \cos \left(\frac{\sqrt{3}t^2}{4} \right) + c_1 \sin \left(\frac{\sqrt{3}t^2}{4} \right) \right)$$

2.412 problem 422

2.412.1 Maple step by step solution 3859

Internal problem ID [7902]
Internal file name [`OUTPUT/6835_Sunday_June_05_2022_05_13_06_PM_37919865/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 422.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - t(t+2)\,y' + (t+2)\,y = 0}$$

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 778: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\,dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1 \left(t\,\mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\,dt \\ &= y_1 \left(\mathrm{e}^t\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\big(t(\mathrm{e}^t)\big)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 t + c_2 \mathrm{e}^t t \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 t + c_2 \mathrm{e}^t t
$$

Verified OK.

2.412.1 Maple step by step solution

Let's solve

$y'' t^2 + (-t^2 - 2t)\, y' + (t+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -2$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 2$

  ◦ $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$y'' t^2 - t(t+2) y' + (t+2) y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  ◦ Shift index using $k- > k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) t^{k+r-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) t^{k+r}$

  ◦ Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r) t^r + \left(\sum_{k=1}^{\infty} (a_k(k+r-1)(k+r-2) - a_{k-1}(k+r-2)) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-2)(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+2}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(t^2*diff(y(t),t$2)-t*(t+2)*diff(y(t),t)+(t+2)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t\, e^t$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-t*(t+2)*y'[t]+(t+2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

2.413 problem 423

2.413.1 Maple step by step solution 3869

Internal problem ID [7903]
Internal file name [`OUTPUT/6836_Sunday_June_05_2022_05_13_08_PM_28631924/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 423.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 780: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-x\, \mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x \left(-x\, \mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x - c_2 x$$

Verified OK.

2.413.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$

  ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$

  ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ○ Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$$

- ◦ Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+e^x c_2$$

✓ Solution by Mathematica
Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

2.414 problem 424

2.414.1 Maple step by step solution 3878

Internal problem ID [7904]
Internal file name [OUTPUT/6837_Sunday_June_05_2022_05_13_11_PM_95611847/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 424.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(x - \frac{3}{16}\right)y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x + \frac{3}{16}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -x + \frac{3}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{16x - 3}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 16x - 3 \\
t &= 16x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 782: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\}\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(1-(1))\\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p}\\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1 - 16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{4\sqrt{x}}} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}}\right) + c_2 \left(x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} - \frac{c_2 x^{\frac{1}{4}} \mathrm{e}^{-2\sqrt{x}}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} - \frac{c_2 x^{\frac{1}{4}} \mathrm{e}^{-2\sqrt{x}}}{2}
$$

Verified OK.

2.414.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-x + \frac{3}{16}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x-3)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(16x-3)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2} \cdot P_{3}(x)\right)\Big|_{x=0} = \frac{3}{16}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^{2}y'' + (-16x + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^{m} \cdot y$ to series expansion for $m = 0..1$

    $x^{m} \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x^{2} \cdot y''$ to series expansion

    $x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 4r)\,(-3 + 4r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_k(4k + 4r - 1)\,(4k + 4r - 3) - 16a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 4r)\,(-3 + 4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k + r - \frac{3}{4}\right)\left(k + r - \frac{1}{4}\right) a_k - 16a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $16\left(k + \frac{1}{4} + r\right)\left(k + \frac{3}{4} + r\right) a_{k+1} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{16a_k}{(4k+1+4r)(4k+3+4r)}$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}$$

- Solution for $r = \frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}, b_{k+1} = \frac{16b_k}{(4k+4)(4k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-(x-1875/10000)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{4}} \sinh\left(2\sqrt{x}\right) + c_2 x^{\frac{1}{4}} \cosh\left(2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]-(x-1875/10000)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2\sqrt{x}}\sqrt[4]{x}\left(2c_1e^{4\sqrt{x}} - c_2\right)$$

2.415 problem 425

2.415.1 Maple step by step solution 3885

Internal problem ID [7905]
Internal file name [`OUTPUT/6838_Sunday_June_05_2022_05_13_13_PM_65271681/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 425.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 784: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.415.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/100)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/100)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.416 problem 426

2.416.1 Maple step by step solution 3892

Internal problem ID [7906]
Internal file name [OUTPUT/6839_Sunday_June_05_2022_05_13_15_PM_47800875/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 426.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - t(t+2)\,y' + (t+2)\,y = 0}$$

Writing the ode as

$$t^2y'' + \left(-t^2 - 2t\right)y' + (t+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - 2t \\ C &= t + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \frac{z(t)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 786: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2t}{t^2}\, dt} \\ &= z_1 e^{\frac{t}{2}+\ln(t)} \\ &= z_1 \left(t\, \mathrm{e}^{\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-2t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+2\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(\mathrm{e}^{t}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(t) + c_2\big(t(\mathrm{e}^t)\big)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 t + c_2 \mathrm{e}^t t \tag{1}
$$

Verification of solutions

$$
y = c_1 t + c_2 \mathrm{e}^t t
$$

Verified OK.

2.416.1 Maple step by step solution

Let's solve

$y''t^2 + (-t^2 - 2t)\,y' + (t + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(t+2)y}{t^2} + \frac{(t+2)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+2)y'}{t} + \frac{(t+2)y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t+2}{t}, P_3(t) = \frac{t+2}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left.(t \cdot P_2(t))\right|_{t=0} = -2$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left.(t^2 \cdot P_3(t))\right|_{t=0} = 2$

  ◦ $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$y'' t^2 - t(t+2)\, y' + (t+2)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k t^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $t^m \cdot y$ to series expansion for $m = 0..1$

$$t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$$

  - Shift index using $k- >k - m$

$$t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$$

  - Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

$$t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, t^{k+r}$$

  - Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-2+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-2) - a_{k-1}(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-2)\,(a_k(k+r-1) - a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+1}(k+r)-a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k}{k+r}$
- Recursion relation for $r=1$

  $a_{k+1}=\frac{a_k}{k+1}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1}=\frac{a_k}{k+1}\right]$
- Recursion relation for $r=2$

  $a_{k+1}=\frac{a_k}{k+2}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1}=\frac{a_k}{k+2}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(t^2*diff(y(t),t$2)-t*(t+2)*diff(y(t),t)+(t+2)*y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t\,e^t$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[t^2*y''[t]-t*(t+2)*y'[t]+(t+2)*y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to t(c_2 e^t + c_1)$$

2.417 problem 427

2.417.1 Maple step by step solution 3902

Internal problem ID [7907]
Internal file name [`OUTPUT/6840_Sunday_June_05_2022_05_13_18_PM_46602948/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 427.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Laguerre]
```

$$\boxed{ty'' - (t+1)\,y' + y = 0}$$

Writing the ode as

$$ty'' + (-t-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 788: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 2t + 3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t-1}{t}\, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\,\mathrm{e}^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1\left(-(t+1)\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-(t+1)\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t + c_2(-t-1)$$

Verified OK.

2.417.1 Maple step by step solution

Let's solve

$t y'' + (-t-1)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+1)y'}{t} + \frac{y}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\bigg|_{t=0} = -1$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - Convert $t \cdot y''$ to series expansion

    $t \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r-1}$

  - Shift index using $k- >k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1(t + 1) + c_2 e^t$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 19

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2(t + 1)$$

2.418 problem 428

2.418.1 Maple step by step solution 3912

Internal problem ID [7908]
Internal file name [OUTPUT/6841_Sunday_June_05_2022_05_13_21_PM_50663195/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 428.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-t+1)\,y''+ty'-y=0}$$

Writing the ode as

$$(-t+1)\,y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t+1 \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4(t-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 4t + 6$$
$$t = 4(t-1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4(t-1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 790: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(t-1)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(t-1)} + \frac{3}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4(t-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(t-1\right)} + \frac{1}{2} \\
&= \frac{t-2}{2t-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(t-1\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\left(t-1\right)^2}\right) + \left(-\frac{1}{2\left(t-1\right)} + \frac{1}{2}\right)^2 - \left(\frac{t^2 - 4t + 6}{4\left(t-1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-1)} + \frac{1}{2}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{-t+1} \, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t-1)}{2}} \\
&= z_1 \left(\sqrt{t-1}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{-t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t-1)}}{(y_1)^2}\, dt \\ &= y_1\left(-t\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-t\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^t - c_2 t \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^t - c_2 t$$

Verified OK.

2.418.1 Maple step by step solution

Let's solve

$(-t+1)\, y'' + t y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t-1} + \frac{ty'}{t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{ty'}{t-1} + \frac{y}{t-1} = 0$

☐ Check to see if $t_0 = 1$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t}{t-1}, P_3(t) = \frac{1}{t-1}\right]$

- $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\Big|_{t=1} = -1$

- $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\Big|_{t=1} = 0$

- $t = 1$is a regular singular point

  Check to see if $t_0 = 1$ is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(t-1) - ty' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1))\,u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (t-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 e^t$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 17

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] == 0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2 t$$

2.419 problem 429

2.419.1 Maple step by step solution 3919

Internal problem ID [7909]
Internal file name [`OUTPUT/6842_Sunday_June_05_2022_05_13_23_PM_12696973/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 429.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 792: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.419.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/100)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/100)*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.420 problem 430

2.420.1 Maple step by step solution 3929

Internal problem ID [7910]
Internal file name [`OUTPUT/6843_Sunday_June_05_2022_05_13_26_PM_71063995/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 430.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{ty'' - (t+1)\, y' + y = 0}$$

Writing the ode as

$$ty'' + (-t-1)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -t-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 794: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t + 3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{1}{2t} \\ &= \frac{t-1}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2 - 2t + 3}{4t^2}\right) \right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-t-1}{t} \, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\ &= z_1 \left(\sqrt{t}\, e^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-t-1}{t}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\
&= y_1\left(-(t+1)\, e^{-t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^t\right) + c_2\left(e^t\left(-(t+1)\, e^{-t}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^t + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 e^t + c_2(-t-1)$$

Verified OK.

2.420.1 Maple step by step solution

Let's solve

$t y'' + (-t-1)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+1)y'}{t} + \frac{y}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 0$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t-1)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

  - Convert $t \cdot y''$ to series expansion

    $t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

  - Shift index using $k- >k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(t*diff(y(t),t$2)-(1+t)*diff(y(t),t)+y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1(t+1) + c_2 \mathrm{e}^t$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[t*y''[t]-(1+t)*y'[t]+y[t] ==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2(t+1)$$

2.421 problem 431

2.421.1 Maple step by step solution 3939

Internal problem ID [7911]
Internal file name [`OUTPUT/6844_Sunday_June_05_2022_05_13_28_PM_96138537/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 431.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-t+1)\, y''+t y'-y=0}$$

Writing the ode as

$$(-t+1)\, y''+t y'-y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t+1 \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t)=r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 4t + 6}{4(t-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 4t + 6$$
$$t = 4(t-1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 4t + 6}{4(t-1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 796: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(t-1)^2$. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2(t-1)} + \frac{3}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{t^3} + \frac{11}{4t^4} + \frac{21}{4t^5} + \frac{15}{2t^6} + \frac{6}{t^7} - \frac{117}{16t^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 4t + 6}{4t^2 - 8t + 4} \\ &= Q + \frac{R}{4t^2 - 8t + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t + 5}{4t^2 - 8t + 4}\right) \\ &= \frac{1}{4} + \frac{-2t + 5}{4t^2 - 8t + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^2 - 4t + 6}{4(t-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(t-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2(t-1)} + \frac{1}{2} \\ &= \frac{t-2}{2t-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(t-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(t-1)^2}\right) + \left(-\frac{1}{2(t-1)} + \frac{1}{2}\right)^2 - \left(\frac{t^2-4t+6}{4(t-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{2(t-1)} + \frac{1}{2}\right) dt} \\ &= \frac{e^{\frac{t}{2}}}{\sqrt{t-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{t}{-t+1}\, dt} \\ &= z_1 e^{\frac{t}{2} + \frac{\ln(t-1)}{2}} \\ &= z_1 \left(\sqrt{t-1}\, e^{\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{-t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t-1)}}{(y_1)^2}\, dt \\ &= y_1\left(-t\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(-t\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^t - c_2 t \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t - c_2 t$$

Verified OK.

2.421.1 Maple step by step solution

Let's solve

$(-t+1)\, y'' + t y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t-1} + \frac{ty'}{t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{ty'}{t-1} + \frac{y}{t-1} = 0$

☐ Check to see if $t_0 = 1$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t}{t-1}, P_3(t) = \frac{1}{t-1}\right]$

- $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\bigg|_{t=1} = -1$

- $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$

- $t = 1$is a regular singular point

  Check to see if $t_0 = 1$ is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(t-1) - ty' + y = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = t-1$

$$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t) = 0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 e^t$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 17

```
DSolve[(1-t)*y''[t]+t*y'[t]-y[t] ==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^t - c_2 t$$

2.422 problem 432

2.422.1 Maple step by step solution 3949

Internal problem ID [7912]
Internal file name [`OUTPUT/6845_Sunday_June_05_2022_05_13_33_PM_71208593/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 432.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 798: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) = 0$$
$$a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.422.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(k a_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,e^{-\frac{x^2}{2}} - \frac{c_2 e^{-\frac{x^2}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x + 2\,e^{\frac{x^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.423 problem 433

Internal problem ID [7913]
Internal file name [`OUTPUT/6846_Sunday_June_05_2022_05_13_36_PM_23148294/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 433.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-4xy'+6y=0}$$

Writing the ode as

$$(x^2+1)\,y''-4xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{8}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 800: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(x-i)^2} + \frac{2}{(x+i)^2} + \frac{2i}{x-i} - \frac{2i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{8}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	2	-1
$-i$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x-i} + \frac{2}{x+i} + (-)(0) \\ &= -\frac{1}{x-i} + \frac{2}{x+i} \\ &= \frac{x-3i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x-i} + \frac{2}{x+i}\right)(0) + \left(\left(\frac{1}{(x-i)^2} - \frac{2}{(x+i)^2}\right) + \left(-\frac{1}{x-i} + \frac{2}{x+i}\right)^2 - \left(-\frac{8}{(x^2+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{x-i} + \frac{2}{x+i}\right) dx} \\ &= \frac{(x^2+1)^2}{(ix+1)^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2+1} \, dx} \\ &= z_1 e^{\ln(x^2+1)} \\ &= z_1 (x^2+1) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^3}{(ix+1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2+1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2\ln(x^2+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^2 - \frac{1}{3}}{(x+i)^3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^3}{(ix+1)^3} \right) + c_2 \left(\frac{(x^2+1)^3}{(ix+1)^3} \left(\frac{x^2 - \frac{1}{3}}{(x+i)^3} \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 (x^2+1)^3}{(ix+1)^3} + \frac{c_2 (x^2+1)^3 \left(x^2 - \frac{1}{3}\right)}{(ix+1)^3 (x+i)^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2+1)^3}{(ix+1)^3} + \frac{c_2 (x^2+1)^3 \left(x^2 - \frac{1}{3}\right)}{(ix+1)^3 (x+i)^3}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve((1+x^2)*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1 \left(-3x^2 + 1\right) + c_2 \left(x^3 - 3x\right)
$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 33

```
DSolve[(1+x^2)*y''[x]-4*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}i\left(c_2\left(3x^2-1\right)+3c_1(x-i)^3\right)$$

2.424 problem 434

2.424.1 Maple step by step solution 3965

Internal problem ID [7914]
Internal file name [`OUTPUT/6847_Sunday_June_05_2022_05_13_38_PM_76621142/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 434.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 801: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\,(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\,(x-1)^2}\right) + \left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4\,(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} - c_2 x$$

Verified OK.

2.424.1 Maple step by step solution

Let's solve

$(1-x)\, y'' + xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

- ☐ Check to see if $x_0 = 1$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$
  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$
  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$
- Multiply by denominators

  $y''(x-1) - xy' + y = 0$
- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1))\, u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 12

```
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 17

```
DSolve[(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.425 problem 435

2.425.1 Maple step by step solution 3975

Internal problem ID [7915]
Internal file name [OUTPUT/6848_Sunday_June_05_2022_05_13_41_PM_58024156/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 435.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2y'' + xy' + 3y = 0}$$

Writing the ode as

$$2y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 20}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 20 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{16} - \frac{5}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 803: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{2x} - \frac{25}{2x^3} - \frac{125}{x^5} - \frac{3125}{2x^7} - \frac{21875}{x^9} - \frac{328125}{x^{11}} - \frac{5156250}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 20}{16} \\
&= Q + \frac{R}{16} \\
&= \left(\frac{x^2}{16} - \frac{5}{4}\right) + (0) \\
&= \frac{x^2}{16} - \frac{5}{4}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{4}\right) - (0) \\
&= -\frac{5}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{4}}{\frac{1}{4}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{16} - \frac{5}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{4}\right) \\ &= -\frac{x}{4} \\ &= -\frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{4}\right)(2x + a_1) + \left(\left(-\frac{1}{4}\right) + \left(-\frac{x}{4}\right)^2 - \left(\frac{x^2}{16} - \frac{5}{4}\right)\right) &= 0 \\ 2 + \frac{a_1 x}{2} + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{\int -\frac{x}{4} dx} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}} \\
&= \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{2} \, dx} \\
&= z_1 e^{-\frac{x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{4}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{4}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2 - 2\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{4}}\right) + c_2 \left(\left(x^2 - 2\right) \mathrm{e}^{-\frac{x^2}{4}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{\left(x^2 - 2\right)^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}} + c_2(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}} + c_2(x^2-2)\,\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2-2)^2}dx\right)$$

Verified OK.

2.425.1 Maple step by step solution

Let's solve

$2y'' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{2} - \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{2} + \frac{3y}{2} = 0$

- Multiply by denominators

  $2y'' + xy' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\,x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(2a_{k+2}(k+2)(k+1) + a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(2k^2 + 6k + 4)\,a_{k+2} + a_k(k+3) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+3)}{2(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(2*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^2}{4}} (x^2 - 2) + c_2 \mathrm{e}^{-\frac{x^2}{4}} (x^2 - 2) \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{(x^2 - 2)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 61

```
DSolve[2*y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8} e^{-\frac{x^2}{4}} \left(\sqrt{\pi} c_2 (x^2 - 2) \operatorname{erfi}\left(\frac{x}{2}\right) + 8c_1 (x^2 - 2) - 2c_2 e^{\frac{x^2}{4}} x \right)$$

2.426 problem 436

2.426.1 Maple step by step solution 3984

Internal problem ID [7916]
Internal file name [`OUTPUT/6849_Sunday_June_05_2022_05_13_44_PM_64242001/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 436.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 805: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha^{\pm}_{\infty}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha^{-}_{\infty} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha^{-}_{\infty} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_{\infty} \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x)\, \mathrm{e}^{-\frac{x^2}{4}} \\
&= x\, \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\, x}{2}\right) x - 2\, \mathrm{e}^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.426.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x^2}{2}} \left(i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x + 2\,\mathrm{e}^{\frac{x^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.427 problem 437

2.427.1 Maple step by step solution 3993

Internal problem ID [7917]
Internal file name [`OUTPUT/6850_Sunday_June_05_2022_05_13_47_PM_61232784/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 437.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$(1-x)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 807: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2} \right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2} \right)(0) + \left(\left(\frac{1}{2(x-1)^2} \right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2} \right) \right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right) dx} \\
&= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1-x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1} \, e^{\frac{x}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} - c_2 x$$

Verified OK.

2.427.1 Maple step by step solution

Let's solve

$(1-x)\, y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 17

```
DSolve[(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.428 problem 438

2.428.1 Maple step by step solution 4003

Internal problem ID [7918]
Internal file name [`OUTPUT/6851_Sunday_June_05_2022_05_13_50_PM_84380099/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 438.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 809: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int -\frac{x}{2} dx} \\
&= (x) \, e^{-\frac{x^2}{4}} \\
&= x \, e^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, e^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2 \, e^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x\,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.428.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2))x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \, e^{-\frac{x^2}{2}} - \frac{c_2 e^{-\frac{x^2}{2}} \left(i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x + 2\,e^{\frac{x^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.429 problem 439

2.429.1 Maple step by step solution 4012

Internal problem ID [7919]
Internal file name [OUTPUT/6852_Sunday_June_05_2022_05_13_53_PM_20333291/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 439.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(-x^2+4)\, y''+xy'+2y=0}$$

Writing the ode as

$$\left(-x^2+4\right) y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+4 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{11x^2 - 24}{4\left(x^2 - 4\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 11x^2 - 24$$
$$t = 4\left(x^2 - 4\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{11x^2 - 24}{4\left(x^2 - 4\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 811: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-4)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{32\,(x+2)} + \frac{5}{16\,(x+2)^2} + \frac{5}{16\,(x-2)^2} + \frac{17}{32\,(x-2)}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{11x^2 - 24}{4\,(x^2-4)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{11}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
2	2	$\{-1,2,5\}$
-2	2	$\{-1,2,5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = -1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (-1))) \\ &= 2 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(2))} + \frac{-1}{(x-(-2))}\right) \\ &= -\frac{1}{2\,(x-2)} - \frac{1}{2\,(x+2)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 2$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 2$, then letting

$$p = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{11x^2 a_1 + 16(a_0+6)\,x + 36a_1}{\left(x^2-4\right)^2} = 0$$

And solving for p gives

$$p = x^2 - 6$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{2x}{x^2-6} - \frac{1}{2\left(x-2\right)} - \frac{1}{2\left(x+2\right)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2x}{x^2-6} - \frac{1}{2\left(x-2\right)} - \frac{1}{2\left(x+2\right)}\right)w + \frac{-11x^4+74x^2-128}{4x^6-56x^4+256x^2-384} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2\left(x^2-6\right)\left(x-2\right)\left(x+2\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx}\\ &= \mathrm{e}^{\int \frac{2\sqrt{3}\,x^2\sqrt{x^2-4}+x^3-8\sqrt{3}\,\sqrt{x^2-4}-2x}{2\left(x^2-6\right)\left(x-2\right)\left(x+2\right)}dx}\\ &= \frac{\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\left(\sqrt{2}\,\sqrt{3}\,x-4\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{\left(4+\sqrt{2}\,\sqrt{3}\,x\right)\sqrt{2}}{2\sqrt{x^2-4}}\right)}{2}}}{\left(x+2\right)^{\frac{1}{4}}\left(x-2\right)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{-x^2+4}\,dx}\\ &= z_1 e^{\frac{\ln\left(x^2-4\right)}{4}}\\ &= z_1\left(\left(x^2-4\right)^{\frac{1}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}}\mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-x^2+4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x^2-4)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\sqrt{x^2-4}\,\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}}\, \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}}{x^2-6}\, dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x^2-6}\,\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \right) + c_2 \left(\sqrt{x^2-6}\,\left(x \right.\right. \\
&\quad \left. +\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\,\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}}\, \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}}}{x^2-6} \right.
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= c_1 \sqrt{x^2-6}\,\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} + c_2 \sqrt{x^2-6}\,\left(x \right. \\
&\quad \left. +\sqrt{x^2-4}\right)^{\sqrt{3}} \mathrm{e}^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} \left(\int \frac{\sqrt{x^2-4}\,\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}}\, \mathrm{e}^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}(}}{x^2-6} \right.
\end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x^2-6}\left(x+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}} + c_2\sqrt{x^2-6}\left(x\right.$$
$$\left.+\sqrt{x^2-4}\right)^{\sqrt{3}} e^{-\frac{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)}{2}-\frac{\operatorname{arctanh}\left(\frac{4+\sqrt{6}\,x}{\sqrt{2x^2-8}}\right)}{2}}\left(\int \frac{\sqrt{x^2-4}\left(x+\sqrt{x^2-4}\right)^{-2\sqrt{3}} e^{\operatorname{arctanh}\left(\frac{\sqrt{6}\,x-4}{\sqrt{2x^2-8}}\right)+\operatorname{arctanh}\left(\right.}}{x^2-6}\right.$$

Verified OK.

2.429.1 Maple step by step solution

Let's solve

$(-x^2+4)\,y''+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{xy'}{x^2-4}+\frac{2y}{x^2-4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{xy'}{x^2-4}-\frac{2y}{x^2-4}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x}{x^2-4}, P_3(x)=-\frac{2}{x^2-4}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-2$

- Multiply by denominators

  $y''(x^2-4)-xy'-2y=0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-u + 2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)(k + r - 1)\, u^{k+r-2+m}$

  ◦ Shift index using $k- > k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)(k + 1 - m + r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(-3 + 2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k + 1 + r)(2k - 1 + 2r) + a_k(k^2 + 2kr + r^2 - 2k - 2r -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-4\left(k - \frac{1}{2} + r\right)(k + 1 + r)\, a_{k+1} + \left(k^2 + (2r - 2)\, k + r^2 - 2r - 2\right) a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k^2+2kr+r^2-2k-2r-2)a_k}{2(2k-1+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+k-\frac{11}{4})a_k}{2(2k+2)(k+\frac{5}{2})}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-2k-2)a_k}{2(2k-1)(k+1)}, b_{k+1} = \frac{(k^2+k-\frac{11}{4})b_k}{2(2k+2)(k+\frac{5}{2})}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
dsolve((4-x^2)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\sqrt{x^2-6}\,\sin\left(\int\frac{\sqrt{-x^2+4}\,\sqrt{3}}{x^2-6}dx\right)+c_2\sqrt{x^2-6}\,\cos\left(\int\frac{\sqrt{-x^2+4}\,\sqrt{3}}{x^2-6}dx\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 58

```
DSolve[(4-x^2)*y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\left(x^2-4\right)^{3/4}\left(c_1P_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)+c_2Q_{-\frac{1}{2}+\sqrt{3}}^{\frac{3}{2}}\left(\frac{x}{2}\right)\right)$$

2.430 problem 440

2.430.1 Maple step by step solution 4019

Internal problem ID [7920]
Internal file name [OUTPUT/6853_Sunday_June_05_2022_05_13_56_PM_34274139/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 440.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(-16x^2+3\right)y = 0}$$

Writing the ode as

$$\begin{align*} 4x^2y'' - 4xy' + \left(-16x^2+3\right)y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 4x^2\\ B &= -4x \tag{3}\\ C &= -16x^2+3 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 813: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4}
$$

Verified OK.

2.430.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (-16x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k - \frac{1}{2} + r\right)\left(k + r - \frac{3}{2}\right) a_k - 16a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $4\left(k + \frac{3}{2} + r\right)\left(k + \frac{1}{2} + r\right) a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+3+2r)(2k+1+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+6)(2k+4)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+4)(2k+2)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+6)(2k+4)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh(2x) + c_2\sqrt{x}\,\cosh(2x)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.431 problem 441

2.431.1 Maple step by step solution 4029

Internal problem ID [7921]
Internal file name [`OUTPUT/6854_Sunday_June_05_2022_05_13_59_PM_42277662/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 441.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x-1\\B&=-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 815: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2(x-1)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^x - c_2 x$$

Verified OK.

2.431.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right) \bigg|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right) \bigg|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$

- Shift index using $k->k +1$

  $u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-1}^{\infty} a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $a_{0} r \left(-2+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(a_{k +1} \left(k +1+r \right) \left(k +r -1\right)-a_{k} \left(k +r -1\right)\right) u^{k +r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r \left(-2+r \right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(k +r -1\right) \left(a_{k +1} \left(k +1+r \right)-a_{k}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k +1} = \frac{a_{k}}{k +1+r}$

- Recursion relation for $r = 0$

  $a_{k +1} = \frac{a_{k}}{k +1}$

- Solution for $r = 0$

  $\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k}, a_{k +1} = \frac{a_{k}}{k +1}\right]$

- Revert the change of variables $u = x -1$

  $\left[y = \sum_{k =0}^{\infty} a_{k} \left(x -1\right)^{k}, a_{k +1} = \frac{a_{k}}{k +1}\right]$

- Recursion relation for $r = 2$

  $a_{k +1} = \frac{a_{k}}{k +3}$

- Solution for $r = 2$

  $\left[y \left(u \right) = \sum_{k =0}^{\infty} a_{k} u^{k +2}, a_{k +1} = \frac{a_{k}}{k +3}\right]$

- Revert the change of variables $u = x -1$

  $\left[y = \sum_{k =0}^{\infty} a_{k} \left(x -1\right)^{k +2}, a_{k +1} = \frac{a_{k}}{k +3}\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.432 problem 442

2.432.1 Maple step by step solution 4036

Internal problem ID [7922]
Internal file name [`OUTPUT/6855_Sunday_June_05_2022_05_14_01_PM_70520777/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 442.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + \left(x^2+2\right)y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 817: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(x \cos(x)) + c_2(x \cos(x) (\tan(x)))$$

Summary

The solution(s) found are the following

$$y = c_1 x \cos(x) + c_2 x \sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x \cos(x) + c_2 x \sin(x)$$

Verified OK.

2.432.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (x^2 + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2 + 2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r) (-2 + r) x^r + a_1 r(-1 + r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (k + r - 1) (k + r - 2) + a_{k-2}) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + r) (-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.433 problem 444

2.433.1 Maple step by step solution 4047

Internal problem ID [7923]
Internal file name [`OUTPUT/6856_Sunday_June_05_2022_05_14_04_PM_67017557/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 444.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0}$$

Writing the ode as

$$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x \\ B &= -x^2+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4(x^2 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 819: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{4x} + \frac{3}{4(x-2)^2} + \frac{3}{4x^2} - \frac{1}{4(x-2)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2} \\
&= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\left(x-2\right)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\left(x-2\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln(x-2)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x-2)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\mathrm{e}^{-x} x^2\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\mathrm{e}^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\left(-\mathrm{e}^{-x} x^2\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

Verified OK.

2.433.1 Maple step by step solution

Let's solve

$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x^2-2)y'}{x(x-2)}-\frac{2(x-1)y}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-2)y'}{x(x-2)}+\frac{2(x-1)y}{x(x-2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)}, P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(x-2)+\left(-x^{2}+2\right) y'+\left(2x-2\right) y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_{k} x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^{m}\cdot y$ to series expansion for $m=0..1$

    $x^{m}\cdot y=\sum_{k=0}^{\infty} a_{k} x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

    $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k} \left(k+r\right) x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m} \left(k+1-m+r\right) x^{k+r}$

  - Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

    $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k} \left(k+r\right) \left(k+r-1\right) x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m} \left(k+2-m+r\right) \left(k+1-m+r\right) x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0} r(-2+r) x^{-1+r}+\left(-2a_{1} (1+r) (-1+r)+a_{0} (1+r) (-2+r)\right) x^{r}+\left(\sum_{k=1}^{\infty} \left(-2a_{k+1} (k+r+\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in \{0, 2\}$

- Each term must be 0

  $-2a_{1} (1+r) (-1+r)+a_{0} (1+r) (-2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1}=$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)(k+r-1)-2(k+1)^2a_{k+2}+(-4ra_{k+2}-a_k)(k+1)-2r^2a_{k+2}-ra_k+3a_k+$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Recursion relation for $r=2$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}, -6a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(2*x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^2 + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 18

```
DSolve[(x^2-2*x)*y''[x]+(2-x^2)*y'[x]+(2*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_2x^2 + c_1e^x$$

2.434 problem 445

2.434.1 Maple step by step solution 4057

Internal problem ID [7924]
Internal file name [`OUTPUT/6857_Sunday_June_05_2022_05_14_07_PM_93240527/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 445.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x+1)\,y''-2y'-(3+2x)\,y=0}$$

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x+1 \\ B &= -2 \\ C &= -2x-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (2x + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 821: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= 1 \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2+8x+6}{4x^2+4x+1} \\
&= Q + \frac{R}{4x^2+4x+1} \\
&= (1) + \left(\frac{4x+5}{4x^2+4x+1}\right) \\
&= 1 + \frac{4x+5}{4x^2+4x+1}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$
\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{1}{1} - 0\right) = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{1}{1} - 0\right) = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(1+x)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{2x+1} \, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1 \left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(x\,\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(x\,\mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x
$$

Verified OK.

2.434.1 Maple step by step solution

Let's solve

$(2x+1)\,y'' - 2y' + (-2x-3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3+2x)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{2x+1} - \frac{(3+2x)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  - Define functions

$\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{3+2x}{2x+1}\right]$

- ○ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$

- ○ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

- ○ $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- • Multiply by denominators

  $(2x+1)\, y'' - 2y' + (-2x-3)\, y = 0$

- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\, y(u) = 0$

- • Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - ○ Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - ○ Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$

  - ○ Shift index using $k- >k + 1$

    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, u^{k+r}$

  - ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- > k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r} + (2a_1(1+r)(-1+r) - 2a_0)\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)(k+r-1) - 2a_k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$2a_1(1+r)(-1+r) - 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$$

- Shift index using $k- > k+1$

$$2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2*x+1)*diff(y(x),x$2)-2*diff(y(x),x)-(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x)=e^{-x}c_1+e^{x}c_2x$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 29

```
DSolve[(2*x+1)*y''[x]-2*y'[x]-(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-x-\frac{1}{2}}\left(c_2e^{2x+1}x+c_1\right)$$

2.435 problem 446

2.435.1 Maple step by step solution 4064

Internal problem ID [7925]
Internal file name [OUTPUT/6858_Sunday_June_05_2022_05_14_10_PM_2096875/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 446.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y''+\left(-8x^2+4x\right)y'+\left(4x^2-4x-1\right)y=0}$$

Writing the ode as

$$4x^2y''+\left(-8x^2+4x\right)y'+\left(4x^2-4x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^2\\B&=-8x^2+4x\\C&=4x^2-4x-1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 823: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx}\\ &= z_1 e^{x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

Verified OK.

2.435.1 Maple step by step solution

Let's solve

$4x^2 y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' - 4x(2x-1)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-1+2r)\, x^r + \left(a_1(3+2r)\,(1+2r) - 4a_0(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r)-4a_0(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=\frac{4a_0}{3+2r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+(-8k-8r+4)a_{k-1}+4a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+(-8k-12-8r)a_{k+1}+4a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$

- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}},a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8},a_1=2a_0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24},a_1=a_0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8},a_1=2a_0,b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+(4*x-8*x^2)*diff(y(x),x)+(4*x^2-4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{\sqrt{x}} + c_2\sqrt{x}\, e^x$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{\sqrt{x}}$$

2.436 problem 447

2.436.1 Maple step by step solution 4071

Internal problem ID [7926]
Internal file name [OUTPUT/6859_Sunday_June_05_2022_05_14_12_PM_81311797/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 447.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 825: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2}$$

Verified OK.

2.436.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

2.437 problem 448

2.437.1 Maple step by step solution 4077

Internal problem ID [7927]
Internal file name [OUTPUT/6860_Sunday_June_05_2022_05_14_14_PM_46908542/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 448.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x(x-1)\,y' + (x^2-2x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + (2x^2-2x)\,y' + (x^2-2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 - 2x \\ C &= x^2 - 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 827: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2-2x}{x^2}\,dx} \\ &= z_1 e^{-x+\ln(x)} \\ &= z_1 \left(x\,\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x+2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} c_2 x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} c_2 x^2
$$

Verified OK.

2.437.1 Maple step by step solution

Let's solve

$x^2 y'' + (2x^2 - 2x)\, y' + (x^2 - 2x + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-2x+2)y}{x^2} - \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x-1)y'}{x} + \frac{(x^2-2x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(x-1)}{x}, P_3(x) = \frac{x^2-2x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 2x(x-1)\, y' + (x^2 - 2x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + (a_1 r(-1+r) + 2a_0(-1+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2) + \right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r)+2a_0(-1+r)=0$

- Solve for the dependent coefficient(s)

$a_1=-\frac{2a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r-1)(k+r-2)+2a_{k-1}k+2a_{k-1}r+a_{k-2}-4a_{k-1}=0$

- Shift index using $k->k+2$

$a_{k+2}(k+1+r)(k+r)+2a_{k+1}(k+2)+2a_{k+1}r+a_k-4a_{k+1}=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{2ka_{k+1}+2a_{k+1}r+a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

$a_{k+2}=-\frac{2ka_{k+1}+a_k+2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r=1$

$\left[y=\sum_{k=0}^{\infty}a_k x^{k+1}, a_{k+2}=-\frac{2ka_{k+1}+a_k+2a_{k+1}}{(k+2)(k+1)}, a_1=-2a_0\right]$

- Recursion relation for $r=2$

$a_{k+2}=-\frac{2ka_{k+1}+a_k+4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r=2$

$\left[y=\sum_{k=0}^{\infty}a_k x^{k+2}, a_{k+2}=-\frac{2ka_{k+1}+a_k+4a_{k+1}}{(k+3)(k+2)}, a_1=-a_0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+2}\right), a_{k+2}=-\frac{2ka_{k+1}+a_k+2a_{k+1}}{(k+2)(k+1)}, a_1=-2a_0, b_{k+2}=-\frac{2kb_{k+1}+b_k+4b_{k+1}}{(k+3)(k+2)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)+2*x*(x-1)*diff(y(x),x)+(x^2-2*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x + e^{-x} c_2 x^2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]+2*x*(x-1)*y'[x]+(x^2-2*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-x} x(c_2 x + c_1)$$

2.438 problem 449

2.438.1 Maple step by step solution 4086

Internal problem ID [7928]
Internal file name [OUTPUT/6861_Sunday_June_05_2022_05_14_16_PM_94543745/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 449.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(2x-1)\,y' + \left(x^2-x-1\right)y = 0}$$

Writing the ode as

$$x^2y'' + \left(-2x^2+x\right)y' + \left(x^2-x-1\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+x \\ C &= x^2-x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 829: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2+x}{x^2} \, dx} \\
&= z_1 e^{x - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{2x - \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 e^x}{x} + \frac{e^x c_2 x}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 e^x}{x} + \frac{e^x c_2 x}{2}
$$

Verified OK.

2.438.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 + x) y' + (x^2 - x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-x-1)y}{x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x^2-x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x^2-x-1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(2x-1)\,y' + (x^2 - x - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\,x^r + (a_1(2+r)\,r - a_0(1+2r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)\,(k+r-1) - a_{k-1}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term must be 0

  $a_1(2+r)\,r - a_0(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{r(2+r)}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + (1-2k-2r)\,a_{k-1} + a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r+1) + (-3-2k-2r)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1}}{(k+3+r)(k+r+1)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, a_1 = a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-x*(2*x-1)*diff(y(x),x)+(x^2-x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{x} + e^x c_2 x$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]-x*(2*x-1)*y'[x]+(x^2-x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right)$$

2.439 problem 450

2.439.1 Maple step by step solution 4097

Internal problem ID [7929]
Internal file name [`OUTPUT/6862_Sunday_June_05_2022_05_14_19_PM_4545229/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 450.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-2x)\,y''+2y'+(2x-3)\,y=0}$$

Writing the ode as

$$(1-2x)\,y''+2y'+(2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-2x\\B&=2\\C&=2x-3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 8x + 6}{(2x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 8x + 6 \\ t &= (2x - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 8x + 6}{(2x - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 831: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x-1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^3} + \frac{11}{32x^4} + \frac{21}{64x^5} + \frac{15}{64x^6} + \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 8x + 6}{4x^2 - 4x + 1} \\ &= Q + \frac{R}{4x^2 - 4x + 1} \\ &= (1) + \left(\frac{-4x + 5}{4x^2 - 4x + 1}\right) \\ &= 1 + \frac{-4x + 5}{4x^2 - 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 8x + 6}{(2x - 1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x - \frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(x - \frac{1}{2}\right)} + 1 \\
&= \frac{2x - 2}{2x - 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4x^2 - 8x + 6}{(2x - 1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{2}\right)} + 1\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{2x - 1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1-2x} \, dx} \\
&= z_1 e^{\frac{\ln(1-2x)}{2}} \\
&= z_1\left(\sqrt{1 - 2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{x} \sqrt{1-2x}}{\sqrt{2x-1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1-2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(1-2x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-x\,\mathrm{e}^{-2x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{x} \sqrt{1-2x}}{\sqrt{2x-1}}\right) + c_2 \left(\frac{\mathrm{e}^{x} \sqrt{1-2x}}{\sqrt{2x-1}} \left(-x\,\mathrm{e}^{-2x}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{x} \sqrt{1-2x}}{\sqrt{2x-1}} - \frac{c_2 x\,\mathrm{e}^{-x} \sqrt{1-2x}}{\sqrt{2x-1}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{x} \sqrt{1-2x}}{\sqrt{2x-1}} - \frac{c_2 x\,\mathrm{e}^{-x} \sqrt{1-2x}}{\sqrt{2x-1}}$$

Verified OK.

2.439.1 Maple step by step solution

Let's solve

$(1-2x)\,y''+2y'+(2x-3)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(2x-3)y}{2x-1}+\frac{2y'}{2x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{2x-1}-\frac{(2x-3)y}{2x-1}=0$

□ Check to see if $x_0=\frac{1}{2}$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2}{2x-1}, P_3(x)=-\frac{2x-3}{2x-1}\right]$

  ◦ $\left(x-\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{2}$

  $\left(\left(x-\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}}=-1$

  ◦ $\left(x-\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}$

  $\left(\left(x-\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}}=0$

  ◦ $x=\frac{1}{2}$is a regular singular point

  Check to see if $x_0=\frac{1}{2}$ is a regular singular point

  $x_0=\frac{1}{2}$

- Multiply by denominators

  $y''(2x-1)-2y'+(-2x+3)\,y=0$

- Change variables using $x=u+\frac{1}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d^2}{du^2}y(u)\right)-2\frac{d}{du}y(u)+(-2u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k- >k + 1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$

- Shift index using $k- >k + 1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-2+r)\, u^{-1+r} + (2a_1(1+r)\,(-1+r) + 2a_0)\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)\,(k+r-1) + 2a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $2a_1(1+r)\,(-1+r) + 2a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)\,(k+r-1) + 2a_k - 2a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $2a_{k+2}(k+2+r)\,(k+r) + 2a_{k+1} - 2a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{-a_{k+1}+a_k}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{-a_{k+1}+a_k}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{-a_{k+1}+a_k}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{-a_{k+1}+a_k}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{-a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{-a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve((1-2*x)*diff(y(x),x$2)+2*diff(y(x),x)+(2*x-3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + e^{-x} c_2 x$$

✓ Solution by Mathematica

Time used: 0.169 (sec). Leaf size: 48

```
DSolve[(1-2*x)*y''[x]+2*y'[x]+(2*x-3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}\sqrt{1-2x}(c_1 e^{2x} - e c_2 x)}{\sqrt{2x-1}}$$

2.440 problem 451

2.440.1 Maple step by step solution 4106

Internal problem ID [7930]
Internal file name [`OUTPUT/6863_Sunday_June_05_2022_05_14_24_PM_26817836/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 451.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2xy'' + (1+4x)\,y' + (2x+1)\,y = 0}$$

Writing the ode as

$$2xy'' + (1+4x)\,y' + (2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= 1+4x \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 833: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} + (-)(0) \\
&= \frac{1}{4x} \\
&= \frac{1}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{4x} dx} \\ &= x^{\frac{1}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1+4x}{2x}\, dx} \\ &= z_1 e^{-x-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1+4x}{2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x-\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(2\sqrt{x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(2\sqrt{x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + 2c_2\sqrt{x}\,\mathrm{e}^{-x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + 2c_2\sqrt{x}\,\mathrm{e}^{-x}
$$

Verified OK.

2.440.1 Maple step by step solution

Let's solve

$2xy'' + (1+4x)\,y' + (2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+4x)y'}{2x} - \frac{(2x+1)y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+4x)y'}{2x} + \frac{(2x+1)y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+4x}{2x}, P_3(x) = \frac{2x+1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2xy'' + (1+4x)\,y' + (2x+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-1+2r)\, x^{-1+r} + (a_1(1+r)\,(1+2r) + a_0(1+4r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(2k+1+2r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(1+r)(1+2r)+a_0(1+4r)=0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{1}{2}+r\right)(k+1+r)a_{k+1}+4a_k k+4a_k r+a_k+2a_{k-1}=0$

- Shift index using $k->k+1$

  $2\left(k+\frac{3}{2}+r\right)(k+2+r)a_{k+2}+4a_{k+1}(k+1)+4ra_{k+1}+a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4ka_{k+1}+4ra_{k+1}+2a_k+5a_{k+1}}{(2k+3+2r)(k+2+r)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}, a_1+a_0=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4ka_{k+1}+2a_k+7a_{k+1}}{(2k+4)\left(k+\frac{5}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4ka_{k+1}+2a_k+7a_{k+1}}{(2k+4)\left(k+\frac{5}{2}\right)}, 3a_1+3a_0=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4ka_{k+1}+2a_k+5a_{k+1}}{(2k+3)(k+2)}, a_1+a_0=0, b_{k+2}=-\frac{4kb_{k+1}+2b_k+7b_k}{(2k+4)\left(k+\frac{5}{2}\right)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(2*x*diff(y(x),x$2)+(4*x+1)*diff(y(x),x)+(2*x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x} c_1 + c_2 \sqrt{x}\, e^{-x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 23

```
DSolve[2*x*y''[x]+(4*x+1)*y'[x]+(2*x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(2 c_2 \sqrt{x} + c_1\right)$$

2.441 problem 452

2.441.1 Maple step by step solution 4115

Internal problem ID [7931]
Internal file name [`OUTPUT/6864_Sunday_June_05_2022_05_14_27_PM_98306537/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 452.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (2x+1)\, y' + (1+x)\, y = 0}$$

Writing the ode as

$$xy'' + (-2x-1)\, y' + (1+x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-1 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 835: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{x} \, dx} \\ &= z_1 e^{x+\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\, e^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2} \, dx \\ &= y_1\left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^2}{2}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\mathrm{e}^x + \frac{x^2\mathrm{e}^x c_2}{2} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1\mathrm{e}^x + \frac{x^2\mathrm{e}^x c_2}{2}$$

Verified OK.

2.441.1 Maple step by step solution

Let's solve

$xy'' + (-2x - 1)\,y' + (1 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+x)y}{x} + \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{x} + \frac{(1+x)y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{1+x}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x - 1)\, y' + (1 + x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2 + r)\, x^{-1+r} + \left(a_1(1 + r)\,(-1 + r) - a_0(-1 + 2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k + r - 1)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $a_1(1+r)(-1+r) - a_0(-1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + a_k(-2k-2r+1) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(-2k-1-2r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+a_{k+1}}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, 3a_1 - 3a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-(2*x+1)*diff(y(x),x)+(x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^2$$

✓ Solution by Mathematica
Time used: 0.023 (sec). Leaf size: 23

```
DSolve[x*y''[x]-(2*x+1)*y'[x]+(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^x \left(c_2 x^2 + 2c_1\right)$$

2.442 problem 453

2.442.1 Maple step by step solution 4122

Internal problem ID [7932]
Internal file name [`OUTPUT/6865_Sunday_June_05_2022_05_14_29_PM_41962991/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 453.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4x(1+x)\,y' + (3+2x)\,y = 0}$$

Writing the ode as

$$4x^2y'' + \left(-4x^2 - 4x\right)y' + (3+2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 - 4x \\ C &= 3 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 837: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\,(\mathrm{e}^x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^x \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^x
$$

Verified OK.

2.442.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 - 4x)\,y' + (3 + 2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y'}{x} - \frac{(3+2x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{(3+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{3+2x}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2 y'' - 4x(1+x)\,y' + (3+2x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right)\left(k+r-\frac{3}{2}\right) = 0$

- Shift index using $k- >k+1$

  $4\left(\left(k+\frac{1}{2}+r\right) a_{k+1}-a_{k}\right) \left(k+r-\frac{1}{2}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2 a_{k}}{2 k+1+2 r}$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2 a_{k}}{2 k+2}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+1}=\frac{2 a_{k}}{2 k+2}\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2 a_{k}}{2 k+4}$
- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{3}{2}}, a_{k+1}=\frac{2 a_{k}}{2 k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_{k} x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{2 a_{k}}{2 k+2}, b_{k+1}=\frac{2 b_{k}}{2 k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[4*x^2*y''[x]-4*x*(x+1)*y'[x]+(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2 e^x + c_1)$$

2.443 problem 454

2.443.1 Maple step by step solution 4129

Internal problem ID [7933]
Internal file name [`OUTPUT/6866_Sunday_June_05_2022_05_14_31_PM_30807990/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 454.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-2x+2)\, y' + (x-2)\, y = 0}$$

Writing the ode as

$$xy'' + (-2x+2)\, y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x+2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 839: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x+2}{x}\,dx} \\
&= z_1 e^{x-\ln(x)} \\
&= z_1 \left(\frac{e^x}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^x}{x} + c_2 e^x \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^x}{x} + c_2 e^x
$$

Verified OK.

2.443.1 Maple step by step solution

Let's solve

$x y'' + (-2x + 2) y' + (x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x+2)\, y' + (x-2)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + (a_1(1+r)\,(2+r) - 2a_0(1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+2+r) - 2a_k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)-2a_0(1+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+2+r)-2a_k k-2a_k r-2a_k+a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)-2a_{k+1}(k+1)-2r a_{k+1}-2a_{k+1}+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{2k a_{k+1}+2r a_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=\frac{2k a_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{2k a_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=\frac{2k a_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=\frac{2k a_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1-2a_0=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=\frac{2k a_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{2k b_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1-2b\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{x} + e^x c_2$$

✓ Solution by Mathematica
Time used: 0.025 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{x}$$

2.444 problem 455

2.444.1 Maple step by step solution 4136

Internal problem ID [7934]
Internal file name [`OUTPUT/6867_Sunday_June_05_2022_05_14_34_PM_58635633/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 455.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 2xy' + 2y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 841: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1(x) + c_2(x(x))\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2x^2 + c_1x \tag{1}$$

Verification of solutions

$$y = c_2x^2 + c_1x$$

Verified OK.

2.444.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{x} - \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 2xy' + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t) + 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^2 + c_1 x$

- Simplify

  $y = x(c_2 x + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_2 x^2 + c_1 x$$

✓ Solution by Mathematica
Time used: 0.011 (sec). Leaf size: 14

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x(c_2 x + c_1)$$

2.445 problem 456

2.445.1 Maple step by step solution 4144

Internal problem ID [7935]
Internal file name [`OUTPUT/6868_Sunday_June_05_2022_05_14_36_PM_80690399/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 456.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (2x+2)\, y' + (x+2)\, y = 0}$$

Writing the ode as

$$xy'' + (-2x-2)\, y' + (x+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 843: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3}
$$

Verified OK.

2.445.1 Maple step by step solution

Let's solve

$xy'' + (-2x - 2)\,y' + (x + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$xy'' + (-2x - 2)\, y' + (x + 2)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3 + r)\, x^{-1+r} + \left(a_1(1 + r)\,(-2 + r) - 2a_0(-1 + r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + 1 + r)\,(k - 2 + r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{0, 3\}$

- Each term must be 0

$a_1(1+r)(-2+r)-2a_0(-1+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k-2+r)-2a_k k-2a_k r+2a_k+a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r=0$

$a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

$a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r=3$

$a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-(2*x+2)*diff(y(x),x)+(x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^3$$

✓ Solution by Mathematica
Time used: 0.026 (sec). Leaf size: 23

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^x\left(c_2x^3 + 3c_1\right)$$

2.446 problem 457

2.446.1 Maple step by step solution 4151

Internal problem ID [7936]
Internal file name [OUTPUT/6869_Sunday_June_05_2022_05_14_39_PM_78667741/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 457.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2 + 2)\,y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2 + 2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 845: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x\cos(x) + c_2 x\sin(x)$$

Verified OK.

2.446.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r)\,(-2 + r)\, x^r + a_1 r(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (k + r - 1)\,(k + r - 2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.447 problem 458

2.447.1 Maple step by step solution 4160

Internal problem ID [7937]
Internal file name [OUTPUT/6870_Sunday_June_05_2022_05_14_41_PM_13621047/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 458.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (1+4x)\,y' + (4x+2)\,y = 0}$$

Writing the ode as

$$xy'' + (-1-4x)\,y' + (4x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1-4x \\ C &= 4x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 847: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1-4x}{x} \, dx} \\ &= z_1 e^{2x + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, \mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-4x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4x + \ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{2x}) + c_2\left(\mathrm{e}^{2x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2}
$$

Verified OK.

2.447.1 Maple step by step solution

Let's solve

$xy'' + (-1-4x)\, y' + (4x+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(2x+1)y}{x} + \frac{(1+4x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+4x)y'}{x} + \frac{2(2x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1+4x}{x}, P_3(x) = \frac{2(2x+1)}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-1 - 4x)\, y' + (4x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$
  - Shift index using $k- > k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2 + r)\, x^{-1+r} + (a_1 (1 + r)\, (-1 + r) - 2a_0(-1 + 2r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k + 1 + r)\, (k + r - 1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term must be 0

  $a_1(1+r)(-1+r) - 2a_0(-1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x$2)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 x^2$$

✓ Solution by Mathematica
Time used: 0.03 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(4*x+1)*y'[x]+(4*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{2x} (c_2 x^2 + 2c_1)$$

2.448 problem 460

2.448.1 Maple step by step solution 4167

Internal problem ID [7938]
Internal file name [`OUTPUT/6871_Sunday_June_05_2022_05_14_44_PM_66674707/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 460.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(-16x^2+3\right) y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2+3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 849: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}\,\mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2\sqrt{x}\,\mathrm{e}^{2x}}{4}
$$

Verified OK.

2.448.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (-16x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = \frac{3}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-16x^2 + 3)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k - 16a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh(2x) + c_2\sqrt{x}\,\cosh(2x)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.449 problem 461

2.449.1 Maple step by step solution 4178

Internal problem ID [7939]
Internal file name [`OUTPUT/6872_Sunday_June_05_2022_05_14_46_PM_13547258/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 461.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x+1)\, xy'' - 2\left(2x^2 - 1\right) y' - 4(1+x)\, y = 0}$$

Writing the ode as

$$\left(2x^2 + x\right) y'' + \left(-4x^2 + 2\right) y' + (-4x - 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 + x \\ B &= -4x^2 + 2 \\ C &= -4x - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 + 8x + 6$$
$$t = (2x + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 851: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 8x + 6}{(2x+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)\,(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(1+x)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{e^{-x}}{\sqrt{2x+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2+2}{2x^2+x}\, dx} \\
&= z_1 e^{x - \ln(x) + \frac{\ln(2x+1)}{2}} \\
&= z_1 \left(\frac{\sqrt{2x+1}\, e^x}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+2}{2x^2+x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x-2\ln(x)+\ln(2x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(x\, \mathrm{e}^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(x\, \mathrm{e}^{2x}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 \mathrm{e}^{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 \mathrm{e}^{2x}$$

Verified OK.

2.449.1 Maple step by step solution

Let's solve

$(2x^2+x)\,y''+(-4x^2+2)\,y'+(-4x-4)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4(1+x)y}{x(2x+1)}+\frac{2(2x^2-1)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(2x^2-1)y'}{x(2x+1)}-\frac{4(1+x)y}{x(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{2(2x^2-1)}{x(2x+1)},P_3(x)=-\frac{4(1+x)}{x(2x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $(2x+1)\,xy''+(-4x^2+2)\,y'+(-4x-4)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=0..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(1+r)\, x^{-1+r}+\left(a_{1}(1+r)\,(2+r)+2a_{0}(1+r)\,(-2+r)\right) x^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$
- Each term must be 0

  $a_{1}(1+r)\,(2+r)+2a_{0}(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+2+r)+2a_{k}(k+r+1)\,(k+r-2)-4a_{k-1}(k+r)=0$
- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+3+r)+2a_{k+1}(k+2+r)\,(k+r-1)-4a_{k}(k+r+1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2\left(k^{2}a_{k+1}+2kra_{k+1}+r^{2}a_{k+1}-2ka_{k}+ka_{k+1}-2ra_{k}+ra_{k+1}-2a_{k}-2a_{k+1}\right)}{(k+2+r)(k+3+r)}$
- Recursion relation for $r=-1$

$a_{k+2} = -\frac{2(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1})}{(k+1)(k+2)}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1})}{(k+1)(k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{2(k^2 a_{k+1} - 2k a_k + k a_{k+1} - 2a_k - 2a_{k+1})}{(k+2)(k+3)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2(k^2 a_{k+1} - 2k a_k + k a_{k+1} - 2a_k - 2a_{k+1})}{(k+2)(k+3)}, 2a_1 - 4a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{2(k^2 a_{k+1} - 2k a_k - k a_{k+1} - 2a_{k+1})}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{2(k^2 b_{k+1} - 2k}{}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((2*x+1)*x*diff(y(x),x$2)-2*(2*x^2-1)*diff(y(x),x)-4*(x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 28

```
DSolve[(2*x+1)*x*y''[x]-2*(2*x^2-1)*y'[x]-4*(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_2 e^{2x+1} x + c_1}{\sqrt{e} x}$$

2.450 problem 462

2.450.1 Maple step by step solution 4189

Internal problem ID [7940]
Internal file name [`OUTPUT/6873_Sunday_June_05_2022_05_14_50_PM_79139677/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 462.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0}$$

Writing the ode as

$$\left(x^{2}-2x\right)y''+\left(-x^{2}+2\right)y'+\left(2x-2\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}-2x \\ B &= -x^{2}+2 \\ C &= 2x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 8x^3 + 24x^2 - 24x + 12$$
$$t = 4\left(x^2 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 853: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 4 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4} - \frac{3}{4x} + \frac{3}{4(x-2)^2} + \frac{3}{4x^2} - \frac{1}{4(x-2)}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\ &= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\ &= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\,(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln(x-2)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{x} \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x-2)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\mathrm{e}^{-x} x^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{x} \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\mathrm{e}^{x} \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} \left(-\mathrm{e}^{-x} x^2\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

Verified OK.

2.450.1 Maple step by step solution

Let's solve

$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x^2-2)y'}{x(x-2)}-\frac{2(x-1)y}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x^2-2)y'}{x(x-2)}+\frac{2(x-1)y}{x(x-2)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{x^2-2}{x(x-2)}, P_3(x)=\frac{2(x-1)}{x(x-2)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(x-2)+(-x^2+2)\,y'+(2x-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..1$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=0..2$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=1..2$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r))\,x^r+\left(\sum_{k=1}^{\infty}(-2a_{k+1}(k+r+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term must be 0

  $-2a_1(1+r)\,(-1+r)+a_0(1+r)\,(-2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2)-2k^2a_{k+1}+(-4ra_{k+1}-a_{k-1})k-2r^2a_{k+1}-a_{k-1}r+3a_{k-1}+2a_{k+1} =$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)(k+r-1)-2(k+1)^2a_{k+2}+(-4ra_{k+2}-a_k)(k+1)-2r^2a_{k+2}-ra_k+3a_k+$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-ka_k+ka_{k+1}-ra_k+ra_{k+1}+2a_k-2a_{k+1}}{2(k^2+2kr+r^2+2k+2r)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+ka_{k+1}+2a_k-2a_{k+1}}{2(k^2+2k)}$

- Recursion relation for $r=2$

  $a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+2}=\frac{k^2a_{k+1}-ka_k+5ka_{k+1}+4a_{k+1}}{2(k^2+6k+8)}, -6a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve((x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)+(2*x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 18

```
DSolve[(x^2-2*x)*y''[x]+(2-x^2)*y'[x]+(2*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_2 x^2 + c_1 e^x$$

2.451 problem 463

2.451.1 Maple step by step solution 4198

Internal problem ID [7941]
Internal file name [`OUTPUT/6874_Sunday_June_05_2022_05_14_53_PM_12286750/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 463.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (1+4x)\,y' + (4x+2)\,y = 0}$$

Writing the ode as

$$xy'' + (-1-4x)\,y' + (4x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1-4x \\ C &= 4x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 855: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1-4x}{x} \, dx} \\ &= z_1 e^{2x+\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1-4x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4x+\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x} \left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 x^2 \mathrm{e}^{2x}}{2}
$$

Verified OK.

2.451.1 Maple step by step solution

Let's solve

$x y'' + (-1 - 4x) y' + (4x + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(2x+1)y}{x} + \frac{(1+4x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+4x)y'}{x} + \frac{2(2x+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1+4x}{x}, P_3(x) = \frac{2(2x+1)}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-1 - 4x)\, y' + (4x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2 + r)\, x^{-1+r} + (a_1(1 + r)\,(-1 + r) - 2a_0(-1 + 2r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k + 1 + r)\,(k + r - 1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$a_1(1+r)(-1+r) - 2a_0(-1+2r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r-1) + a_k(-4k-4r+2) + 4a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)(k+r) + a_{k+1}(-4k-2-4r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2(2ka_{k+1}+2ra_{k+1}-2a_k+a_{k+1})}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{2(2ka_{k+1}-2a_k+a_{k+1})}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2(2ka_{k+1}-2a_k+5a_{k+1})}{(k+4)(k+2)}, 3a_1 - 6a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x$2)-(4*x+1)*diff(y(x),x)+(4*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{2x} c_2 x^2$$

✓ Solution by Mathematica
Time used: 0.026 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(4*x+1)*y'[x]+(4*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{2x} \left(c_2 x^2 + 2c_1\right)$$

2.452 problem 464

2.452.1 Maple step by step solution 4208

Internal problem ID [7942]
Internal file name [`OUTPUT/6875_Sunday_June_05_2022_05_14_56_PM_91222050/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 464.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0}$$

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1\\ B &= -3x-2\\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2 - 108x + 54 \\ t &= 4(3x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4(3x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 857: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x-1)^2$. There is a pole at $x = \frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4\left(x-\frac{1}{3}\right)^2} - \frac{3}{2\left(x-\frac{1}{3}\right)}$$

For the pole at $x = \frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{1}{9x^3} + \frac{11}{108x^4} + \frac{7}{108x^5} + \frac{5}{162x^6} + \frac{2}{243x^7} - \frac{13}{3888x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\ &= Q + \frac{R}{36x^2 - 24x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\ &= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{-6 + 9x}{6x - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x - 1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x-2}{3x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(3x-1)}{2}} \\
&= z_1 \left(\sqrt{3x - 1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2} \, dx \\ &= y_1 \left(-e^{-3x} x\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} \left(-e^{-3x} x\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} - c_2 x \, e^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} - c_2 x \, e^{-x}$$

Verified OK.

2.452.1 Maple step by step solution

Let's solve

$(3x - 1) y'' + (-3x - 2) y' + (-6x + 8) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2(3x-4)y}{3x-1} + \frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3x+2)y'}{3x-1} - \frac{2(3x-4)y}{3x-1} = 0$

□ Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{3x+2}{3x-1}, P_3(x) = -\frac{2(3x-4)}{3x-1}\right]$

- ○ $\left(x-\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right) \cdot P_2(x)\right)\Big|_{x=\frac{1}{3}} = -1$

- ○ $\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)\right)\Big|_{x=\frac{1}{3}} = 0$

- ○ $x = \frac{1}{3}$is a regular singular point

  Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

  $x_0 = \frac{1}{3}$

- Multiply by denominators

  $(3x-1)\, y'' + (-3x-2)\, y' + (-6x+8)\, y = 0$

- Change variables using $x = u + \frac{1}{3}$ so that the regular singular point is at $u = 0$

  $3u\left(\frac{d^2}{du^2}y(u)\right) + (-3u-3)\left(\frac{d}{du}y(u)\right) + (-6u+6)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(-2+r)\, u^{-1+r} + (3a_1(1+r)\,(-1+r) - 3a_0(-2+r))\, u^{r} + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+1+r)\,(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$3a_1(1+r)\,(-1+r) - 3a_0(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$3a_{k+1}(k+1+r)\,(k+r-1) + a_k(-3k-3r+6) - 6a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$3a_{k+2}(k+2+r)\,(k+r) + a_{k+1}(-3k+3-3r) - 6a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k a_{k+1} + r a_{k+1} + 2a_k - a_{k+1}}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{k a_{k+1} + 2a_k - a_{k+1}}{(k+2)k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((3*x-1)*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-x} c_2 x$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 35

```
DSolve[(3*x-1)*y''[x]-(3*x+2)*y'[x]-(6*x-8)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}(c_1 e^{3x} + 2e c_2 x)}{\sqrt{2}}$$

2.453 problem 465

2.453.1 Maple step by step solution 4215

Internal problem ID [7943]
Internal file name [OUTPUT/6876_Sunday_June_05_2022_05_14_59_PM_8866877/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 465.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+x)^2 y'' - 2(1+x) y' - (x^2+2x-1) y = 0}$$

Writing the ode as

$$(1+x)^2 y'' + (-2x-2) y' + (-x^2-2x+1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (1+x)^2 \\ B &= -2x-2 \\ C &= -x^2-2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 859: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{(1+x)^2}\,dx} \\
&= z_1 e^{\ln(1+x)} \\
&= z_1(1+x)
\end{aligned}$$

Which simplifies to

$$y_1 = (1+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{(1+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\left(1+x\right) \mathrm{e}^{-x}\right)+c_2\left(\left(1+x\right) \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2 x}}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \left(1+x\right) \mathrm{e}^{-x}+\frac{c_2 \left(1+x\right) \mathrm{e}^{x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \left(1+x\right) \mathrm{e}^{-x}+\frac{c_2 \left(1+x\right) \mathrm{e}^{x}}{2}$$

Verified OK.

2.453.1 Maple step by step solution

Let's solve
$\left(1+x\right)^{2} y'' +\left(-2 x -2\right) y' +\left(-x^{2}-2 x +1\right) y =0$

- Highest derivative means the order of the ODE is 2
y''
- Isolate 2nd derivative
$y'' =\frac{\left(x^{2}+2 x -1\right) y}{\left(1+x \right)^{2}}+\frac{2 y'}{1+x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear
$y'' -\frac{2 y'}{1+x}-\frac{\left(x^{2}+2 x -1\right) y}{\left(1+x \right)^{2}}=0$

☐ Check to see if $x_{0}=-1$ is a regular singular point
  - Define functions
  $\left[P_{2}\left(x \right)=-\frac{2}{1+x}, P_{3}\left(x \right)=-\frac{x^{2}+2 x -1}{\left(1+x \right)^{2}}\right]$
  - $\left(1+x \right)\cdot P_{2}\left(x \right)$ is analytic at $x =-1$
  $\left(\left(1+x \right)\cdot P_{2}\left(x \right)\right)\bigg|_{x =-1}=-2$
  - $\left(1+x \right)^{2}\cdot P_{3}\left(x \right)$ is analytic at $x =-1$

$$\left(\left(1+x\right)^{2}\cdot P_{3}(x)\right)\Big|_{x=-1}=2$$

- $x=-1$is a regular singular point

  Check to see if $x_0=-1$ is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $\left(1+x\right)^{2} y''+\left(-2x-2\right) y'+\left(-x^{2}-2x+1\right) y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $u^{2}\left(\frac{d^{2}}{du^{2}}y(u)\right)-2u\left(\frac{d}{du}y(u)\right)+\left(-u^{2}+2\right) y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m}\cdot y(u)$ to series expansion for $m=0..2$

  $u^{m}\cdot y(u)=\sum_{k=0}^{\infty} a_{k} u^{k+r+m}$

- Shift index using $k->k-m$

  $u^{m}\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_{k}\left(k+r\right) u^{k+r}$

- Convert $u^{2}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion

  $u^{2}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}\left(k+r\right)\left(k+r-1\right) u^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}\left(-1+r\right)\left(-2+r\right) u^{r}+a_{1} r\left(-1+r\right) u^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}\left(k+r-1\right)\left(k+r-2\right)-a_{k-2}\right) u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\left(-1+r\right)\left(-2+r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) - a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}(k+1+r)(k+r) - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 =$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x+1)^2*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)-(x^2+2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sinh(x)(x+1) + c_2 \cosh(x)(x+1)$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 147

```
DSolve[(x+1)^2*y''[x]-2*(x+1)*x*y'[x]-(x^2+2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to 2^{\frac{1}{2}i\left(\sqrt{7}+i\right)} e^{-\left(\left(\sqrt{2}-1\right)(x+1)\right)} (x+1)^{\frac{1}{2}i\left(\sqrt{7}+i\right)} \left(c_1 \text{HypergeometricU}\left(\frac{1}{2}\left(1-\sqrt{2}+i\sqrt{7}\right), 1+i\sqrt{7}, 2\sqrt{2}(x+1)\right) + c_2 L_{\frac{1}{2}\left(-1+\sqrt{2}-i\sqrt{7}\right)}^{i\sqrt{7}}\left(2\sqrt{2}(x+1)\right)\right)$$

2.454 problem 466

2.454.1 Maple step by step solution 4222

Internal problem ID [7944]
Internal file name [`OUTPUT/6877_Sunday_June_05_2022_05_15_01_PM_50291176/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 466.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 861: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx}\\ &= z_1 e^{x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{e^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x
$$

Verified OK.

2.454.1 Maple step by step solution

Let's solve

$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_{0} = 0$ is a regular singular point

  $x_{0} = 0$

- Multiply by denominators

  $4x^{2}y'' - 4x(2x-1)\,y' + (4x^{2}-4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k} x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^{m} \cdot y$ to series expansion for $m = 0..2$

    $x^{m} \cdot y = \sum_{k=0}^{\infty} a_{k} x^{k+r+m}$

  - Shift index using $k- > k-m$

    $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^{m} \cdot y'$ to series expansion for $m = 1..2$

    $x^{m} \cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^{2} \cdot y''$ to series expansion

    $x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+2r)\,(-1+2r)\, x^{r} + \left(a_{1}(3+2r)\,(1+2r) - 4a_{0}(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(2k+2r+1)\,(2k -\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r)-4a_0(1+2r)=0$
- Solve for the dependent coefficient(s)

  $a_1=\frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+(-8k-8r+4)a_{k-1}+4a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+(-8k-12-8r)a_{k+1}+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}}, a_{k+2}=\frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1=a_0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right), a_{k+2}=\frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1=2a_0, b_{k+2}=\frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+(4*x-8*x^2)*diff(y(x),x)+(4*x^2-4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{\sqrt{x}}$$

2.455 problem 467

2.455.1 Maple step by step solution 4229

Internal problem ID [7945]
Internal file name [`OUTPUT/6878_Sunday_June_05_2022_05_15_04_PM_16929851/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 467.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 863: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2}(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2}$$

Verified OK.

2.455.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2+7k+12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(4*x-8*x^2)*y'[x]+(4*x^2-4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^x (c_2 x + c_1)}{\sqrt{x}}$$

2.456 problem 468

2.456.1 Maple step by step solution 4238

Internal problem ID [7946]
Internal file name [`OUTPUT/6879_Sunday_June_05_2022_05_15_06_PM_13156584/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 468.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x+1)\,y''-2y'-(3+2x)\,y=0}$$

Writing the ode as

$$(2x+1)\,y''-2y'+(-2x-3)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x+1\\B&=-2\\C&=-2x-3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 8x + 6 \\ t &= (2x + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 865: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 8x + 6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)(1) \\
&= -\frac{1}{2\left(x+\frac{1}{2}\right)} - 1 \\
&= -\frac{2(1+x)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\ &= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{2x+1}\, dx} \\ &= z_1 e^{\frac{\ln(2x+1)}{2}} \\ &= z_1\left(\sqrt{2x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{2x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(x\,\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(x\,\mathrm{e}^{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 x
$$

Verified OK.

2.456.1 Maple step by step solution

Let's solve

$(2x+1)\,y'' - 2y' + (-2x-3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3+2x)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{2x+1} - \frac{(3+2x)y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{3+2x}{2x+1}\right]$

- ○ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$

- ○ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

- ○ $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- • Multiply by denominators

  $(2x+1)\, y'' - 2y' + (-2x-3)\, y = 0$

- • Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\, y(u) = 0$

- • Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - ○ Shift index using $k- >k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - ○ Convert $\frac{d}{du}y(u)$ to series expansion

    $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$

  - ○ Shift index using $k- >k+1$

    $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, u^{k+r}$

  - ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-2+r)\, u^{-1+r} + (2a_1(1+r)(-1+r) - 2a_0)\, u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+r)(k+r-1) - 2a_k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $2a_1(1+r)(-1+r) - 2a_0 = 0$
- Each term in the series must be 0, giving the recursion relation

  $2a_{k+1}(k+1+r)(k+r-1) - 2a_k - 2a_{k-1} = 0$
- Shift index using $k->k+1$

  $2a_{k+2}(k+2+r)(k+r) - 2a_{k+1} - 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+2)k}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2} = \frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1 - 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve((2*x+1)*diff(y(x),x$2)-2*diff(y(x),x)-(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + e^{x}c_2 x$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 29

```
DSolve[(2*x+1)*y''[x]-2*y'[x]-(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x-\frac{1}{2}}\left(c_2 e^{2x+1} x + c_1\right)$$

2.457 problem 469

2.457.1 Maple step by step solution 4247

Internal problem ID [7947]
Internal file name [`OUTPUT/6880_Sunday_June_05_2022_05_15_09_PM_26980054/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 469.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (2x+2)\, y' + (x+2)\, y = 0}$$

Writing the ode as

$$xy'' + (-2x-2)\, y' + (x+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 867: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3}
$$

Verified OK.

2.457.1 Maple step by step solution

Let's solve

$xy'' + (-2x - 2)\,y' + (x + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x - 2)\, y' + (x + 2)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3 + r)\, x^{-1+r} + \left(a_1(1 + r)\, (-2 + r) - 2a_0(-1 + r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k + 1 + r)\, (k - 2 + r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)\,(-2+r)-2a_0(-1+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r)-2a_k k-2a_k r+2a_k+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)\,(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-(2*x+2)*diff(y(x),x)+(x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^3$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 29

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(x+2)*y[x]==6*x^3*Exp[x],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{6} e^x \left(9x^4 + 2c_2 x^3 + 6c_1\right)$$

2.458 problem 470

2.458.1 Maple step by step solution 4254

Internal problem ID [7948]
Internal file name [`OUTPUT/6881_Sunday_June_05_2022_05_15_12_PM_10887431/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 470.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2+2)\,y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 869: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}$$

Verification of solutions

$$y = c_1 x\cos(x) + c_2 x\sin(x)$$

Verified OK.

2.458.1 Maple step by step solution

Let's solve

$x^2y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r)\,(-2 + r)\, x^r + a_1 r(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (k + r - 1)\,(k + r - 2) + a_{k-2})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.459 problem 472

2.459.1 Maple step by step solution 4261

Internal problem ID [7949]
Internal file name [OUTPUT/6882_Sunday_June_05_2022_05_15_14_PM_40233347/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 472.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + \left(-16x^2 + 3\right)y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + \left(-16x^2 + 3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= -16x^2 + 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 871: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\,\mathrm{e}^{-2x}\right) + c_2 \left(\sqrt{x}\,\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2 \sqrt{x}\,\mathrm{e}^{2x}}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-2x} + \frac{c_2 \sqrt{x}\,\mathrm{e}^{2x}}{4}$$

Verified OK.

2.459.1 Maple step by step solution

Let's solve

$4x^2 y'' - 4xy' + (-16x^2 + 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x^2-3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(16x^2-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{16x^2-3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-16x^2 + 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 2r)\,(-3 + 2r)\, x^r + a_1(1 + 2r)\,(-1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r - 1)\,(2k + 2r - 3) - 1\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 2r)\,(-3 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1 + 2r)\,(-1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k - 16a_{k-2} = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{16a_k}{(2k+1+2r)(2k+3+2r)}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = \frac{16a_k}{(2k+4)(2k+6)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = \frac{16a_k}{(2k+2)(2k+4)}, a_1 = 0, b_{k+2} = \frac{16b_k}{(2k+4)(2k+6)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(3-16*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh(2x) + c_2\sqrt{x}\,\cosh(2x)$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 32

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(3-16*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}\sqrt{x}\left(c_2e^{4x} + 4c_1\right)$$

2.460 problem 473

2.460.1 Maple step by step solution 4268

Internal problem ID [7950]
Internal file name [`OUTPUT/6883_Sunday_June_05_2022_05_15_17_PM_15777019/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 473.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4xy' + (4x^2+3)\,y = 0}$$

Writing the ode as

$$4x^2y'' - 4xy' + (4x^2+3)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 4x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 873: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos(x)\sqrt{x}\right) + c_2\left(\cos(x)\sqrt{x}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x)\sqrt{x} + c_2 \sin(x)\sqrt{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos(x)\sqrt{x} + c_2 \sin(x)\sqrt{x}
$$

Verified OK.

2.460.1 Maple step by step solution

Let's solve

$4x^2y'' - 4xy' + (4x^2+3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2+3)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(4x^2+3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{4x^2+3}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{3}{4}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2y''-4xy'+(4x^2+3)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2\cdot y''$ to series expansion

    $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\,x^r+a_1(1+2r)\,(-1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_k(2k+2r-1)\,(2k+2r-3)+4\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-3+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2},\frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,(-1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(k+r-\frac{1}{2}\right)a_k+4a_{k-2}=0$
- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right)a_{k+2}+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(2k+1+2r)(2k+3+2r)}$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+2)(2k+4)}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{(2k+2)(2k+4)}, a_1=0\right]$
- Recursion relation for $r=\frac{3}{2}$

  $a_{k+2}=-\frac{4a_k}{(2k+4)(2k+6)}$
- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k+6)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+2}=-\frac{4a_k}{(2k+2)(2k+4)}, a_1=0, b_{k+2}=-\frac{4b_k}{(2k+4)(2k+6)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sin(x) + c_2\sqrt{x}\,\cos(x)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+(4*x^2+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-ix}\sqrt{x}\left(2c_1 - ic_2e^{2ix}\right)$$

2.461 problem 474

2.461.1 Maple step by step solution 4275

Internal problem ID [7951]
Internal file name [OUTPUT/6884_Sunday_June_05_2022_05_15_19_PM_67325111/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 474.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' - y(x^2 - 2) = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (-x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 875: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\,\mathrm{e}^{-x}\right) + c_2 \left(x\,\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x} c_2 x}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x} c_2 x}{2}
$$

Verified OK.

2.461.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (-x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y(x^2-2)}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} - \frac{y(x^2-2)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -\frac{x^2-2}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (-x^2 + 2)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ○ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ○ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1 + r)\,(-2 + r)\, x^r + a_1 r(-1 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r - 1)\,(k + r - 2) - a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + r)\,(-2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) - a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)(k+r) - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = \frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)-(x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sinh(x) + c_2 x \cosh(x)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-2*x*y'[x]-(x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} x + \frac{1}{2} c_2 e^x x$$

2.462 problem 475

2.462.1 Maple step by step solution 4282

Internal problem ID [7952]
Internal file name [`OUTPUT/6885_Sunday_June_05_2022_05_15_21_PM_60549857/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 475.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2x(1+x)\,y' + (x^2+2x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + (-2x^2-2x)\,y' + (x^2+2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2 - 2x \\ C &= x^2 + 2x + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 877: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-2x}{x^2}\,dx} \\
&= z_1 e^{x+\ln(x)} \\
&= z_1 (x\,\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\,\mathrm{e}^x) + c_2(x\,\mathrm{e}^x(x))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^x c_1 x + x^2 \mathrm{e}^x c_2 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^x c_1 x + x^2 \mathrm{e}^x c_2
$$

Verified OK.

2.462.1 Maple step by step solution

Let's solve

$x^2 y'' + (-2x^2 - 2x)\,y' + (x^2 + 2x + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2x+2)y}{x^2} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x^2+2x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x^2+2x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2x(1+x) y' + (x^2+2x+2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)(-2+r) x^r + (a_1 r(-1+r) - 2a_0(-1+r)) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)(k+r-2) -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

$a_1 r(-1+r) - 2a_0(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0}{r}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2) - 2a_{k-1}k - 2a_{k-1}r + a_{k-2} + 4a_{k-1} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+1+r)(k+r) - 2a_{k+1}(k+2) - 2a_{k+1}r + a_k + 4a_{k+1} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2a_{k+1}r-a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}, a_1 = a_0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}, a_1 = 2a_0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+3)(k+2)}, b_1 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+1)*diff(y(x),x)+(x^2+2*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 x + c_2 e^x x^2$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ix} x (c_1 \operatorname{HypergeometricU}(-i, 0, -2ix) + c_2 L_i^{-1}(-2ix))$$

2.463 problem 476

2.463.1 Maple step by step solution 4289

Internal problem ID [7953]
Internal file name [`OUTPUT/6886_Sunday_June_05_2022_05_15_23_PM_60658752/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 476.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2x(x+2)\,y' + (x^2+4x+6)\,y = 0}$$

Writing the ode as

$$x^2y'' + (-2x^2-4x)\,y' + (x^2+4x+6)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-4x \\ C &= x^2+4x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 879: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-4x}{x^2}\,dx} \\ &= z_1 e^{x+2\ln(x)} \\ &= z_1 \left(x^2 \mathrm{e}^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x+4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x (x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^2 \mathrm{e}^x c_1 + c_2 x^3 \mathrm{e}^x \tag{1}
$$

Verification of solutions

$$
y = x^2 \mathrm{e}^x c_1 + c_2 x^3 \mathrm{e}^x
$$

Verified OK.

2.463.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-2x^2 - 4x\right) y' + \left(x^2 + 4x + 6\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{\left(x^2+4x+6\right) y}{x^2} + \frac{2(x+2) y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x+2) y'}{x} + \frac{\left(x^2+4x+6\right) y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2(x+2)}{x}, P_3(x) = \frac{x^2+4x+6}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -4$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 6$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - 2x(x+2)\,y' + (x^2+4x+6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r + (a_1(-1+r)\,(-2+r) - 2a_0(-2+r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 3\}$

- Each term must be 0

$$a_1(-1+r)(-2+r)-2a_0(-2+r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=\frac{2a_0}{-1+r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)(k+r-3)-2a_{k-1}k-2a_{k-1}r+a_{k-2}+6a_{k-1}=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+r)(k+r-1)-2a_{k+1}(k+2)-2a_{k+1}r+a_k+6a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{2ka_{k+1}+2a_{k+1}r-a_k-2a_{k+1}}{(k+r)(k+r-1)}$$

- Recursion relation for $r=2$

$$a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+2},a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)},a_1=2a_0\right]$$

- Recursion relation for $r=3$

$$a_{k+2}=\frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+3},a_{k+2}=\frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+3)(k+2)},a_1=a_0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k+2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+3}\right),a_{k+2}=\frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+2)(k+1)},a_1=2a_0,b_{k+2}=\frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+3)(k+2)},b_1 =$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-2*x*(x+2)*diff(y(x),x)+(x^2+4*x+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 + c_2 e^x x^3$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]-2*x*(x+2)*y'[x]+(x^2+4*x+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^x x^2 (c_2 x + c_1)$$

2.464 problem 477

2.464.1 Maple step by step solution 4296

Internal problem ID [7954]
Internal file name [OUTPUT/6887_Sunday_June_05_2022_05_15_26_PM_4018711/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 477.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4xy' + \left(x^2 + 6\right) y = 0}$$

Writing the ode as

$$x^2y'' - 4xy' + \left(x^2 + 6\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 881: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1\left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2 \cos(x)\right) + c_2\left(x^2 \cos(x)\left(\tan(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \cos(x) + c_2 \sin(x)\, x^2 \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \cos(x) + c_2 \sin(x)\, x^2$$

Verified OK.

2.464.1 Maple step by step solution

Let's solve

$x^2 y'' - 4xy' + (x^2+6)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+6)y}{x^2} + \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{(x^2+6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-4xy'+(x^2+6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r+a_1(-1+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3)+a_{k-2})\,x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{2,3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)(k+r-3)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+r)(k+r-1)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+r)(k+r-1)}$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 \sin(x) + c_2 \cos(x) x^2$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-4*x*y'[x]+(x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-ix} x^2 \left(2c_1 - ic_2 e^{2ix}\right)$$

2.465 problem 478

2.465.1 Maple step by step solution 4306

Internal problem ID [7955]
Internal file name [`OUTPUT/6888_Sunday_June_05_2022_05_15_28_PM_48994474/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 478.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 - 4x + 6 \\
t &= 4(x-1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 883: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\,(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^x - c_2 x$$

Verified OK.

2.465.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

2.466 problem 479

2.466.1 Maple step by step solution 4313

Internal problem ID [7956]
Internal file name [`OUTPUT/6889_Sunday_June_05_2022_05_15_31_PM_49880107/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 479.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^{2}y''-4x(1+x)\,y'+(3+2x)\,y=0}$$

Writing the ode as

$$4x^{2}y''+\left(-4x^{2}-4x\right)y'+(3+2x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^{2} \\ B &= -4x^{2}-4x \\ C &= 3+2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 885: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\,(\mathrm{e}^{x})\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x}
$$

Verified OK.

2.466.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 - 4x)\, y' + (3 + 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y'}{x} - \frac{(3+2x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{(3+2x)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{3+2x}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2 y'' - 4x(1+x)\, y' + (3+2x)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k + 1 - m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$4\left(\left(k+r-\frac{1}{2}\right) a_k - a_{k-1}\right) \left(k+r-\frac{3}{2}\right) = 0$

- Shift index using $k->k+1$

  $4\left(\left(k+\frac{1}{2}+r\right)a_{k+1}-a_k\right)\left(k+r-\frac{1}{2}\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k}{2k+1+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{2a_k}{2k+2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=\frac{2a_k}{2k+2}\right]$

- Recursion relation for $r=\frac{3}{2}$

  $a_{k+1}=\frac{2a_k}{2k+4}$

- Solution for $r=\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{3}{2}}, a_{k+1}=\frac{2a_k}{2k+4}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{3}{2}}\right), a_{k+1}=\frac{2a_k}{2k+2}, b_{k+1}=\frac{2b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*(x+1)*diff(y(x),x)+(2*x+3)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[4*x^2*y''[x]-4*x*(x+1)*y'[x]+(2*x+3)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2e^x + c_1)$$

2.467 problem 480

2.467.1 Maple step by step solution 4323

Internal problem ID [7957]
Internal file name [OUTPUT/6890_Sunday_June_05_2022_05_15_35_PM_24331395/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 480.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x-1)\,y''-(3x+2)\,y'-(6x-8)\,y=0}$$

Writing the ode as

$$(3x-1)\,y''+(-3x-2)\,y'+(-6x+8)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x-1 \\ B &= -3x-2 \\ C &= -6x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81x^2 - 108x + 54}{4(3x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81x^2 - 108x + 54 \\ t &= 4(3x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{81x^2 - 108x + 54}{4(3x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 887: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t=4(3x-1)^2$. There is a pole at $x=\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L=[1,2]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r=\frac{9}{4}+\frac{3}{4\left(x-\frac{1}{3}\right)^2}-\frac{3}{2\left(x-\frac{1}{3}\right)}$$

For the pole at $x=\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty)=0$ then

$$v=\frac{-O_r(\infty)}{2}=\frac{0}{2}=0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v=x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r}\approx\frac{3}{2}-\frac{1}{2x}+\frac{1}{9x^3}+\frac{11}{108x^4}+\frac{7}{108x^5}+\frac{5}{162x^6}+\frac{2}{243x^7}-\frac{13}{3888x^8}+\dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a=\frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{81x^2 - 108x + 54}{36x^2 - 24x + 4} \\
&= Q + \frac{R}{36x^2 - 24x + 4} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-54x + 45}{36x^2 - 24x + 4}\right) \\
&= \frac{9}{4} + \frac{-54x + 45}{36x^2 - 24x + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -54. Dividing this by leading coefficient in t which is 36 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{81x^2 - 108x + 54}{4\left(3x - 1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{3}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2} \\
&= \frac{-6 + 9x}{6x - 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\left(x - \frac{1}{3}\right)^2}\right) + \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right)^2 - \left(\frac{81x^2 - 108x + 54}{4\,(3x - 1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x - \frac{1}{3}\right)} + \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{3x - 1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x - 2}{3x - 1}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(3x - 1)}{2}} \\
&= z_1 \left(\sqrt{3x - 1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{3x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(3x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-\mathrm{e}^{-3x} x\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}\left(-\mathrm{e}^{-3x} x\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} - c_2 x\, \mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} - c_2 x\, \mathrm{e}^{-x}$$

Verified OK.

2.467.1 Maple step by step solution

Let's solve

$(3x - 1)\, y'' + (-3x - 2)\, y' + (-6x + 8)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2(3x-4)y}{3x-1} + \frac{(3x+2)y'}{3x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(3x+2)y'}{3x-1} - \frac{2(3x-4)y}{3x-1} = 0$

□ Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

  - Define functions

  $\left[P_2(x) = -\frac{3x+2}{3x-1}, P_3(x) = -\frac{2(3x-4)}{3x-1}\right]$

  - $\left(x-\frac{1}{3}\right) \cdot P_2(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right) \cdot P_2(x)\right)\bigg|_{x=\frac{1}{3}} = -1$

  - $\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{1}{3}$

  $\left(\left(x-\frac{1}{3}\right)^2 \cdot P_3(x)\right)\bigg|_{x=\frac{1}{3}} = 0$

  - $x = \frac{1}{3}$is a regular singular point

  Check to see if $x_0 = \frac{1}{3}$ is a regular singular point

  $x_0 = \frac{1}{3}$

- Multiply by denominators

$(3x-1)\, y'' + (-3x-2)\, y' + (-6x+8)\, y = 0$

- Change variables using $x = u + \frac{1}{3}$ so that the regular singular point is at $u = 0$

$3u\left(\frac{d^2}{du^2}y(u)\right) + (-3u-3)\left(\frac{d}{du}y(u)\right) + (-6u+6)\, y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-2+r)\,u^{-1+r}+(3a_1(1+r)\,(-1+r)-3a_0(-2+r))\,u^{r}+\left(\sum_{k=1}^{\infty}(3a_{k+1}(k+1+r)\,(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $3r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- Each term must be 0

  $3a_1(1+r)\,(-1+r)-3a_0(-2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $3a_{k+1}(k+1+r)\,(k+r-1)+a_k(-3k-3r+6)-6a_{k-1}=0$

- Shift index using $k->k+1$

  $3a_{k+2}(k+2+r)\,(k+r)+a_{k+1}(-3k+3-3r)-6a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{ka_{k+1}+ra_{k+1}+2a_k-a_{k+1}}{(k+2+r)(k+r)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{(k+2)k}$

- Recursion relation for $r=2$

$a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

- Revert the change of variables $u = x - \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{3}\right)^{k+2}, a_{k+2} = \frac{k a_{k+1} + 2 a_k + a_{k+1}}{(k+4)(k+2)}, 9a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((3*x-1)*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)-(6*x-8)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2x} + e^{-x} c_2 x$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 35

```
DSolve[(3*x-1)*y''[x]-(3*x+2)*y'[x]-(6*x-8)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-\frac{1}{2}}\left(c_1 e^{3x} + 2 e c_2 x\right)}{\sqrt{2}}$$

2.468 problem 481

2.468.1 Maple step by step solution 4334

Internal problem ID [7958]
Internal file name [`OUTPUT/6891_Sunday_June_05_2022_05_15_38_PM_67645724/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 481.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+2)\,y''+xy'+3y=0}$$

Writing the ode as

$$(x+2)\,y''+xy'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 12x - 20}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 12x - 20 \\ t &= 4(x+2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 12x - 20}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 889: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{4}{x+2} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{4}{x} - \frac{6}{x^2} - \frac{72}{x^3} - \frac{556}{x^4} - \frac{5440}{x^5} - \frac{55088}{x^6} - \frac{586688}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 12x - 20}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-16x - 24}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-16x - 24}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -16. Dividing this by leading coefficient in t which is 4 gives -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 0\right) &&= -4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 0\right) &&= 4
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 12x - 20}{4\left(x+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-4	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 4 - (2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{x+2} - \frac{1}{2} \\
&= -\frac{x-2}{2(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{2}{x+2} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{2}{(x+2)^2}\right) + \left(\frac{2}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 12x - 20}{4(x+2)^2}\right)\right) = 0
$$

$$
\frac{(a_1 + 6)x + 2a_0 + 2a_1 + 4}{x+2} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4, a_1 = -6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 6x + 4
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6x + 4\right) e^{\int \left(\frac{2}{x+2} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 6x + 4\right) e^{-\frac{x}{2} + 2\ln(x+2)} \\
&= \left(x^2 - 6x + 4\right) (x+2)^2 e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-6x+4\right)(x+2)^3\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{-(x^2-6x+4)\,(x+2)^3\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(x+2)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2-6x+4\right)(x+2)^3\,\mathrm{e}^{-x}\right) + c_2\Bigg(\left(x^2-6x+4\right)(x \\
&\quad +2)^3\,\mathrm{e}^{-x}\left(\frac{-(x^2-6x+4)\,(x+2)^3\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^4-x^3-18x^2-22x+8)}{240\,(x^2-6x+4)\,(x+2)^3}\right)\Bigg)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-6x+4)(x+2)^3 e^{-x} + c_2\left(-\frac{(x^2-6x+4)(x+2)^3 e^{-x-2}\operatorname{expIntegral}_1(-x-2)}{240} - \frac{x^4}{240} + \frac{x^3}{240} + \frac{3x^2}{40} + \frac{11x}{120} - \frac{1}{30}\right) \quad (1)$$

Verification of solutions

$$y = c_1(x^2-6x+4)(x+2)^3 e^{-x} + c_2\left(-\frac{(x^2-6x+4)(x+2)^3 e^{-x-2}\operatorname{expIntegral}_1(-x-2)}{240} - \frac{x^4}{240} + \frac{x^3}{240} + \frac{3x^2}{40} + \frac{11x}{120} - \frac{1}{30}\right)$$

Verified OK.

2.468.1 Maple step by step solution

Let's solve

$(x+2)\,y'' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x+2} - \frac{xy'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{x+2} + \frac{3y}{x+2} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x}{x+2}, P_3(x) = \frac{3}{x+2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = -2$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x=-2$

  $$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\,y''+xy'+3y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $$u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)+3y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)+a_k(k+r+3)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k-2+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$
- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-2)}$
- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}$
- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$
- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+6)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 115

```
dsolve((2+x)*diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x^5 - 20x^3 - 40x^2 + 32) - \frac{c_2(\text{expIntegral}_1(-2-x)\,e^{-2}x^5 + e^x x^4 - 20\,e^{-2}\,\text{expIntegral}_1(-2-x)\,x^3 - e^x x^3 - 40\,e^{-2}\,\text{expIntegral}}{240}$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 81

```
DSolve[(2+x)*y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{960}e^{-x-1}\big(c_2(x^2 - 6x + 4)(x+2)^3\,\text{ExpIntegralEi}(x+2) + 3840c_1(x^2 - 6x + 4)(x+2)^3 - c_2 e^{x+2}(x^4 - x^3 - 18x^2 - 22x + 8)\big)$$

2.469 problem 482

Internal problem ID [7959]
Internal file name [OUTPUT/6892_Sunday_June_05_2022_05_15_42_PM_81061552/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 482.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)\,y''+x(4+x)\,y'+(-x+2)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+4x\right)y'+(-x+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+4x \\ C &= -x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 36}{4x(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x + 36 \\ t &= 4x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 36}{4x(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 891: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{x} + \frac{35}{4(x-1)^2} - \frac{9}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+36}{4x(x-1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x+36}{4x\left(x-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{x} - \frac{5}{2\left(x-1\right)} + (-)\left(0\right) \\
&= \frac{1}{x} - \frac{5}{2\left(x-1\right)} \\
&= \frac{1}{x} - \frac{5}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x} - \frac{5}{2(x-1)}\right)(2x + a_1) + \left(\left(-\frac{1}{x^2} + \frac{5}{2(x-1)^2}\right) + \left(\frac{1}{x} - \frac{5}{2(x-1)}\right)^2 - \left(\frac{-x+36}{4x(x-1)^2}\right)\right) = 0$$

$$\frac{(a_1 - 6)\, x + 4a_0 - 2a_1}{x(x-1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 6x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^2 + 6x + 3)\, e^{\int \left(\frac{1}{x} - \frac{5}{2(x-1)}\right) dx} \\
&= (x^2 + 6x + 3)\, e^{\ln(x) - \frac{5\ln(x-1)}{2}} \\
&= \frac{(x^2 + 6x + 3)\, x}{(x-1)^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+4x}{-x^3+x^2}\, dx} \\
&= z_1 e^{-2\ln(x) + \frac{5\ln(x-1)}{2}} \\
&= z_1 \left(\frac{(x-1)^{\frac{5}{2}}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2+6x+3}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)+5\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{1}{9x} + \frac{152x+138}{9x^2+54x+27}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2+6x+3}{x}\right) + c_2 \left(\frac{x^2+6x+3}{x}\left(\ln(x) + \frac{1}{9x} + \frac{152x+138}{9x^2+54x+27}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(1+3(x^3+6x^2+3x)\ln(x)+51x^2+48x)}{3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(1+3(x^3+6x^2+3x)\ln(x)+51x^2+48x)}{3x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2+6x+3)}{x} + \frac{c_2(3x^3\ln(x)+18x^2\ln(x)+9x\ln(x)+51x^2+48x+1)}{3x^2}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 53

```
DSolve[x^2*(1-x)*y''[x]+x*(4+x)*y'[x]+(2-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_1x(x^2+6x+3) - c_2(51x^2+3(x^2+6x+3)\,x\log(x)+48x+1)}{3x^2}$$

2.470 problem 483

2.470.1 Maple step by step solution 4351

Internal problem ID [7960]
Internal file name [OUTPUT/6893_Sunday_June_05_2022_05_15_45_PM_7147512/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 483.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''+x(2x+1)\,y'-(6x+4)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(2x^2+x\right)y'+(-6x-4)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x)\\ B &= 2x^2+x\\ C &= -6x-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 + 40x + 15}{4(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 + 40x + 15$$
$$t = 4(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 + 40x + 15}{4(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 892: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{2x} - \frac{1}{4(1+x)^2} + \frac{15}{4x^2} - \frac{5}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 + 40x + 15}{4(x^2+x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^2 + 40x + 15}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x + 2} + \frac{5}{2x} + (0) \\
&= \frac{1}{2x + 2} + \frac{5}{2x} \\
&= \frac{6x + 5}{2x\,(1 + x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0) + 2\left(\frac{1}{2x + 2} + \frac{5}{2x}\right)(0) + \left(\left(-\frac{1}{2\,(1 + x)^2} - \frac{5}{2x^2}\right) + \left(\frac{1}{2x + 2} + \frac{5}{2x}\right)^2 - \left(\frac{24x^2 + 40x + 15}{4\,(x^2 + x)^2}\right)\right) = 0 \\
0 = 0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x+2} + \frac{5}{2x}\right) dx} \\
&= \sqrt{1 + x}\, x^{\frac{5}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2 + x}{x^2 (1 + x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(1 + x)}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\, x^{\frac{5}{2}}}{\sqrt{x(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x) \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\sqrt{1+x}\, x^{\frac{5}{2}}}{\sqrt{x(1+x)}} \right) + c_2 \left(\frac{\sqrt{1+x}\, x^{\frac{5}{2}}}{\sqrt{x(1+x)}} \left(-\frac{1}{4x^4} - \frac{1}{2x^2} + \ln(x) + \frac{1}{3x^3} + \frac{1}{x} - \ln(1+x) \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{1+x}\, x^{\frac{5}{2}}}{\sqrt{x(1+x)}} + \frac{c_2 \sqrt{1+x}\, (12\ln(x)\, x^4 - 12\ln(1+x)\, x^4 + 12x^3 - 6x^2 + 4x - 3)}{12x^{\frac{3}{2}} \sqrt{x(1+x)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{1+x}\, x^{\frac{5}{2}}}{\sqrt{x(1+x)}} + \frac{c_2 \sqrt{1+x}\, (12\ln(x)\, x^4 - 12\ln(1+x)\, x^4 + 12x^3 - 6x^2 + 4x - 3)}{12x^{\frac{3}{2}} \sqrt{x(1+x)}}$$

Verified OK.

2.470.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+(2x^2+x)\,y'+(-6x-4)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2(3x+2)y}{x^2(1+x)}-\frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x+1)y'}{x(1+x)}-\frac{2(3x+2)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x+1}{x(1+x)},P_3(x)=-\frac{2(3x+2)}{x^2(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''+x(2x+1)\,y'+(-6x-4)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(2u^2-3u+1)\left(\frac{d}{du}y(u)\right)+(-6u+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - a_0(2r^2+r-2)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k^2+4kr+2r^2+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term must be 0

$$a_1(1+r)^2 - a_0(2r^2+r-2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2-k-6)\, a_{k-1} + (-2k^2-k+2)\, a_k + a_{k+1}(k+1)^2 = 0$$

- Shift index using $k->k+1$

$$\left((k+1)^2-k-7\right) a_k + \left(-2(k+1)^2-k+1\right) a_{k+1} + a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 5k a_{k+1} - 6a_k - a_{k+1}}{(k+2)^2}, a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1+2*x)*diff(y(x),x)-(4+6*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 - \frac{c_2(12\ln(x+1)x^4 - 12x^4\ln(x) - 12x^3 + 6x^2 - 4x + 3)}{12x^2}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 52

```
DSolve[x^2*(1+x)*y''[x]+x*(1+2*x)*y'[x]-(4+6*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_1 x^2 + \frac{c_2(12x^4 \log(x) - 12x^4 \log(x+1) + 12x^3 - 6x^2 + 4x - 3)}{12x^2}$$

2.471 problem 484

2.471.1 Maple step by step solution 4361

Internal problem ID [7961]
Internal file name [OUTPUT/6894_Sunday_June_05_2022_05_15_48_PM_6378677/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 484.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2x^2+1)\, y''+x(2x^2+4)\, y'+2(1-x^2)\, y=0}$$

Writing the ode as

$$\left(2x^4+x^2\right) y''+\left(2x^3+4x\right) y'+\left(-2x^2+2\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+x^2 \\ B &= 2x^3+4x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 9}{(2x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 9$$
$$t = (2x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 9}{(2x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 894: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{21}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{15i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{15i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 9}{(2x^2+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 9}{(2x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (-)\,(0) \\
&= -\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\
&= -\frac{3x}{2x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(1) + \left(\left(\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{4\left(x-\frac{i\sqrt{2}}{2}\right)}-\frac{3}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)dx} \\
&= (x)\, \frac{1}{(4x^2+2)^{\frac{3}{4}}} \\
&= \frac{x}{(4x^2+2)^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x}{2x^4+x^2}\, dx} \\
&= z_1 e^{-2\ln(x)+\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1 \left(\frac{(2x^2+1)^{\frac{3}{4}}}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{\frac{1}{4}}}{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x}{2x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)+\frac{3\ln\left(2x^2+1\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{2}\,(2x^2-2)\sqrt{2x^2+1}+6\,\operatorname{arcsinh}\left(\sqrt{2}\,x\right)x}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2^{\frac{1}{4}}}{2x}\right) + c_2\left(\frac{2^{\frac{1}{4}}}{2x}\left(\frac{\sqrt{2}\,(2x^2-2)\sqrt{2x^2+1}+6\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)x}{x}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1 2^{\frac{1}{4}}}{2x} + \frac{c_2 2^{\frac{1}{4}}\left(\sqrt{2}\,(x^2-1)\sqrt{2x^2+1}+3\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)x\right)}{x^2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 2^{\frac{1}{4}}}{2x} + \frac{c_2 2^{\frac{1}{4}}\left(\sqrt{2}\,(x^2-1)\sqrt{2x^2+1}+3\,\mathrm{arcsinh}\left(\sqrt{2}\,x\right)x\right)}{x^2}
$$

Verified OK.

2.471.1 Maple step by step solution

Let's solve

$(2x^4+x^2)\,y''+(2x^3+4x)\,y'+(-2x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2(x^2-1)y}{x^2(2x^2+1)}-\frac{2(x^2+2)y'}{x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x^2+2)y'}{x(2x^2+1)}-\frac{2(x^2-1)y}{x^2(2x^2+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(x^2+2)}{x(2x^2+1)}, P_3(x)=-\frac{2(x^2-1)}{x^2(2x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=4$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$

- $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x^2+1)\,y''+2(x^2+2)\,xy'+(-2x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m=0..2$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m=1..3$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m=2..4$

  $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(1+r)\,x^r+a_1(3+r)\,(2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r+2)\,(k+r+1)+2a_{k-2}(k+r-1\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + 2a_{k-2}(k+r-1)(k-3+r) = 0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)(k+3+r) + 2a_k(k+r+1)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k(k+r+1)(k+r-1)}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k k(k-2)}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{2a_k(k-1)(k-3)}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{2b_k k(k-2)}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 59

```
dsolve(x^2*(1+2*x^2)*diff(y(x),x$2)+x*(4+2*x^2)*diff(y(x),x)+2*(1-x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1}{x} + \frac{c_2\sqrt{2}\left(\sqrt{2}\,\sqrt{2x^2+1}\,x^2 + 3\,\text{arcsinh}\left(\sqrt{2}\,x\right)x - \sqrt{2}\,\sqrt{2x^2+1}\right)}{2x^2}$$

✓ Solution by Mathematica
Time used: 0.091 (sec). Leaf size: 77

```
DSolve[x^2*(1+2*x^2)*y''[x]+x*(4+2*x^2)*y'[x]+2*(1-x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to -\frac{c_2\sqrt{2x^2+1}}{x^2} + c_2\sqrt{2x^2+1} - \frac{3c_2\log\left(\sqrt{2x^2+1}-\sqrt{2}x\right)}{\sqrt{2}x} + \frac{c_1}{x}$$

2.472 problem 485

2.472.1 Maple step by step solution 4371

Internal problem ID [7962]
Internal file name [`OUTPUT/6895_Sunday_June_05_2022_05_15_51_PM_69351415/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 485.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+2)\,y''+2x(x^2+5)\,y'+2(-x^2+3)\,y=0}$$

Writing the ode as

$$(x^4+2x^2)\,y''+(2x^3+10x)\,y'+(-2x^2+6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+2x^2 \\ B &= 2x^3+10x \\ C &= -2x^2+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^4 - 5x^2 + 3 \\ t &= (x^3 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 896: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + 2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{21}{16\left(x - i\sqrt{2}\right)^2} + \frac{21}{16\left(x + i\sqrt{2}\right)^2} + \frac{11i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} - \frac{11i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^4 - 5x^2 + 3}{(x^3 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i\sqrt{2}$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 2 - (0) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} \\
&= \frac{3}{x^3 + 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - \frac{3}{4\left(x - i\sqrt{2}\right)} - \frac{3}{4\left(x + i\sqrt{2}\right)} \right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} + \frac{3}{4\left(x - i\sqrt{2}\right)^2} + \frac{3}{4\left(x + i\sqrt{2}\right)^2} \right) + \right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x^2+8\right) \mathrm{e}^{\int \left(\frac{3}{2x}-\frac{3}{4\left(x-i\sqrt{2}\right)}-\frac{3}{4\left(x+i\sqrt{2}\right)}\right)dx} \\
&= \left(x^2+8\right) \mathrm{e}^{\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+2\right)}{4}} \\
&= \frac{\left(x^2+8\right) x^{\frac{3}{2}}}{\left(x^2+2\right)^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+10x}{x^4+2x^2}\,dx} \\
&= z_1 e^{\frac{3\ln\left(x^2+2\right)}{4}-\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(x^2+2\right)^{\frac{3}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+8}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+10x}{x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln\left(x^2+2\right)}{2}-5\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{-x^2\sqrt{2}\left(x^2+8\right)\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)+\left(2x^2-8\right)\sqrt{x^2+2}}{64x^2\left(x^2+8\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{x^2+8}{x}\right) + c_2\left(\frac{x^2+8}{x}\left(\frac{-x^2\sqrt{2}\,(x^2+8)\,\text{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) + (2x^2-8)\sqrt{x^2+2}}{64x^2\,(x^2+8)}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+8)}{x} + \frac{c_2\left(-x^2\sqrt{2}\,(x^2+8)\,\text{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) + (2x^2-8)\sqrt{x^2+2}\right)}{64x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+8)}{x} + \frac{c_2\left(-x^2\sqrt{2}\,(x^2+8)\,\text{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right) + (2x^2-8)\sqrt{x^2+2}\right)}{64x^3}$$

Verified OK.

2.472.1 Maple step by step solution

Let's solve

$(x^4+2x^2)\,y'' + (2x^3+10x)\,y' + (-2x^2+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(x^2-3)y}{x^2(x^2+2)} - \frac{2(x^2+5)y'}{x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x^2+5)y'}{x(x^2+2)} - \frac{2(x^2-3)y}{x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2(x^2+5)}{(x^2+2)x}, P_3(x) = -\frac{2(x^2-3)}{x^2(x^2+2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 5$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 3$$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$x^2(x^2+2)\, y'' + 2x(x^2+5)\, y' + (-2x^2+6)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - ◦ Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0(3+r)(1+r)x^r + 2a_1(4+r)(2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2(3+r)(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -1\}$

- Each term must be 0

  $2a_1(4+r)(2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2a_k(k+r+3)(k+r+1) + a_{k-2}(k+r)(k-3+r) = 0$

- Shift index using $k- >k+2$

  $2a_{k+2}(k+5+r)(k+r+3) + a_k(k+r+2)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+2)(k+r-1)}{2(k+5+r)(k+r+3)}$

- Recursion relation for $r = -3$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0\right]$

- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k(k+1)(k-2)}{2(k+4)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k(k-1)(k-4)}{2(k+2)k}, a_1 = 0, b_{k+2} = -\frac{b_k(k+1)(k-2)}{2(k+4)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 85

```
dsolve(x^2*(2+x^2)*diff(y(x),x$2)+2*x*(x^2+5)*diff(y(x),x)+2*(3-x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1(x^2+8)}{x} - \frac{c_2\sqrt{2}\left(\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)x^4 - \sqrt{2}\,\sqrt{x^2+2}\,x^2 + 8\operatorname{arctanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)x^2 + 4\sqrt{2}\,\sqrt{x^2+2}\right)}{64x^3}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 88

```
DSolve[x^2*(2+x^2)*y''[x]+2*x*(x^2+5)*y'[x]+2*(3-x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{-\sqrt{2}c_2(x^2+8)\,x^2\operatorname{arctanh}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right) + 64c_1x^4 + 2x^2\left(c_2\sqrt{x^2+2} + 256c_1\right) - 8c_2\sqrt{x^2+2}}{64x^3}$$

2.473 problem 486

Internal problem ID [7963]
Internal file name [`OUTPUT/6896_Sunday_June_05_2022_05_15_54_PM_5132414/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 486.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''+6xy'+6y=0}$$

Writing the ode as

$$(x^2+1)\,y''+6xy'+6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 898: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)(0) + \left(\left(\frac{1}{2\left(x-i\right)^{2}}-\frac{3}{2\left(x+i\right)^{2}}\right)+\left(-\frac{1}{2\left(x-i\right)}+\frac{3}{2\left(x+i\right)}\right)^{2}-\left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{\left(x^{2}+1\right)^{\frac{3}{2}}}{\left(ix+1\right)^{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^{2}+1}\, dx} \\
&= z_1 e^{-\frac{3\ln\left(x^{2}+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^{2}+1\right)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\left(ix+1\right)^{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-3\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{1}{(ix+1)^2}\right) + c_2 \left(\frac{1}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{(ix+1)^2} + \frac{c_2 x}{(x-i)^2 (x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{(ix+1)^2} + \frac{c_2 x}{(x-i)^2 (x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve((1+x^2)*diff(y(x),x$2)+6*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(x^2+1)^2} + \frac{c_2(x^2-1)}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 29

```
DSolve[(1+x^2)*y''[x]+6*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x - c_1(x-i)^2}{(x^2+1)^2}$$

2.474 problem 487

Internal problem ID [7964]
Internal file name [`OUTPUT/6897_Sunday_June_05_2022_05_15_57_PM_42552890/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 487.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+2xy'-2y=0}$$

Writing the ode as

$$(x^2+1)\,y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 3}{(x^2 + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 3$$
$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 3}{(x^2 + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 899: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{5i}{4(x-i)} + \frac{5i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2+3}{(x^2+1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2+3}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} + (0) \\
&= \frac{1}{2x - 2i} + \frac{1}{2x + 2i} \\
&= \frac{x}{x^2 + 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(1) + \left(\left(-\frac{1}{2(x - i)^2} - \frac{1}{2(x + i)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{2x^2 + 3}{(x^2 + 1)^2} - \frac{2(x^2 + 1)\, a_0}{(-x + i)^2 (x + i)^2}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= (x)\sqrt{x^2 + 1} \\
&= \sqrt{x^2 + 1}\, x
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(-\frac{1}{x} - \arctan\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(-\frac{1}{x} - \arctan\left(x\right)\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 x + c_2(-\arctan\left(x\right)x - 1) \tag{1}
$$

Verification of solutions

$$y = c_1 x + c_2(-\arctan(x) x - 1)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2(\arctan(x) x + 1)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 48

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} i(2c_1 x - c_2 x \log(1 - ix) + c_2 x \log(1 + ix) + 2ic_2)$$

2.475 problem 488

Internal problem ID [7965]
Internal file name [`OUTPUT/6898_Sunday_June_05_2022_05_16_00_PM_82824629/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 488.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\, y''-8xy'+20y=0}$$

Writing the ode as

$$\begin{align*}(x^2+1)\, y''-8xy'+20y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+1\\ B&=-8x \tag{3}\\ C&=20\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{align*}r&=\frac{s}{t} \tag{5}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-24}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -24 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{24}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 900: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{24}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x-i} + \frac{3}{x+i} + (-)\,(0) \\ &= -\frac{2}{x-i} + \frac{3}{x+i} \\ &= \frac{x-5i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-i} + \frac{3}{x+i}\right)(0) + \left(\left(\frac{2}{(x-i)^2} - \frac{3}{(x+i)^2}\right) + \left(-\frac{2}{x-i} + \frac{3}{x+i}\right)^2 - \left(-\frac{24}{(x^2+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x-i}+\frac{3}{x+i}\right)dx} \\ &= \frac{(x^2+1)^3}{(ix+1)^5} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x}{x^2+1}\, dx} \\ &= z_1 e^{2\ln(x^2+1)} \\ &= z_1 \left((x^2+1)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^5}{(ix+1)^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-8x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{4\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1\left(\frac{x^4-2x^2+\frac{1}{5}}{(x+i)^5}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{(x^2+1)^5}{(ix+1)^5}\right) + c_2\left(\frac{(x^2+1)^5}{(ix+1)^5}\left(\frac{x^4-2x^2+\frac{1}{5}}{(x+i)^5}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^5}{(ix+1)^5} + \frac{c_2(x^2+1)^5\left(x^4-2x^2+\frac{1}{5}\right)}{(ix+1)^5(x+i)^5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^5}{(ix+1)^5} + \frac{c_2(x^2+1)^5\left(x^4-2x^2+\frac{1}{5}\right)}{(ix+1)^5(x+i)^5}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((1+x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)+20*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(5x^4 - 10x^2 + 1) + c_2(x^5 - 10x^3 + 5x)$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 38

```
DSolve[(1+x^2)*y''[x]-8*x*y'[x]+20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}ic_2(5x^4 - 10x^2 + 1) + c_1(1 + ix)^5$$

2.476 problem 489

2.476.1 Maple step by step solution 4401

Internal problem ID [7966]
Internal file name [`OUTPUT/6899_Sunday_June_05_2022_05_16_03_PM_51314943/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 489.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-8xy'-12y=0}$$

Writing the ode as

$$(1-x^2)\,y''-8xy'-12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1-x^2\\B&=-8x\\C&=-12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8}{\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8 \\ t &= \left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8}{\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 901: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{(1+x)^2} + \frac{2}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{1+x}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	2	-1
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x-1} + \frac{2}{1+x} + (-)(0) \\
&= -\frac{1}{x-1} + \frac{2}{1+x} \\
&= \frac{-3+x}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{x-1}+\frac{2}{1+x}\right)(0)+\left(\left(\frac{1}{(x-1)^2}-\frac{2}{(1+x)^2}\right)+\left(-\frac{1}{x-1}+\frac{2}{1+x}\right)^2-\left(\frac{8}{(x^2-1)^2}\right)\right)=0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x-1}+\frac{2}{1+x}\right) dx} \\
&= \frac{(1+x)^2}{x-1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x}{1-x^2}\, dx} \\
&= z_1 e^{-2\ln(x-1)-2\ln(1+x)} \\
&= z_1\left(\frac{1}{(x-1)^2(1+x)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{(x-1)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8x}{1-x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4\ln(x-1)-4\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-3x^2-1}{3(1+x)^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{(x-1)^3}\right) + c_2 \left(\frac{1}{(x-1)^3} \left(\frac{-3x^2-1}{3(1+x)^3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3(x-1)^3(1+x)^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{(x-1)^3} + \frac{c_2(-3x^2-1)}{3(x-1)^3(1+x)^3}$$

Verified OK.

2.476.1 Maple step by step solution

Let's solve

$(1-x^2)\, y'' - 8xy' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{8xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{8x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $$((1+x) \cdot P_2(x))\bigg|_{x=-1} = 4$$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' + 8xy' + 12y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $$(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (8u - 8)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1}(k+1+r)(k+r+4) + a_k(k+r+4)(k+r+3))\, u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\, a_{k+1} + a_k(k+r+3))(k+r+4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+3)}{2(k+1+r)}$

- Recursion relation for $r = -3$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = \frac{a_k k}{2(k-2)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+3)}{2(k+1)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+3)}{2(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((1-x^2)*diff(y(x),x$2)-8*x*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x^2+1)}{(x-1)^3 (x+1)^3} + \frac{c_2(x^3+3x)}{(x-1)^3 (x+1)^3}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 37

```
DSolve[(1-x^2)*y''[x]-8*x*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{3c_1(x-1)^3 - c_2(3x^2+1)}{3(x^2-1)^3}$$

2.477 problem 490

Internal problem ID [7967]
Internal file name [OUTPUT/6900_Sunday_June_05_2022_05_16_06_PM_55223659/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 490.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+1)\,y''+7xy'+2y=0}$$

Writing the ode as

$$(2x^2+1)\,y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+1 \\ B &= 7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5x^2 + 6 \\ t &= 4\left(2x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 903: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x+\frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x-\frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x+\frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} + (0) \\
&= \frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}} \\
&= \frac{x}{4x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{1}{8x-4i\sqrt{2}} + \frac{1}{8x+4i\sqrt{2}}\right) dx} \\
&= (x)\left(4x^2+2\right)^{\frac{1}{8}} \\
&= x\left(4x^2+2\right)^{\frac{1}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2\,(2x^2+1)^{\frac{3}{4}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve((1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{(2x^2+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 66

```
DSolve[(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{(2x^2+1)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{(2x^2+1)^{3/4}\,\text{Gamma}\left(\frac{1}{4}\right)}$$

2.478 problem 491

2.478.1 Maple step by step solution 4419

Internal problem ID [7968]
Internal file name [OUTPUT/6901_Sunday_June_05_2022_05_16_09_PM_20643768/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 491.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-5xy'-4y=0}$$

Writing the ode as

$$(1-x^2)\,y''-5xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1-x^2\\ B&=-5x \\ C&=-4 \end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= 4\left(x^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 904: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(1+x)^2} + \frac{5}{16(x-1)^2} - \frac{7}{16(x-1)} + \frac{7}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2+6}{4\left(x^2-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
-1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)} + (-)\left(0\right) \\ &= -\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)} \\ &= -\frac{x}{2x^2-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-1\right)^2} + \frac{1}{4\left(1+x\right)^2}\right) + \left(-\frac{1}{4\left(x-1\right)} - \frac{1}{4\left(1+x\right)}\right)^2 - \left(\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, e^{\int \left(-\frac{1}{4(x-1)} - \frac{1}{4(1+x)}\right) dx} \\ &= (x)\, e^{-\frac{\ln(x-1)}{4} - \frac{\ln(1+x)}{4}} \\ &= \frac{x}{\left(x-1\right)^{\frac{1}{4}} \left(1+x\right)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x-1)}{4}-\frac{5\ln(1+x)}{4}} \\
&= z_1\left(\frac{1}{(x-1)^{\frac{5}{4}}(1+x)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{5}{4}}(1+x)^{\frac{5}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x-1)}{2}-\frac{5\ln(1+x)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x-\sqrt{x^2-1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{5}{4}}(1+x)^{\frac{5}{4}}}\right) \\
&\quad + c_2\left(\frac{x}{(x^2-1)^{\frac{1}{4}}(x-1)^{\frac{5}{4}}(1+x)^{\frac{5}{4}}}\left(\frac{\ln\left(x+\sqrt{x^2-1}\right)x-\sqrt{x^2-1}}{x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} + \frac{c_2\left(\ln\left(x+\sqrt{x^2-1}\right) x - \sqrt{x^2-1}\right)}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}} + \frac{c_2\left(\ln\left(x+\sqrt{x^2-1}\right) x - \sqrt{x^2-1}\right)}{(x^2-1)^{\frac{1}{4}} (x-1)^{\frac{5}{4}} (1+x)^{\frac{5}{4}}}$$

Verified OK.

2.478.1 Maple step by step solution

Let's solve

$(1-x^2)\, y'' - 5xy' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5xy'}{x^2-1} - \frac{4y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5xy'}{x^2-1} + \frac{4y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5x}{x^2-1}, P_3(x) = \frac{4}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = \frac{5}{2}$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $\left(x^2-1\right) y''+5 x y'+4 y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $\left(u^2-2 u\right)\left(\frac{d^2}{d u^2} y(u)\right)+(5 u-5)\left(\frac{d}{d u} y(u)\right)+4 y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot\left(\frac{d}{d u} y(u)\right)$ to series expansion for $m=0..1$

    $u^m \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r) u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot\left(\frac{d}{d u} y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$

  - Convert $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)$ to series expansion for $m=1..2$

    $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1) u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m \cdot\left(\frac{d^2}{d u^2} y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(3+2 r) u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)(2 k+5+2 r)+a_k(k+r+2)^2\right) u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(3+2 r)=0$

- Values of r that satisfy the indicial equation

  $r \in\left\{0,-\frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)^2-2 a_{k+1}(k+1+r)\left(k+\frac{5}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r+2)^2}{(k+1+r)(2 k+5+2 r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}\right]$$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}$

- Solution for $r = -\frac{3}{2}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{3}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{a_k(k+2)^2}{(k+1)(2k+5)}, b_{k+1} = \frac{b_k\left(k+\frac{1}{2}\right)^2}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 46

```
dsolve((1-x^2)*diff(y(x),x$2)-5*x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(\ln\left(x+\sqrt{x^2-1}\right)x-\sqrt{x^2-1}\right)}{(x^2-1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 52

```
DSolve[(1-x^2)*y''[x]-5*x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2\sqrt{x^2-1}-c_2 x\log\left(\sqrt{x^2-1}-x\right)+c_1 x}{(x^2-1)^{3/2}}$$

2.479 problem 492

Internal problem ID [7969]
Internal file name [`OUTPUT/6902_Sunday_June_05_2022_05_16_13_PM_13319785/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 492.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-10xy'+28y=0}$$

Writing the ode as

$$\begin{align*}(x^2+1)\,y''-10xy'+28y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+1\\ B&=-10x \tag{3}\\ C&=28\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{align*}r&=\frac{s}{t} \tag{5}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 33}{\left(x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 33 \\ t &= \left(x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 33}{\left(x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 906: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4(x-i)^2} + \frac{35}{4(x+i)^2} + \frac{31i}{4(x-i)} - \frac{31i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 33}{(x^2+1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 33}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-i$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{5}{2(x-i)} + \frac{7}{2(x+i)} + (0) \\ &= -\frac{5}{2(x-i)} + \frac{7}{2(x+i)} \\ &= \frac{x-6i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x-i\right)} + \frac{7}{2\left(x+i\right)}\right)(1) + \left(\left(\frac{5}{2\left(x-i\right)^2} - \frac{7}{2\left(x+i\right)^2}\right) + \left(-\frac{5}{2\left(x-i\right)} + \frac{7}{2\left(x+i\right)}\right)^2 - \Big(\frac{2x^{\cdots}}{(x^{2}\cdots}$$

$$-\frac{2(6i + a_0)\left(x^{\cdots}\right.}{(-x+i)^2\left(x \cdots\right.}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 6i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 6i)\, \mathrm{e}^{\int \left(-\frac{5}{2(x-i)} + \frac{7}{2(x+i)}\right) dx} \\
&= (x - 6i)\, \mathrm{e}^{\frac{\ln\left(x^2+1\right)}{2} - 6i \arctan(x)} \\
&= \frac{(-x + 6i)\left(x^2 + 1\right)^{\frac{7}{2}}}{(-x + i)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x}{x^2+1} \, dx} \\
&= z_1 e^{\frac{5 \ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\left(x^2 + 1\right)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(-x+6i)\left(x^2+1\right)^6}{(-x+i)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-10x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{5\ln(x^2+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{35x^4-42x^2+3}{105\,(-x+6i)\,(x+i)^6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(-x+6i)\left(x^2+1\right)^6}{(-x+i)^6}\right) + c_2 \left(\frac{(-x+6i)\left(x^2+1\right)^6}{(-x+i)^6} \left(\frac{35x^4-42x^2+3}{105\,(-x+6i)\,(x+i)^6}\right)\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1(-x+6i)\left(x^2+1\right)^6}{(-x+i)^6} + \frac{c_2\left(x^2+1\right)^6\left(35x^4-42x^2+3\right)}{105\,(-x+i)^6\,(x+i)^6} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1(-x+6i)\left(x^2+1\right)^6}{(-x+i)^6} + \frac{c_2\left(x^2+1\right)^6\left(35x^4-42x^2+3\right)}{105\,(-x+i)^6\,(x+i)^6}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve((1+x^2)*diff(y(x),x$2)-10*x*diff(y(x),x)+28*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(1+\frac{35}{3}x^4-14x^2\right)+c_2\left(x^7+21x^5-105x^3+35x\right)$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 40

```
DSolve[(1+x^2)*y''[x]-10*x*y'[x]+28*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{105}c_2\left(35x^4-42x^2+3\right)-c_1(x-i)^6(x+6i)$$

2.480 problem 493

2.480.1 Maple step by step solution 4436

Internal problem ID [7970]
Internal file name [OUTPUT/6903_Sunday_June_05_2022_05_16_16_PM_79941929/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 493.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 907: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4} - \frac{3}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\,\mathrm{e}^{\int -\frac{x}{2} dx} \\
&= (x)\,\mathrm{e}^{-\frac{x^2}{4}} \\
&= x\,\mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.480.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,e^{-\frac{x^2}{2}} - \frac{c_2 e^{-\frac{x^2}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x + 2\,e^{\frac{x^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.481 problem 495

2.481.1 Maple step by step solution 4445

Internal problem ID [7971]
Internal file name [`OUTPUT/6904_Sunday_June_05_2022_05_16_19_PM_16685877/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 495.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2-8x+11)\,y''-16(x-2)\,y'+36y=0}$$

Writing the ode as

$$(2x^2-8x+11)\,y''+(-16x+32)\,y'+36y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2-8x+11 \\ B &= -16x+32 \\ C &= 36 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 8x^2 - 32x - 100$$
$$t = (2x^2 - 8x + 11)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^2 - 32x - 100}{(2x^2 - 8x + 11)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 909: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(2x^2 - 8x + 11\right)^2$. There is a pole at $x = 2 + \frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = 2 - \frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} + \frac{6}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2} + \frac{5i\sqrt{6}}{3\left(x - 2 - \frac{i\sqrt{6}}{2}\right)} - \frac{5i\sqrt{6}}{3\left(x - 2 + \frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = 2 + \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = 2 - \frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^2 - 32x - 100}{\left(2x^2 - 8x + 11\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{8x^2-32x-100}{\left(2x^2-8x+11\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$2+\frac{i\sqrt{6}}{2}$	2	0	3	-2
$2-\frac{i\sqrt{6}}{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$\begin{aligned}d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\ &= 2-(1)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} + (0) \\
&= -\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}} \\
&= \frac{-5i\sqrt{6} + 2x - 4}{2x^2 - 8x + 11}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}} + \frac{3}{x - 2 + \frac{i\sqrt{6}}{2}}\right)(1) + \left(\left(\frac{2}{\left(x - 2 - \frac{i\sqrt{6}}{2}\right)^2} - \frac{3}{\left(x - 2 + \frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\frac{2}{x - 2 - \frac{i\sqrt{6}}{2}}\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{5i\sqrt{6}}{2} - 2\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 2 - \frac{5i\sqrt{6}}{2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\int \left(-\frac{2}{x-2-\frac{i\sqrt{6}}{2}}+\frac{3}{x-2+\frac{i\sqrt{6}}{2}}\right)dx} \\
&= \left(x-2-\frac{5i\sqrt{6}}{2}\right) \mathrm{e}^{\frac{\ln\left(4x^2-16x+22\right)}{2}-5i\arctan\left(\frac{(2x-4)\sqrt{6}}{6}\right)} \\
&= \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^3\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x+32}{2x^2-8x+11}\,dx} \\
&= z_1 e^{2\ln\left(2x^2-8x+11\right)} \\
&= z_1\left(\left(2x^2-8x+11\right)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5\sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x+32}{2x^2-8x+11}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln\left(2x^2-8x+11\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{-\frac{16}{3}x^3+32x^2-\frac{296}{5}x+\frac{496}{15}}{\left(2x-4+i\sqrt{6}\right)^5\left(5i\sqrt{6}-2x+4\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5 \sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \right) \\
&\quad + c_2 \left(\frac{9\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5 \sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \left(\frac{-\frac{16}{3}x^3+32x^2-\frac{296}{5}x+\frac{496}{15}}{\left(2x-4+i\sqrt{6}\right)^5\left(5i\sqrt{6}-2x+4\right)} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{9c_1\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5 \sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \\
& - \frac{12c_2\left(2x^2-8x+11\right)^5 \sqrt{6}\left(10x^3-60x^2+111x-62\right)}{5\left(2ix-4i-\sqrt{6}\right)^5\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{9c_1\left(5\sqrt{6}+2ix-4i\right)\left(2x^2-8x+11\right)^5 \sqrt{6}}{2\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5} \\
& - \frac{12c_2\left(2x^2-8x+11\right)^5 \sqrt{6}\left(10x^3-60x^2+111x-62\right)}{5\left(2ix-4i-\sqrt{6}\right)^5\left(-\sqrt{6}\,x+2\sqrt{6}+3i\right)^5}
\end{aligned}
$$

Verified OK.

2.481.1 Maple step by step solution

Let's solve

$\left(2x^2-8x+11\right)y''+\left(-16x+32\right)y'+36y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{36y}{2x^2-8x+11}+\frac{16(x-2)y'}{2x^2-8x+11}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{16(x-2)y'}{2x^2-8x+11}+\frac{36y}{2x^2-8x+11}=0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{16(x-2)}{2x^2-8x+11}, P_3(x) = \frac{36}{2x^2-8x+11}\right]$
  - $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$
  - $\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x-2+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=2-\frac{\mathrm{I}\sqrt{6}}{2}} = 0$
  - $x = 2-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 2-\frac{\mathrm{I}\sqrt{6}}{2}$
- Multiply by denominators

  $(2x^2-8x+11)\,y''+(-16x+32)\,y'+36y = 0$
- Change variables using $x = u+2-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u = 0$

  $(2u^2-2\,\mathrm{I}u\sqrt{6})\left(\frac{d^2}{du^2}y(u)\right)+(-16u+8\,\mathrm{I}\sqrt{6})\left(\frac{d}{du}y(u)\right)+36y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-2 \mathrm{I} \sqrt{6}\, r \left(-5+r \right) a_{0} u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(-2 \mathrm{I} \sqrt{6}\, \left(k +1+r \right) \left(k -4+r \right) a_{k +1}+2 a_{k} \left(k +r -3\right) \left(k +r -\right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-2 \mathrm{I} \sqrt{6}\, r \left(-5+r \right) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2 \mathrm{I} \sqrt{6}\, \left(k +1+r \right) \left(k -4+r \right) a_{k +1}+2 a_{k} \left(k +r -3\right) \left(k +r -6\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1} = \frac{-\frac{\mathrm{I}}{6} a_{k} \left(k^{2}+2 k r +r^{2}-9 k -9 r +18\right) \sqrt{6}}{k^{2}+2 k r +r^{2}-3 k -3 r -4}$$

- Recursion relation for $r = 0$; series terminates at $k = 3$

$$a_{k +1} = \frac{-\frac{\mathrm{I}}{6} a_{k} \left(k^{2}-9 k +18\right) \sqrt{6}}{k^{2}-3 k -4}$$

- Apply recursion relation for $k = 0$

$$a_{1} = \frac{3 \mathrm{I}}{4} a_{0} \sqrt{6}$$

- Apply recursion relation for $k = 1$

$$a_{2} = \frac{5 \mathrm{I}}{18} a_{1} \sqrt{6}$$

- Express in terms of a_{0}

$$a_{2} = -\frac{5 a_{0}}{4}$$

- Apply recursion relation for $k = 2$

$$a_{3} = \frac{\mathrm{I}}{9} a_{2} \sqrt{6}$$

- Express in terms of a_{0}

$$a_{3} = -\frac{5 \mathrm{I}}{36} a_{0} \sqrt{6}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y \left(u \right) = a_{0} \cdot \left(1+\frac{3 \mathrm{I} \sqrt{6}\, u}{4}-\frac{5 u^{2}}{4}-\frac{5 \mathrm{I} \sqrt{6}\, u^{3}}{36}\right)$$

- Revert the change of variables $u = x -2+\frac{\mathrm{I} \sqrt{6}}{2}$

$$\left[y = -\frac{\mathrm{I}}{72} a_{0} \sqrt{6}\, \left(10 x^{3}-60 x^{2}+111 x -62\right)\right]$$

- Recursion relation for $r = 5$; series terminates at $k = 1$

  $a_{k+1} = \frac{-\frac{I}{6}a_k(k^2+k-2)\sqrt{6}}{k^2+7k+6}$
- Apply recursion relation for $k = 0$

  $a_1 = \frac{I}{18}a_0\sqrt{6}$
- Terminating series solution of the ODE for $r = 5$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 + \frac{I\sqrt{6}\,u}{18}\right)$
- Revert the change of variables $u = x - 2 + \frac{I\sqrt{6}}{2}$

  $\left[y = a_0\left(\frac{5}{6} + \frac{I(x-2)\sqrt{6}}{18}\right)\right]$
- Combine solutions and rename parameters

  $\left[y = -\frac{Ia_0\sqrt{6}\,(10x^3-60x^2+111x-62)}{72} + b_0\left(\frac{5}{6} + \frac{I(x-2)\sqrt{6}}{18}\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((11-8*x+2*x^2)*diff(y(x),x$2)-16*(x-2)*diff(y(x),x)+36*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{31}{5} + x^3 - 6x^2 + \frac{111}{10}x\right) + c_2\left(x^6 - 12x^5 + \frac{165}{2}x^4 - \frac{16577}{8} - \frac{5445}{4}x^2 + 3267x\right)$$

✓ Solution by Mathematica

Time used: 0.898 (sec). Leaf size: 91

```
DSolve[(11-8*x+2*x^2)*y''[x]-16*(x-2)*y'[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{15} i c_2 \left(10x^3 - 60x^2 + 111x - 62\right) + \frac{c_1 \left(2x + 5i\sqrt{6} - 4\right) \left(2(x-4)x + 11\right)^2 \left(2ix + \sqrt{6} - 4i\right)^3}{2\left(-2ix + \sqrt{6} + 4i\right)^2}$$

2.482 problem 496

2.482.1 Maple step by step solution 4456

Internal problem ID [7972]
Internal file name [`OUTPUT/6905_Sunday_June_05_2022_05_16_22_PM_8860585/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 496.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + (-3 + x) y' + 3y = 0}$$

Writing the ode as

$$y'' + (-3 + x) y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 + x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 911: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2} - \frac{5}{2x} - \frac{15}{2x^2} - \frac{115}{4x^3} - \frac{495}{4x^4} - \frac{2285}{4x^5} - \frac{11055}{4x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= -\frac{3}{2} + \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4} - \frac{3}{2}x + \frac{1}{4}x^2$$

This shows that the coefficient of 1 in the above is $\frac{9}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6x - 1}{4} \\
&= Q + \frac{R}{4} \\
&= \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right) + (0) \\
&= -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{4}\right) - \left(\frac{9}{4}\right) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{3}{2} + \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{3}{2} + \frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(-\frac{3}{2} + \frac{x}{2}\right) \\ &= \frac{3}{2} - \frac{x}{2} \\ &= \frac{3}{2} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2} - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(\frac{3}{2} - \frac{x}{2}\right)^2 - \left(-\frac{1}{4} + \frac{1}{4}x^2 - \frac{3}{2}x\right)\right) = 0$$

$$(x+3)\,a_1 + 6x + 2a_0 + 2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = -6\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6x + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{\int \left(\frac{3}{2} - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= \left(x^2 - 6x + 8\right) \mathrm{e}^{-\frac{x(x-6)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3+x}{1} \, dx} \\
&= z_1 e^{\frac{3}{2}x - \frac{1}{4}x^2} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(x-6)}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2 - 6x + 8\right) \mathrm{e}^{-\frac{x(x-6)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3+x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3x - \frac{1}{2}x^2}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x(x-6)}{2}}}{\left(x^2 - 6x + 8\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 6x + 8\right) \mathrm{e}^{-\frac{x(x-6)}{2}} \right) + c_2 \left(\left(x^2 - 6x + 8\right) \mathrm{e}^{-\frac{x(x-6)}{2}} \left(\int \frac{\mathrm{e}^{\frac{x(x-6)}{2}}}{\left(x^2 - 6x + 8\right)^2} dx \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^2-6x+8)\,e^{-\frac{x(x-6)}{2}} + c_2(x^2-6x+8)\,e^{-\frac{x(x-6)}{2}}\left(\int \frac{e^{\frac{x(x-6)}{2}}}{(x^2-6x+8)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-6x+8)\,e^{-\frac{x(x-6)}{2}} + c_2(x^2-6x+8)\,e^{-\frac{x(x-6)}{2}}\left(\int \frac{e^{\frac{x(x-6)}{2}}}{(x^2-6x+8)^2}dx\right)$$

Verified OK.

2.482.1 Maple step by step solution

Let's solve

$y''+(-3+x)\,y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m=0..1$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ◦ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$

- ◦ Convert y'' to series expansion

  $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k->k+2$

  $y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_{k+1}(k+1) + a_k(k+3)\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + \left(a_k - 3a_{k+1} + 3a_{k+2}\right) k + 3a_k - 3a_{k+1} + 2a_{k+2} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k k - 3a_{k+1} k + 3a_k - 3a_{k+1}}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 68

```
dsolve(diff(y(x),x$2)+(x-3)*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{1}{2}x^2+3x}(x^2-6x+8) + c_2 e^{-\frac{1}{2}x^2+3x}(x^2-6x+8)\left(\int \frac{e^{\frac{1}{2}x^2-3x}}{(x-2)^2(x-4)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.275 (sec). Leaf size: 90

```
DSolve[y''[x]+(x-3)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-\frac{1}{2}(x-6)x-8}\left(e^{7/2}\sqrt{2\pi}c_2(x^2-6x+8)\operatorname{erfi}\left(\frac{x-3}{\sqrt{2}}\right) + 4e^8 c_1(x^2-6x+8) - 2c_2 e^{\frac{1}{2}(x-4)^2+x}(x-3)\right)$$

2.483 problem 497

2.483.1 Maple step by step solution 4465

Internal problem ID [7973]
Internal file name [`OUTPUT/6906_Sunday_June_05_2022_05_16_25_PM_90020348/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 497.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-8x+14)\,y''-8(x-4)\,y'+20y=0}$$

Writing the ode as

$$(x^2-8x+14)\,y''+(-8x+32)\,y'+20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-8x+14 \\ B &= -8x+32 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48}{\left(x^2 - 8x + 14\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48$$
$$t = \left(x^2 - 8x + 14\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48}{\left(x^2 - 8x + 14\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 913: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 8x + 14)^2$. There is a pole at $x = 4 + \sqrt{2}$ of order 2. There is a pole at $x = 4 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x - 4 + \sqrt{2})^2} + \frac{6}{(x - 4 - \sqrt{2})^2} + \frac{3\sqrt{2}}{x - 4 + \sqrt{2}} - \frac{3\sqrt{2}}{x - 4 - \sqrt{2}}$$

For the pole at $x = 4 + \sqrt{2}$ let b be the coefficient of $\frac{1}{(x-4-\sqrt{2})^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = 4 - \sqrt{2}$ let b be the coefficient of $\frac{1}{(x-4+\sqrt{2})^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48}{(x^2 - 8x + 14)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$4+\sqrt{2}$	2	0	3	-2
$4-\sqrt{2}$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} + (-)(0) \\
&= -\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}} \\
&= \frac{x-4-5\sqrt{2}}{x^2-8x+14}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right)(0) + \left(\left(\frac{2}{\left(x-4-\sqrt{2}\right)^2} - \frac{3}{\left(x-4+\sqrt{2}\right)^2}\right) + \left(-\frac{2}{x-4-\sqrt{2}} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{2}{x-4-\sqrt{2}} + \frac{3}{x-4+\sqrt{2}}\right) dx} \\ &= \frac{\left(x-4+\sqrt{2}\right)^3}{\left(-x+4+\sqrt{2}\right)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-8x+32}{x^2-8x+14} \, dx} \\ &= z_1 e^{2\ln(x^2-8x+14)} \\ &= z_1 \left(\left(x^2-8x+14\right)^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x+32}{x^2-8x+14}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x^2-8x+14)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-5x^4+80x^3-500x^2+1440x-1604}{5\left(x-4+\sqrt{2}\right)^5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2}\right) \\
&\quad + c_2 \left(\frac{\left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2}\left(\frac{-5x^4+80x^3-500x^2+1440x-1604}{5\left(x-4+\sqrt{2}\right)^5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 \left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2} \\
&\quad - \frac{c_2 \left(x^4-16x^3+100x^2-288x+\frac{1604}{5}\right)\left(x^2-8x+14\right)^2}{\left(x-4-\sqrt{2}\right)^2 \left(x-4+\sqrt{2}\right)^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 \left(x-4+\sqrt{2}\right)^3 \left(x^2-8x+14\right)^2}{\left(-x+4+\sqrt{2}\right)^2} \\
&\quad - \frac{c_2 \left(x^4-16x^3+100x^2-288x+\frac{1604}{5}\right)\left(x^2-8x+14\right)^2}{\left(x-4-\sqrt{2}\right)^2 \left(x-4+\sqrt{2}\right)^2}
\end{aligned}
$$

Verified OK.

2.483.1 Maple step by step solution

Let's solve

$(x^2-8x+14)\,y''+(-8x+32)\,y'+20y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{20y}{x^2-8x+14}+\frac{8(x-4)y'}{x^2-8x+14}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{8(x-4)y'}{x^2-8x+14}+\frac{20y}{x^2-8x+14}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{8(x-4)}{x^2-8x+14}, P_3(x)=\frac{20}{x^2-8x+14}\right]$

  ◦ $(x-4+\sqrt{2})\cdot P_2(x)$ is analytic at $x=4-\sqrt{2}$

  $\left((x-4+\sqrt{2})\cdot P_2(x)\right)\Big|_{x=4-\sqrt{2}}=0$

  ◦ $(x-4+\sqrt{2})^2\cdot P_3(x)$ is analytic at $x=4-\sqrt{2}$

  $\left((x-4+\sqrt{2})^2\cdot P_3(x)\right)\Big|_{x=4-\sqrt{2}}=0$

  ◦ $x=4-\sqrt{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=4-\sqrt{2}$

- Multiply by denominators

  $(x^2-8x+14)\,y''+(-8x+32)\,y'+20y=0$

- Change variables using $x=u+4-\sqrt{2}$ so that the regular singular point is at $u=0$

  $(u^2-2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right)+(-8u+8\sqrt{2})\left(\frac{d}{du}y(u)\right)+20y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\, r(r-5)\, a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)(k+r-4)\, a_{k+1} + a_k(k+r-4)(k+r-5)\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\, r(r-5) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 5\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-5)\right)(k+r-4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)\sqrt{2}}{4(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)\sqrt{2}}{4(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{5a_0\sqrt{2}}{4}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{2}$

- Express in terms of a_0

$a_2 = \frac{5a_0}{4}$

- Apply recursion relation for $k = 2$

$a_3 = -\frac{a_2\sqrt{2}}{4}$

- Express in terms of a_0

$a_3 = -\frac{5a_0\sqrt{2}}{16}$

- Apply recursion relation for $k = 3$

$a_4 = -\frac{a_3\sqrt{2}}{8}$

- Express in terms of a_0

$a_4 = \frac{5a_0}{64}$

- Apply recursion relation for $k = 4$

$a_5 = -\frac{a_4\sqrt{2}}{20}$

- Express in terms of a_0

$a_5 = -\frac{a_0\sqrt{2}}{256}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{5u\sqrt{2}}{4} + \frac{5u^2}{4} - \frac{5\sqrt{2}\,u^3}{16} + \frac{5u^4}{64} - \frac{\sqrt{2}\,u^5}{256}\right)$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

$\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right)\right]$

- Recursion relation for $r = 5$

$a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}$

- Solution for $r = 5$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Revert the change of variables $u = x - 4 + \sqrt{2}$

$\left[y = \sum_{k=0}^{\infty} a_k \left(x - 4 + \sqrt{2}\right)^{k+5}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+6)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0\left(\frac{(-x^5+20x^4-180x^3+880x^2-2260x+2384)\sqrt{2}}{256} + \frac{5x^4}{128} - \frac{5x^3}{8} + \frac{125x^2}{32} - \frac{45x}{4} + \frac{401}{32}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x - 4 + \right.\right.\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((x^2-8*x+14)*diff(y(x),x$2)-8*(x-4)*diff(y(x),x)+20*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{1604}{5} + x^4 - 16x^3 + 100x^2 - 288x\right) + c_2(x^5 - 140x^3 + 1120x^2 - 3500x + 4032)$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 77

```
DSolve[(x^2-8*x+14)*y''[x]+8*(x-4)*y'[x]+20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 P^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right) + c_2 Q^3_{\frac{1}{2}i\left(i+\sqrt{31}\right)}\left(\frac{x-4}{\sqrt{2}}\right)}{(x^2-8x+14)^{3/2}}$$

2.484 problem 498

2.484.1 Maple step by step solution 4475

Internal problem ID [7974]
Internal file name [`OUTPUT/6907_Sunday_June_05_2022_05_16_28_PM_13060829/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 498.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+4x+5)\,y''-20(1+x)\,y'+60y=0}$$

Writing the ode as

$$(2x^2+4x+5)\,y''+(-20x-20)\,y'+60y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+4x+5 \\ B &= -20x-20 \\ C &= 60 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-210}{(2x^2 + 4x + 5)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -210 \\ t &= (2x^2 + 4x + 5)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{210}{(2x^2 + 4x + 5)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 915: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+4x+5)^2$. There is a pole at $x = -1+\frac{i\sqrt{6}}{2}$ of order 2. There is a pole at $x = -1-\frac{i\sqrt{6}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} + \frac{35}{4\left(x+1+\frac{i\sqrt{6}}{2}\right)^2} + \frac{35i\sqrt{6}}{12\left(x+1-\frac{i\sqrt{6}}{2}\right)} - \frac{35i\sqrt{6}}{12\left(x+1+\frac{i\sqrt{6}}{2}\right)}$$

For the pole at $x = -1+\frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1-\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = -1-\frac{i\sqrt{6}}{2}$ let b be the coefficient of $\frac{1}{\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{210}{\left(2x^2+4x+5\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-1+\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-1-\frac{i\sqrt{6}}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} + (-)\,(0) \\ &= -\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)} \\ &= \frac{-6i\sqrt{6}+2x+2}{2x^2+4x+5} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)^2} - \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)^2}\right) + \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{5}{2\left(x+1-\frac{i\sqrt{6}}{2}\right)} + \frac{7}{2\left(x+1+\frac{i\sqrt{6}}{2}\right)}\right) dx} \\
&= \frac{27\sqrt{2}\,(2x^2+4x+5)^{\frac{7}{2}}}{\left(3+i\,(1+x)\sqrt{6}\right)^6}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x-20}{2x^2+4x+5}\, dx} \\
&= z_1 e^{\frac{5\ln(2x^2+4x+5)}{2}} \\
&= z_1\left((2x^2+4x+5)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = -\frac{(2x^2+4x+5)^6\sqrt{2}}{27\left(i - \frac{(1+x)\sqrt{3}\sqrt{2}}{3}\right)^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x-20}{2x^2+4x+5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5\ln(2x^2+4x+5)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-16x^5 - 80x^4 - 80x^3 + 80x^2 + 124x + 28}{\left(2x + 2 + i\sqrt{6}\right)^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(-\frac{\left(2x^2 + 4x + 5\right)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6}\right) \\
&\quad + c_2 \left(-\frac{\left(2x^2 + 4x + 5\right)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} \left(\frac{-16x^5 - 80x^4 - 80x^3 + 80x^2 + 124x + 28}{\left(2x + 2 + i\sqrt{6}\right)^6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-\frac{c_1 \left(2x^2 + 4x + 5\right)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} \\
&+ \frac{4c_2 \left(2x^2 + 4x + 5\right)^6 \sqrt{2}\, (4x^5 + 20x^4 + 20x^3 - 20x^2 - 31x - 7)}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6 \left(2x + 2 + i\sqrt{3}\,\sqrt{2}\right)^6}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = -\frac{c_1 \left(2x^2 + 4x + 5\right)^6 \sqrt{2}}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6} + \frac{4c_2 \left(2x^2 + 4x + 5\right)^6 \sqrt{2}\, (4x^5 + 20x^4 + 20x^3 - 20x^2 - 31x - 7)}{27 \left(i - \frac{(1+x)\sqrt{3}\,\sqrt{2}}{3}\right)^6 \left(2x + 2 + i\sqrt{3}\,\sqrt{2}\right)^6}$$

Verified OK.

2.484.1 Maple step by step solution

Let's solve

$\left(2x^2+4x+5\right)y''+\left(-20x-20\right)y'+60y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{60y}{2x^2+4x+5}+\frac{20(1+x)y'}{2x^2+4x+5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{20(1+x)y'}{2x^2+4x+5}+\frac{60y}{2x^2+4x+5}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{20(1+x)}{2x^2+4x+5}, P_3(x)=\frac{60}{2x^2+4x+5}\right]$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

  ◦ $\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

    $\left(\left(x+1+\frac{\mathrm{I}\sqrt{6}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-1-\frac{\mathrm{I}\sqrt{6}}{2}}=0$

  ◦ $x=-1-\frac{\mathrm{I}\sqrt{6}}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1-\frac{\mathrm{I}\sqrt{6}}{2}$

- Multiply by denominators

  $\left(2x^2+4x+5\right)y''+\left(-20x-20\right)y'+60y=0$

- Change variables using $x=u-1-\frac{\mathrm{I}\sqrt{6}}{2}$ so that the regular singular point is at $u=0$

  $\left(2u^2-2\,\mathrm{I}u\sqrt{6}\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(-20u+10\,\mathrm{I}\sqrt{6}\right)\left(\frac{d}{du}y(u)\right)+60y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2\,\mathrm{I}\sqrt{6}\,(r-6)\,r a_0 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2\,\mathrm{I}\sqrt{6}\,(k+r-5)\,(k+1+r)\,a_{k+1} + 2a_k(k+r-5)\,(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2\,\mathrm{I}\sqrt{6}\,(r-6)\,r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 6\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2(k+r-5)\left(\mathrm{I}(k+1+r)\,a_{k+1}\sqrt{6} - a_k(k+r-6)\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k+r-6)\sqrt{6}}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 6$

  $$a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}$$

- Recursion relation that defines the terminating series solution of the ODE for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{5} a_k u^k, a_{k+1} = \frac{-\frac{\mathrm{I}}{6}a_k(k-6)\sqrt{6}}{k+1}\right]$$

- Revert the change of variables $u = x + 1 + \frac{\mathrm{I}\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{5} a_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^k, a_{k+1} = \frac{-\frac{I}{6} a_k (k-6)\sqrt{6}}{k+1}\right]$$

- Recursion relation for $r = 6$

$$a_{k+1} = \frac{-\frac{I}{6} a_k k\sqrt{6}}{k+7}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+1} = \frac{-\frac{I}{6} a_k k\sqrt{6}}{k+7}\right]$$

- Revert the change of variables $u = x+1+\frac{I\sqrt{6}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}, a_{k+1} = \frac{-\frac{I}{6} a_k k\sqrt{6}}{k+7}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{5} a_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+1+\frac{I\sqrt{6}}{2}\right)^{k+6}\right), a_{k+1} = \frac{-\frac{I}{6} a_k (k-6)\sqrt{6}}{k+1}, b_{k+1} = \frac{-\frac{I}{6} b_k k\sqrt{6}}{k+7}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve((2*x^2+4*x+5)*diff(y(x),x$2)-20*(x+1)*diff(y(x),x)+60*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{7}{4}+x^5+5x^4+5x^3-5x^2-\frac{31}{4}x\right) + c_2\left(x^6+\frac{155}{8}-\frac{75}{2}x^4-100x^3-\frac{225}{4}x^2+30x\right)$$

✓ Solution by Mathematica

Time used: 0.917 (sec). Leaf size: 83

```
DSolve[(2*x^2+4*x+5)*y''[x]-20*(x+1)*y'[x]+60*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{(2x^2+4x+5)^{5/2}\left(4c_2(4x^5+20x^4+20x^3-20x^2-31x-7)+c_1\left(2ix+\sqrt{6}+2i\right)^6\right)}{(4x^2+8x+10)^{5/2}}$$

2.485 problem 499

2.485.1 Maple step by step solution 4485

Internal problem ID [7975]
Internal file name [OUTPUT/6908_Sunday_June_05_2022_05_16_33_PM_54899190/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 499.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^3+1)\, y''+7x^2y'+9yx=0}$$

Writing the ode as

$$(x^3+1)\, y''+7x^2y'+9yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+1 \\ B &= 7x^2 \\ C &= 9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x(x^3+8)}{4\left(x^3+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x\left(x^3+8\right)$$
$$t = 4\left(x^3+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{x(x^3+8)}{4\left(x^3+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 917: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+1)^2$. There is a pole at $x=-1$ of order 2. There is a pole at $x=\frac{1}{2}-\frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x=\frac{1}{2}+\frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n=1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{7}{36\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} + \frac{7}{36\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{36}+\frac{5i\sqrt{3}}{36}}{x-\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{-\frac{5}{36}-\frac{5i\sqrt{3}}{36}}{x-\frac{1}{2}+\frac{i\sqrt{3}}{2}} + \frac{5}{18(1+x)} + \frac{7}{36(1+x)^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x=\frac{1}{2}-\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x=\frac{1}{2}+\frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{x(x^3+8)}{4\left(x^3+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} + (-)(0) \\ &= -\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)} \\ &= -\frac{x^2}{2x^3+2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right)(1) + \left(\left(\frac{1}{6(1+x)^2} + \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2} + \right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{6(1+x)} - \frac{1}{6\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - \frac{1}{6\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{\ln(1+x)}{6} - \frac{\ln\left(4x^2-4x+4\right)}{6}} \\
&= \frac{x 2^{\frac{2}{3}}}{2\,(1+x)^{\frac{1}{6}}\,(x^2-x+1)^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2}{x^3+1}\, dx} \\
&= z_1 e^{-\frac{7\ln\left(x^3+1\right)}{6}} \\
&= z_1 \left(\frac{1}{(x^3+1)^{\frac{7}{6}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{7x^2}{x^3+1}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-\frac{7\ln(x^3+1)}{3}}}{(y_1)^2}\,dx$$
$$= y_1 \left(\int \frac{(x^3+1)^{\frac{1}{3}}\, 2^{\frac{2}{3}}}{x^2} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}}\right) + c_2 \left(\frac{2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}} \left(\int \frac{(x^3+1)^{\frac{1}{3}}\, 2^{\frac{2}{3}}}{x^2} dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}} + \frac{c_2 2^{\frac{1}{3}} x \left(\int \frac{(x^3+1)^{\frac{1}{3}}}{x^2} dx\right)}{(x^3+1)^{\frac{4}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 2^{\frac{2}{3}} x}{2\,(x^3+1)^{\frac{4}{3}}} + \frac{c_2 2^{\frac{1}{3}} x \left(\int \frac{(x^3+1)^{\frac{1}{3}}}{x^2} dx\right)}{(x^3+1)^{\frac{4}{3}}}$$

Verified OK.

2.485.1 Maple step by step solution

Let's solve

$(x^3+1)\, y'' + 7x^2 y' + 9yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{7x^2 y'}{x^3+1} - \frac{9xy}{x^3+1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{7x^2 y'}{x^3+1} + \frac{9xy}{x^3+1} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{7x^2}{x^3+1}, P_3(x) = \frac{9x}{x^3+1}\right]$$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $$\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $$x_0 = -1$$

- Multiply by denominators

$$(x^3+1)\, y'' + 7x^2 y' + 9yx = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(u^3 - 3u^2 + 3u)\left(\frac{d^2}{du^2} y(u)\right) + (7u^2 - 14u + 7)\left(\frac{d}{du} y(u)\right) + (9u - 9)\, y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..3$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}r(4+3r)\,u^{-1+r}+\left(a_{1}(1+r)\,(7+3r)-a_{0}(3r^{2}+11r+9)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(3k+7\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r(4+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{4}{3}\right\}$$

- Each term must be 0

$$a_{1}(1+r)\,(7+3r)-a_{0}(3r^{2}+11r+9)=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(3k+7+3r)-a_{k}(3k^{2}+6kr+3r^{2}+11k+11r+9)+a_{k-1}(k+2+r)^{2}=0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(3k+10+3r)-a_{k+1}\left(3(k+1)^{2}+6(k+1)\,r+3r^{2}+11k+20+11r\right)+a_{k}(k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{k^{2}a_{k}-3k^{2}a_{k+1}+2kra_{k}-6kra_{k+1}+r^{2}a_{k}-3r^{2}a_{k+1}+6ka_{k}-17ka_{k+1}+6ra_{k}-17ra_{k+1}+9a_{k}-23a_{k+1}}{(k+2+r)(3k+10+3r)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{k^{2}a_{k}-3k^{2}a_{k+1}+6ka_{k}-17ka_{k+1}+9a_{k}-23a_{k+1}}{(k+2)(3k+10)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{k^{2}a_{k}-3k^{2}a_{k+1}+6ka_{k}-17ka_{k+1}+9a_{k}-23a_{k+1}}{(k+2)(3k+10)},7a_{1}-9a_{0}=0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 6k a_k - 17k a_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1 - 9a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{4}{3}}, a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + \frac{10}{3} k a_k - 9k a_{k+1} + \frac{25}{9} a_k - \frac{17}{3} a_{k+1}}{\left(k+\frac{2}{3}\right)(3k+6)}, -a_1 + \frac{a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{k^2 a_k - 3k^2 a_{k+1} + 6k a_k - 17k a_{k+1} + 9a_k - 23a_{k+1}}{(k+2)(3k+10)}, 7a_1\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve((1+x^3)*diff(y(x),x$2)+7*x^2*diff(y(x),x)+9*x*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1x}{\left(x^3+1\right)^{\frac{4}{3}}}+\frac{c_2x\left(\int\frac{\left((x+1)\left(x^2-x+1\right)\right)^{\frac{1}{3}}}{x^2}dx\right)}{\left(x^3+1\right)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 0.798 (sec). Leaf size: 118

```
DSolve[(1+x^3)*y''[x]+7*x^2*y'[x]+9*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2\sqrt{3}c_2x\arctan\left(\frac{\sqrt{3}x}{2\sqrt[3]{x^3+1}+x}\right)-6c_2\sqrt[3]{x^3+1}-2c_2x\log\left(\sqrt[3]{x^3+1}-x\right)+c_2x\log\left(\sqrt[3]{x^3+1}x+(x^3-\right.}{6\left(x^3+1\right)^{4/3}}$$

2.486 problem 500

2.486.1 Maple step by step solution 4499

Internal problem ID [7976]
Internal file name [`OUTPUT/6909_Sunday_June_05_2022_05_16_37_PM_27176108/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 500.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^5+1)\,y''+14y'x^4+10yx^3=0}$$

Writing the ode as

$$(2x^5+1)\,y''+14y'x^4+10yx^3=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^5+1 \\ B &= 14x^4 \\ C &= 10x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^3(5x^5+6) \\ t &= (2x^5+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^3(5x^5+6)}{(2x^5+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 919: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 10 - 8 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^5+1)^2$. There is a pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = -\frac{2^{\frac{4}{5}}}{2}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ of order 2. There is a pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \text{Expression too large to display}$$

For the pole at $x = \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{2^{\frac{4}{5}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{2^{\frac{4}{5}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}$$

For the pole at $x = \frac{2^{\frac{4}{5}}}{8} - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + \frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8} + \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8} - \frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{100}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{10} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{10}
\end{aligned}$$

For the pole at $x=-\frac{2^{\frac{4}{5}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2^{\frac{4}{5}}}{2}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

For the pole at $x=\frac{2^{\frac{4}{5}}}{8}-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{\frac{4}{5}}}{8}+\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

For the pole at $x=\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$ let b be the coefficient of $\frac{1}{\left(x-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{2^{\frac{4}{5}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)^{2}}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{21}{100}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{10} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{3}{10}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty=0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{3x^3(5x^5+6)}{(2x^5+1)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^3(5x^5+6)}{(2x^5+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{2^{\frac{4}{5}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}}{8}-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8}+\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$-\frac{2^{\frac{4}{5}}}{2}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}}{8}-\frac{2^{\frac{4}{5}}\sqrt{5}}{8}-\frac{i\sqrt{5-\sqrt{5}}\,\sqrt{5}\,2^{\frac{3}{10}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$
$\frac{2^{\frac{4}{5}}\sqrt{5}}{8}+\frac{2^{\frac{4}{5}}}{8}-\frac{i2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}$	2	0	$\frac{7}{10}$	$\frac{3}{10}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^-\right) \\ &= \frac{5}{2} - \left(\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^-}{x-c_4}\right)$$

$$= \frac{3}{10\left(x - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{2^{\frac{4}{5}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\sqrt{5}\,2^{\frac{3}{10}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{\frac{4}{5}}}{2}\right.}$$

$$= \frac{3}{10\left(x - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{2^{\frac{4}{5}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\sqrt{5}\,2^{\frac{3}{10}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{\frac{4}{5}}}{2}\right.}$$

$$= \frac{3x^4}{2x^5+1}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = p e^{\int \omega\, dx}$$

$$= (x)\, e^{\int \left(\frac{3}{10\left(x - \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{2^{\frac{4}{5}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{4}\right)} + \frac{3}{10\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} - \frac{i\sqrt{5-\sqrt{5}}\sqrt{5}\,2^{\frac{3}{10}}}{8} - \frac{i 2^{\frac{3}{10}}\sqrt{5-\sqrt{5}}}{8}\right)} + \frac{3}{10\left(x + \frac{2^{\frac{4}{5}}}{2}\right)} + \frac{}{10\left(x - \frac{2^{\frac{4}{5}}}{8} + \frac{2^{\frac{4}{5}}\sqrt{5}}{8} + i\right.}\right.}$$

$$= (x)\, e^{\frac{3\ln\left(2^{\frac{4}{5}} + 2x\right)}{10} + \frac{3\ln\left(32\, 2^{\frac{3}{5}} - 16x 2^{\frac{4}{5}}\sqrt{5} - 16x 2^{\frac{4}{5}} + 64x^2\right)}{10} + \frac{3\ln\left(32\, 2^{\frac{3}{5}} + 16x 2^{\frac{4}{5}}\sqrt{5} - 16x 2^{\frac{4}{5}} + 64x^2\right)}{10} - \frac{3i\arctan\left(\frac{8x - 2^{\frac{4}{5}} + 2^{\frac{4}{5}}\sqrt{5}}{-\sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}\sqrt{5} - \sqrt{5-\sqrt{5}}\,2^{\frac{3}{10}}}\right.}{10}}$$

$$= 4x\left(2^{\frac{4}{5}} + 2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}} + 4x^2 + 2\, 2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}} + 4x^2 + 2\, 2^{\frac{3}{5}}\right)^{\frac{3}{10}}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{14x^4}{2x^5+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^5+1\right)}{10}} \\
&= z_1 \left(\frac{1}{\left(2x^5+1\right)^{\frac{7}{10}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{4x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{14x^4}{2x^5+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^5+1\right)}{5}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{1}{5}}}{32x^2\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}}\left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{5}}\left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{5}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{4x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} \right)$$

$$+ c_2 \left(\frac{4x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} \left(\int \frac{}{32x^2 \left(\cdots \right.} \right.\right.$$

Summary

The solution(s) found are the following

$$y = \frac{4c_1 x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} + \frac{c_2 x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{3}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(\int \frac{}{\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}} \left(- \cdots \right.} \right.}{8\left(2x^5+1\right)^{\frac{7}{10}}} \tag{1}$$

Verification of solutions

$$y = \frac{4c_1 x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{2}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}}}{\left(2x^5+1\right)^{\frac{7}{10}}} + \frac{c_2 x\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{10}} 2^{\frac{3}{5}} \left(-x\left(\sqrt{5}+1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(x\left(\sqrt{5}-1\right)2^{\frac{4}{5}}+4x^2+2\,2^{\frac{3}{5}}\right)^{\frac{3}{10}} \left(\int \frac{}{\left(2^{\frac{4}{5}}+2x\right)^{\frac{3}{5}} \left(- \cdots \right.} \right.}{8\left(2x^5+1\right)^{\frac{7}{10}}}$$

Verified OK.

2.486.1 Maple step by step solution

Let's solve

$(2x^5+1)\, y''+14y'x^4+10yx^3=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{14x^4y'}{2x^5+1}-\frac{10x^3y}{2x^5+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{14x^4y'}{2x^5+1}+\frac{10x^3y}{2x^5+1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{14x^4}{2x^5+1}, P_3(x)=\frac{10x^3}{2x^5+1}\right]$

  ◦ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

  $\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

  ◦ $\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}\right.$

  $\left(\left(x-\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)\right.\right.$

  ◦ $x=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$ is

  Check to see if x_0 is a regular singular point

  $x_0=\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}+\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}+\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\sin\left(\frac{\pi}{5}\right)\right)2^{\frac{4}{5}}$

- Multiply by denominators

$(2x^5+1)\, y''+14y'x^4+10yx^3=0$

- Change variables using $x = u + \left(\left(\left(-\frac{\sqrt{5}}{8} - \frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right) - \frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right) 2^{\frac{4}{5}} + \mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\cos\left(\frac{\pi}{5}\right)}{8}\right.\right.$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..3$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- > k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..4$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 0..5$

    $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$

  - Shift index using $k- > k + 2 - m$

    $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 \cos\left(\frac{\pi}{5}\right) - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - \mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 + 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2 \sin\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) + 1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 \cos\left(\frac{\pi}{5}\right) - 4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^3 - \mathrm{I}\sin\left(\frac{\pi}{5}\right)^3 + 3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^2 \sin\right.$

- Values of r that satisfy the indicial equation

  $r = r$

- The coefficients of each power of u must be 0

$$\left[-\left(\mathrm{I}\sin\left(\frac{\pi}{5}\right)+\cos\left(\frac{\pi}{5}\right)+1\right)\left(4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)^{3}\cos\left(\frac{\pi}{5}\right)-4\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)^{3}-\mathrm{I}\sin\left(\frac{\pi}{5}\right)^{3}+3\,\mathrm{I}\cos\left(\frac{\pi}{5}\right)^{2}\mathrm{s}\right.\right.$$

- Solve for the dependent coefficient(s)
- Each term in the series must be 0, giving the recursion relation

$$\frac{5\,\mathrm{I}\left(-a_{k+1}\left(\sqrt{5}+1\right)\left(k+r+\frac{7}{5}\right)(k+r+1)2^{\frac{1}{5}}-\left(\sqrt{5}+1\right)\left(k^{2}+\left(2r+\frac{3}{5}\right)k+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k-2}2^{\frac{4}{5}}+4a_{k}2^{\frac{2}{5}}\left(k^{2}+\left(2r+\frac{9}{5}\right)k+r^{2}+\frac{9r}{5}+\frac{1}{2}\right)+4\,2^{\frac{3}{5}}\left(\right.\right.}{4}$$

- Shift index using $k->k+3$

$$\frac{5\,\mathrm{I}\left(-a_{k+4}\left(\sqrt{5}+1\right)\left(k+\frac{22}{5}+r\right)(k+4+r)2^{\frac{1}{5}}-\left(\sqrt{5}+1\right)\left((k+3)^{2}+\left(2r+\frac{3}{5}\right)(k+3)+r^{2}+\frac{3r}{5}-\frac{11}{5}\right)a_{k+1}2^{\frac{4}{5}}+4a_{k+3}2^{\frac{2}{5}}\left((k+3)^{2}+\left(2r+\frac{9}{5}\right)(k\right.\right.}{4}$$

- Recursion relation that defines series solution to ODE

$$a_{k+4}=-\frac{2\left(-78\,\mathrm{I}\sqrt{5-\sqrt{5}}\,a_{k+3}2^{\frac{9}{10}}r-10\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}k^{2}-10\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}r^{2}+5\,\mathrm{I}\sqrt{25-5\sqrt{5}}\,2^{\frac{3}{10}}a_{k+1}k^{2}+5\,\mathrm{I}\sqrt{25-5\sqrt{5}}\,2^{\frac{3}{10}}\right.}{}$$

- Recursion relation for $r=r$

$$a_{k+4}=-\frac{2\left(-78\,\mathrm{I}\sqrt{5-\sqrt{5}}\,a_{k+3}2^{\frac{9}{10}}r-10\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}k^{2}-10\,\mathrm{I}\sin\left(\frac{\pi}{5}\right)a_{k+1}2^{\frac{4}{5}}r^{2}+5\,\mathrm{I}\sqrt{25-5\sqrt{5}}\,2^{\frac{3}{10}}a_{k+1}k^{2}+5\,\mathrm{I}\sqrt{25-5\sqrt{5}}\,2^{\frac{3}{10}}\right.}{}$$

- Solution for $r=r$
- Revert the change of variables $u=x-\left(\left(\left(-\frac{\sqrt{5}}{8}-\frac{1}{8}\right)\cos\left(\frac{\pi}{5}\right)-\frac{\sqrt{2}\,\sqrt{5-\sqrt{5}}\,\sin\left(\frac{\pi}{5}\right)}{8}\right)2^{\frac{4}{5}}-\mathrm{I}\left(\frac{\sqrt{2}\,\sqrt{5-}}{}\right.\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve((1+2*x^5)*diff(y(x),x$2)+14*x^4*diff(y(x),x)+10*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^5+1)^{\frac{2}{5}}} + \frac{c_2 x \left(\int \frac{1}{(2x^5+1)^{\frac{3}{5}} x^2} dx\right)}{(2x^5+1)^{\frac{2}{5}}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(1+2*x^5)*y''[x]+14*x^4*y'[x]+10*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.487 problem 501

2.487.1 Maple step by step solution 4510

Internal problem ID [7977]
Internal file name [OUTPUT/6910_Sunday_June_05_2022_05_16_41_PM_78236690/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 501.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x^6 + 7yx^5 = 0}$$

Writing the ode as

$$y'' + y'x^6 + 7yx^5 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^6 \\ C &= 7x^5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^5(x^7 - 16)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^5(x^7 - 16) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^5(x^7 - 16)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 921: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 12 \\ &= -12 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -12 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -12$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{12}{2} = 6$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{6} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^6$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^6}{2} - \frac{4}{x} - \frac{16}{x^8} - \frac{128}{x^{15}} - \frac{1280}{x^{22}} - \frac{14336}{x^{29}} - \frac{172032}{x^{36}} - \frac{2162688}{x^{43}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 6$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{6} a_i x^i \\ &= \frac{x^6}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^5 = x^5$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{12}}{4}$$

This shows that the coefficient of x^5 in the above is 0. Now we need to find the coefficient of x^5 in r. How this is done depends on if $v = 0$ or not. Since $v = 6$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^5 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^5(x^7 - 16)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^{12} - 4x^5\right) + (0) \\ &= \frac{1}{4}x^{12} - 4x^5 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -4. Now b can be found.

$$\begin{aligned} b &= (-4) - (0) \\ &= -4 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^6}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-4}{\frac{1}{2}} - 6\right) &= -7 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-4}{\frac{1}{2}} - 6\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^5(x^7 - 16)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-12	$\frac{x^6}{2}$	-7	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-) \left(\frac{x^6}{2} \right) \\ &= -\frac{x^6}{2} \\ &= -\frac{x^6}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^6}{2}\right)(1) + \left((-3x^5) + \left(-\frac{x^6}{2}\right)^2 - \left(\frac{x^5(x^7 - 16)}{4}\right)\right) = 0$$
$$x^5 a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x)\, \mathrm{e}^{\int -\frac{x^6}{2} dx} \\ &= (x)\, \mathrm{e}^{-\frac{x^7}{14}} \\ &= x\, \mathrm{e}^{-\frac{x^7}{14}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^6}{1} \, dx} \\ &= z_1 e^{-\frac{x^7}{14}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^7}{14}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^7}{7}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^6}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^7}{7}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-7\, \mathrm{e}^{\frac{x^7}{7}} (-x^7)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}} \left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{7 (-x^7)^{\frac{6}{7}}\, x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^7}{7}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^7}{7}}\left(\frac{-7\,\mathrm{e}^{\frac{x^7}{7}}\left(-x^7\right)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}\,x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^7}{7}} + \frac{c_2\left(-7\left(-x^7\right)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^7}{7}} + \frac{c_2\left(-7\left(-x^7\right)^{\frac{6}{7}} + x^7 7^{\frac{6}{7}}\mathrm{e}^{-\frac{x^7}{7}}\left(\Gamma\left(\frac{6}{7}\right) - \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right)\right)\right)}{7\left(-x^7\right)^{\frac{6}{7}}}
$$

Verified OK.

2.487.1 Maple step by step solution

Let's solve

$y'' + y'x^6 + 7yx^5 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^5 \cdot y$ to series expansion

    $x^5 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+5}$

  ◦ Shift index using $k- >k-5$

    $x^5 \cdot y = \sum_{k=5}^{\infty} a_{k-5} x^k$

- Convert $x^6 \cdot y'$ to series expansion

  $$x^6 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+5}$$

- Shift index using $k- >k-5$

  $$x^6 \cdot y' = \sum_{k=5}^{\infty} a_{k-5}(k-5)\, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=5}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-5}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0, 30a_6 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)\,(ka_{k+2} + a_{k-5} + a_{k+2}) = 0$$

- Shift index using $k- >k+5$

  $$(k+7)\,((k+5)\,a_{k+7} + a_k + a_{k+7}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+7} = -\frac{a_k}{k+6}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve(diff(y(x),x$2)+x^6*diff(y(x),x)+7*x^5*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^7}{7}} x + \frac{7 c_2 (-1)^{\frac{6}{7}} \mathrm{e}^{-\frac{x^7}{7}} \left(-\Gamma\left(\frac{6}{7}\right) x^7 + (-x^7)^{\frac{6}{7}} 7^{\frac{1}{7}} \mathrm{e}^{\frac{x^7}{7}} + \Gamma\left(\frac{6}{7}, -\frac{x^7}{7}\right) x^7\right)}{(-x^7)^{\frac{6}{7}}}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 53

```
DSolve[y''[x]+x^6*y'[x]+7*x^5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{49} e^{-\frac{x^7}{7}} \left(49 c_1 x - 7^{6/7} c_2 \sqrt[7]{-x^7} \Gamma\left(-\frac{1}{7}, -\frac{x^7}{7}\right)\right)$$

2.488 problem 502

Internal problem ID [7978]
Internal file name [`OUTPUT/6911_Sunday_June_05_2022_05_16_45_PM_37466547/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 502.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^8+1)\, y''-16y'x^7+72yx^6=0}$$

Writing the ode as

$$(x^8+1)\, y''-16y'x^7+72yx^6=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^8+1\\ B &= -16x^7\\ C &= 72x^6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-128x^6}{(x^8+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -128x^6 \\ t &= (x^8+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{128x^6}{(x^8+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 923: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 16 - 6 \\ &= 10 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^8+1)^2$. There is a pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. There is a pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 10 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 10 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$\begin{aligned} r = {} & \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\ & + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} \\ & + \frac{2}{\left(x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} + \frac{2}{\left(x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^2} \\ & + \frac{2}{\left(x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} \\ & + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} + \frac{2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}} \\ & + \frac{2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2+\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2+\sqrt{2}}}{2}} + \frac{2\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^7}{x - \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}} \end{aligned}$$

For the pole at $x = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$

in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} + \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x = -\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

For the pole at $x=-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\,\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

For the pole at $x=\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{\sqrt{2}\,\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=2 \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

Since the order of r at ∞ is $10>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{128x^6}{(x^8+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}+\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$-\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1
$\frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2}+\frac{\sqrt{2-\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
10	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^- + \alpha_{c_4}^- + \alpha_{c_5}^- + \alpha_{c_6}^- + \alpha_{c_7}^- + \alpha_{c_8}^+\right) \\ &= 1 - (-5) \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + \left((-)[\sqrt{r}]_{c_4} + \frac{\alpha_{c_4}^{-}}{x-c_4}\right)$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= -\frac{1}{x - \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x - \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} - \frac{i\sqrt{2-\sqrt{2}}}{2}} - \frac{1}{x + \frac{\sqrt{2-\sqrt{2}}}{2} - \frac{i\sqrt{2-\sqrt{2}}\sqrt{2}}{2} -}$$

$$= \frac{\left((3x^6 - 3ix^4 - 3ix^2 - 3)\sqrt{2} - 3(x^2+1)\left((-1+i)x^4 + 1 + i\right)\right)\sqrt{2-\sqrt{2}} - 3\Big(\left((-1+i)x^4 + 1 + i\right)}{2\left(x^2 - x\sqrt{2-\sqrt{2}} + 1\right)\left(x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(-x\left(1+\sqrt{2}\right)\right.}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) and Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{i\sqrt{2} - 1 + i}{1 + i + i\sqrt{2}}, a_1 = \frac{\left(\frac{12}{7} - \frac{12i}{7}\right)\sqrt{2}}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}}, a_2 = \frac{\frac{15}{7} - \frac{15i}{7} + \frac{15\sqrt{2}}{7}}{1 + i + i\sqrt{2}}, a_3 = \frac{32}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{}}}\right.$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 - i + \sqrt{2}\right)}{7\left(1 + i + i\sqrt{}\right.}$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega\, dx}$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 - i + \right.}{7\left(1 + i + \right.}\right.$$

$$= \left(x^6 + \frac{\left(\frac{12}{7} + \frac{12i}{7}\right)\sqrt{2}\,x^5}{\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 + i + \sqrt{2}\right)x^4}{7\left(1 + i + i\sqrt{2}\right)} + \frac{32x^3}{7\left(1 + i + i\sqrt{2}\right)\sqrt{2-\sqrt{2}}} + \frac{15\left(1 - i + \right.}{7\left(1 + i + \right.}\right.$$

$$= -\frac{\left(\left(\left(ix^6 + \frac{15}{7}x^4 + \frac{15}{7}x^2 - i\right)\sqrt{2} + (1+i)x^6 + \left(\frac{15}{7} + \frac{15i}{7}\right)x^4 + \left(\frac{15}{7} - \frac{15i}{7}\right)x^2 + 1 - i\right)\sqrt{2-\sqrt{2}} + \right.}{256\sqrt{2-\sqrt{2}}\left(-1 + i - \sqrt{2}\right)\left(x\sqrt{2-\sqrt{2}} + x^2 + 1\right)\left(-x\left(1+\sqrt{2}\right)\sqrt{2-\sqrt{2}} + x^2 + 1\right)\Big(}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-16x^7}{x^8+1}\,dx} \\
&= z_1 e^{\ln(x^8+1)} \\
&= z_1\left(x^8+1\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{7i\sqrt{2}\,x^9\sqrt{2-\sqrt{2}}+7i\sqrt{2-\sqrt{2}}\,x^9+7\sqrt{2-\sqrt{2}}\,x^9-9i\sqrt{2}\,x^8-9\sqrt{2}\,x^8-9i\sqrt{2-\sqrt{2}}\,\sqrt{2}\,x-9i\sqrt{2-}}{224\sqrt{2}\,\sqrt{2-\sqrt{2}}-224i\sqrt{2-\sqrt{2}}+224\sqrt{2-\sqrt{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-16x^7}{x^8+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x^8+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-1032192x^8+802816}{108\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3\left(\left(\left(-x^4+\frac{4}{3}ix^2+i\right)\sqrt{2}+(-1+i)\,x^4+\left(-\frac{4}{3}+\frac{4i}{3}\right)x^2+1\right.\right.} \right.
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{7i\sqrt{2}\,x^9\sqrt{2-\sqrt{2}}+7i\sqrt{2-\sqrt{2}}\,x^9+7\sqrt{2-\sqrt{2}}\,x^9-9i\sqrt{2}\,x^8-9\sqrt{2}\,x^8-9i\sqrt{2-\sqrt{2}}\,\sqrt{2}\,x-9i}{224\sqrt{2}\,\sqrt{2-\sqrt{2}}-224i\sqrt{2-\sqrt{2}}+224\sqrt{2-\sqrt{2}}}\right. \\
&\quad + c_2\left(\frac{7i\sqrt{2}\,x^9\sqrt{2-\sqrt{2}}+7i\sqrt{2-\sqrt{2}}\,x^9+7\sqrt{2-\sqrt{2}}\,x^9-9i\sqrt{2}\,x^8-9\sqrt{2}\,x^8-9i\sqrt{2-\sqrt{2}}\,\sqrt{2}\,x-9}{224\sqrt{2}\,\sqrt{2-\sqrt{2}}-224i\sqrt{2-\sqrt{2}}+224\sqrt{2-\sqrt{2}}}\right.
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(7i\sqrt{2}\,x^9\sqrt{2-\sqrt{2}}+7i\sqrt{2-\sqrt{2}}\,x^9+7\sqrt{2-\sqrt{2}}\,x^9-9i\sqrt{2}\,x^8-9\sqrt{2}\,x^8-9i\sqrt{2-\sqrt{2}}\,\sqrt{2}\,x-9i\sqrt{2}\right.}{224\sqrt{2}\,\sqrt{2-\sqrt{2}}-224i\sqrt{2-\sqrt{2}}+224\sqrt{2-\sqrt{2}}} - \frac{256c_2\left(x^8-\frac{7}{9}\right)\left(\left(-1+i-\sqrt{2}\right)x\left(x^8-\frac{9}{7}\right)\sqrt{2-\sqrt{2}}+\left(\frac{9}{7}-\frac{9i}{7}\right)\left(\right.\right.}{\sqrt{2-\sqrt{2}}\,\left(-1+i-\sqrt{2}\right)\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3\left(\frac{6\left(1-i+\sqrt{2}\,(ix^4+\frac{4}{3}x^2+1)+(1+i)x^4+(\frac{4}{3}+\frac{4i}{3})x^2\right)x\sqrt{}}{7}\right.} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(7i\sqrt{2}\,x^9\sqrt{2-\sqrt{2}}+7i\sqrt{2-\sqrt{2}}\,x^9+7\sqrt{2-\sqrt{2}}\,x^9-9i\sqrt{2}\,x^8-9\sqrt{2}\,x^8-9i\sqrt{2-\sqrt{2}}\,\sqrt{2}\,x-9i\sqrt{2}\right.}{224\sqrt{2}\,\sqrt{2-\sqrt{2}}-224i\sqrt{2-\sqrt{2}}+224\sqrt{2-\sqrt{2}}} - \frac{256c_2\left(x^8-\frac{7}{9}\right)\left(\left(-1+i-\sqrt{2}\right)x\left(x^8-\frac{9}{7}\right)\sqrt{2-\sqrt{2}}+\left(\frac{9}{7}-\frac{9i}{7}\right)\left(\right.\right.}{\sqrt{2-\sqrt{2}}\,\left(-1+i-\sqrt{2}\right)\left(\left(-1+i-\sqrt{2}\right)\sqrt{2-\sqrt{2}}+2x\right)^3\left(\frac{6\left(1-i+\sqrt{2}\,(ix^4+\frac{4}{3}x^2+1)+(1+i)x^4+(\frac{4}{3}+\frac{4i}{3})x^2\right)x\sqrt{}}{7}\right.}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1+x^8)*diff(y(x),x$2)-16*x^7*diff(y(x),x)+72*x^6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \left(-\frac{7}{9} + x^8\right) + c_2 \left(x^9 - \frac{9}{7}x\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(1+x^8)*y''[x]-16*x^7*y'[x]+72*x^6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

2.489 problem 503

2.489.1 Maple step by step solution 4529

Internal problem ID [7979]
Internal file name [`OUTPUT/6912_Sunday_June_05_2022_05_17_05_PM_39183091/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 503.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + y'x^5 + 6yx^4 = 0}$$

Writing the ode as

$$y'' + y'x^5 + 6yx^4 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^5 \\ C &= 6x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4(x^6 - 14)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4(x^6 - 14) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4(x^6 - 14)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 924: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 10 \\ &= -10 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -10 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -10$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{10}{2} = 5$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{5} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^5$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^5}{2} - \frac{7}{2x} - \frac{49}{4x^7} - \frac{343}{4x^{13}} - \frac{12005}{16x^{19}} - \frac{117649}{16x^{25}} - \frac{2470629}{32x^{31}} - \frac{27176919}{32x^{37}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 5$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{5} a_i x^i \\ &= \frac{x^5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^4 = x^4$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^{10}}{4}$$

This shows that the coefficient of x^4 in the above is 0. Now we need to find the coefficient of x^4 in r. How this is done depends on if $v = 0$ or not. Since $v = 5$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x^4 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4(x^6 - 14)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^{10} - \frac{7}{2}x^4\right) + (0) \\
&= \frac{1}{4}x^{10} - \frac{7}{2}x^4
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{7}{2}\right) - (0) \\
&= -\frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^5}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &= -6 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 5\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4(x^6 - 14)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-10	$\frac{x^5}{2}$	-6	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x^5}{2}\right) \\
&= -\frac{x^5}{2} \\
&= -\frac{x^5}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{x^5}{2}\right)(1) + \left(\left(-\frac{5x^4}{2}\right) + \left(-\frac{x^5}{2}\right)^2 - \left(\frac{x^4(x^6-14)}{4}\right)\right) &= 0 \\
x^4 a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x) \, \mathrm{e}^{\int -\frac{x^5}{2} dx} \\ &= (x) \, \mathrm{e}^{-\frac{x^6}{12}} \\ &= x \, \mathrm{e}^{-\frac{x^6}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^5}{1} \, dx} \\ &= z_1 e^{-\frac{x^6}{12}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^6}{12}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x \, \mathrm{e}^{-\frac{x^6}{6}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^5}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^6}{6}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-6 \, \mathrm{e}^{\frac{x^6}{6}} \left(-x^6\right)^{\frac{5}{6}} + 6^{\frac{5}{6}} x^6 \left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right) \right)}{6 \left(-x^6\right)^{\frac{5}{6}} x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^6}{6}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^6}{6}}\left(\frac{-6\,\mathrm{e}^{\frac{x^6}{6}}\left(-x^6\right)^{\frac{5}{6}} + 6^{\frac{5}{6}} x^6\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}} x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^6}{6}} + \frac{c_2\left(-6(-x^6)^{\frac{5}{6}} + x^6 6^{\frac{5}{6}}\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^6}{6}} + \frac{c_2\left(-6(-x^6)^{\frac{5}{6}} + x^6 6^{\frac{5}{6}}\mathrm{e}^{-\frac{x^6}{6}}\left(\Gamma\left(\frac{5}{6}\right) - \Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right)\right)\right)}{6\left(-x^6\right)^{\frac{5}{6}}}
$$

Verified OK.

2.489.1 Maple step by step solution

Let's solve

$y'' + y'x^5 + 6yx^4 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^4 \cdot y$ to series expansion

    $x^4 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+4}$

  ◦ Shift index using $k- >k-4$

    $x^4 \cdot y = \sum_{k=4}^{\infty} a_{k-4} x^k$

- Convert $x^5 \cdot y'$ to series expansion

  $$x^5 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+4}$$

- Shift index using $k- > k - 4$

  $$x^5 \cdot y' = \sum_{k=4}^{\infty} a_{k-4}(k-4) \, x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- > k + 2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$$

  Rewrite ODE with series expansions

  $$20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=4}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_{k-4}(k+2)\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0, 20a_5 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+2)(ka_{k+2} + a_{k-4} + a_{k+2}) = 0$$

- Shift index using $k- > k + 4$

  $$(k+6)((k+4)a_{k+6} + a_k + a_{k+6}) = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{a_k}{k+5}, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
dsolve(diff(y(x),x$2)+x^5*diff(y(x),x)+6*x^4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^6}{6}} x - \frac{2c_2 \mathrm{e}^{-\frac{x^6}{6}} \left(6^{\frac{2}{3}} \sqrt{3}\, \sqrt{2}\, (-x^6)^{\frac{5}{6}} \mathrm{e}^{\frac{x^6}{6}} + 6\Gamma\left(\frac{5}{6}, -\frac{x^6}{6}\right) x^6 - 6\Gamma\left(\frac{5}{6}\right) x^6\right)}{(-x^6)^{\frac{5}{6}} \left(\sqrt{3} + i\right)}$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 53

```
DSolve[y''[x]+x^5*y'[x]+6*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36} e^{-\frac{x^6}{6}} \left(36 c_1 x - 6^{5/6} c_2 \sqrt[6]{-x^6}\, \Gamma\left(-\frac{1}{6}, -\frac{x^6}{6}\right)\right)$$

2.490 problem 504

2.490.1 Maple step by step solution 4539

Internal problem ID [7980]
Internal file name [OUTPUT/6913_Sunday_June_05_2022_05_17_09_PM_45393394/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 504.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x+1)\,y''+xy'+2y=0}$$

Writing the ode as

$$(3x+1)\,y''+xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x+1\\ B&=x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 24x - 6}{4 (3x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 24x - 6$$
$$t = 4(3x + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 24x - 6}{4 (3x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 926: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x+1)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{324\left(x+\frac{1}{3}\right)^2} - \frac{37}{54\left(x+\frac{1}{3}\right)}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{19}{324}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{18} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{18}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} - \frac{37}{18x} - \frac{319}{27x^2} - \frac{11831}{81x^3} - \frac{2157901}{972x^4} - \frac{110035199}{2916x^5} - \frac{1501983319}{2187x^6} - \frac{85889060456}{6561x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 24x - 6}{36x^2 + 24x + 4} \\
&= Q + \frac{R}{36x^2 + 24x + 4} \\
&= \left(\frac{1}{36}\right) + \left(\frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}\right) \\
&= \frac{1}{36} + \frac{-\frac{74x}{3} - \frac{55}{9}}{36x^2 + 24x + 4}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-\frac{74}{3}$. Dividing this by leading coefficient in t which is 36 gives $-\frac{37}{54}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{37}{54}\right) - (0) \\
&= -\frac{37}{54}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= -\frac{37}{18} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{37}{54}}{\frac{1}{6}} - 0\right) &&= \frac{37}{18}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 24x - 6}{4\,(3x+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$-\frac{1}{3}$	2	0	$\frac{19}{18}$	$-\frac{1}{18}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$-\frac{37}{18}$	$\frac{37}{18}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{37}{18}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{37}{18} - \left(\frac{19}{18}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} + (-)\left(\frac{1}{6}\right) \\
&= \frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6} \\
&= -\frac{x-6}{2\,(3x+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)(1) + \left(\left(-\frac{19}{18\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right)^2 - \left(\frac{x^2 - 24x - 6}{4\,(3x+1)^2}\right)\right) = 0
$$

$$
\frac{a_0 + 6}{3x+1} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -6\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 6
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-6)\,\mathrm{e}^{\int \left(\frac{19}{18\left(x+\frac{1}{3}\right)} - \frac{1}{6}\right) dx} \\
&= (x-6)\,\mathrm{e}^{-\frac{x}{6} + \frac{19\ln(3x+1)}{18}} \\
&= (x-6)\,(3x+1)^{\frac{19}{18}}\,\mathrm{e}^{-\frac{x}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{3x+1}\,dx} \\
&= z_1 e^{-\frac{x}{6}+\frac{\ln(3x+1)}{18}} \\
&= z_1\left((3x+1)^{\frac{1}{18}}\, \mathrm{e}^{-\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-6)\,(3x+1)^{\frac{10}{9}}\, \mathrm{e}^{-\frac{x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{3x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}+\frac{\ln(3x+1)}{9}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-6)\,(3x+1)^{\frac{10}{9}}\, \mathrm{e}^{-\frac{x}{3}}\right) + c_2\left((x-6)\,(3x+1)^{\frac{10}{9}}\, \mathrm{e}^{-\frac{x}{3}}\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-6)\,(3x+1)^{\frac{10}{9}}\, \mathrm{e}^{-\frac{x}{3}} + c_2(x-6)\,(3x+1)^{\frac{10}{9}}\, \mathrm{e}^{-\frac{x}{3}}\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2\,(3x+1)^{\frac{19}{9}}}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1(x-6)\left(3x+1\right)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}+c_2(x-6)\left(3x+1\right)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}\left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{\left(x-6\right)^{2}\left(3x+1\right)^{\frac{19}{9}}}dx\right)$$

Verified OK.

2.490.1 Maple step by step solution

Let's solve

$\left(3x+1\right)y''+xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{3x+1}-\frac{xy'}{3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{3x+1}+\frac{2y}{3x+1}=0$

□ Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x}{3x+1}, P_3(x)=\frac{2}{3x+1}\right]$

  ◦ $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=-\frac{1}{9}$

  ◦ $\left(x+\frac{1}{3}\right)^{2}\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

    $\left(\left(x+\frac{1}{3}\right)^{2}\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

  ◦ $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $\left(3x+1\right)y''+xy'+2y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

$$3u\left(\frac{d^2}{du^2}y(u)\right)+\left(u-\frac{1}{3}\right)\left(\frac{d}{du}y(u)\right)+2y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$$

- ○ Shift index using $k->k+1$

$$u\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{a_0 r(-10+9r)u^{-1+r}}{3}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(9k-1+9r)}{3}+a_k(k+r+2)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{r(-10+9r)}{3}=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{10}{9}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k-\frac{1}{9}+r\right)(k+1+r)\,a_{k+1}+a_k(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{3a_k(k+r+2)}{(9k-1+9r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=-\frac{3a_k(k+2)}{(9k-1)(k+1)}$$

- Solution for $r=0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^k, a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}\right]$$

- Recursion relation for $r = \frac{10}{9}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}$$

- Solution for $r = \frac{10}{9}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}, a_{k+1} = -\frac{3a_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{3}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{3}\right)^{k+\frac{10}{9}}\right), a_{k+1} = -\frac{3a_k(k+2)}{(9k-1)(k+1)}, b_{k+1} = -\frac{3b_k\left(k+\frac{28}{9}\right)}{(9k+9)\left(k+\frac{19}{9}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve((1+3*x)*diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}(x-6) + c_2(3x+1)^{\frac{10}{9}} \mathrm{e}^{-\frac{x}{3}}(x-6) \left(\int \frac{\mathrm{e}^{\frac{x}{3}}}{(x-6)^2 (3x+1)^{\frac{19}{9}}} dx\right)$$

✓ Solution by Mathematica

Time used: 0.9 (sec). Leaf size: 124

```
DSolve[(1+3*x)*y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{x}{3}-\frac{1}{9}}\left(1520c_1\sqrt[9]{3x+1}(3x^2-17x-6)-2^{8/9}c_2e^{\frac{x}{3}+\frac{1}{9}}(9x^2-48x-26)+2^{8/9}3^{7/9}c_2\sqrt[9]{-3x-1}(3x^2-17\right.}{380\ 2^{17/18}}$$

2.491 problem 505

2.491.1 Maple step by step solution 4550

Internal problem ID [7981]
Internal file name [OUTPUT/6914_Sunday_June_05_2022_05_17_13_PM_10232347/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 505.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0}$$

Writing the ode as

$$(3x^2+x+1)\,y''+(2+15x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2+x+1 \\ B &= 2+15x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^2 - 12x - 18$$
$$t = 4\left(3x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 928: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(3x^2 + x + 1)^2$. There is a pole at $x = \frac{i\sqrt{11}}{6} - \frac{1}{6}$ of order 2. There is a pole at $x = -\frac{i\sqrt{11}}{6} - \frac{1}{6}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{3i\sqrt{11}}{88} + \frac{27}{88}}{\left(x - \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2} + \frac{-\frac{3i\sqrt{11}}{88} + \frac{27}{88}}{\left(x + \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2} + \frac{57i\sqrt{11}}{242\left(x - \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)} - \frac{57i\sqrt{11}}{242\left(x + \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)}$$

For the pole at $x = \frac{i\sqrt{11}}{6} - \frac{1}{6}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3i\sqrt{11}}{88} + \frac{27}{88}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 + 66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 + 66i\sqrt{11}}}{44}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{11}}{6} - \frac{1}{6}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3i\sqrt{11}}{88} + \frac{27}{88}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{1078 - 66i\sqrt{11}}}{44} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{1078 - 66i\sqrt{11}}}{44}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^2 - 12x - 18}{4\left(3x^2 + x + 1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-9x^2-12x-18}{4\left(3x^2+x+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{i\sqrt{11}}{6}-\frac{1}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078+66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078+66i\sqrt{11}}}{44}$
$-\frac{i\sqrt{11}}{6}-\frac{1}{6}$	2	0	$\frac{1}{2}+\frac{\sqrt{1078-66i\sqrt{11}}}{44}$	$\frac{1}{2}-\frac{\sqrt{1078-66i\sqrt{11}}}{44}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{1}{2}-\left(-\frac{1}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\omega = \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty$$
$$= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x - \frac{i\sqrt{11}}{6} + \frac{1}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{i\sqrt{11}}{6} + \frac{1}{6}} + (-)(0)$$
$$= \frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x - \frac{i\sqrt{11}}{6} + \frac{1}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{i\sqrt{11}}{6} + \frac{1}{6}}$$
$$= -\frac{3x}{6x^2 + 2x + 2}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x - \frac{i\sqrt{11}}{6} + \frac{1}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{i\sqrt{11}}{6} + \frac{1}{6}}\right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{\left(x - \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2} - \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{\left(x + \frac{i\sqrt{11}}{6} + \frac{1}{6}\right)^2}\right) + \left(\frac{1}{2}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = p e^{\int \omega \, dx}$$
$$= (x)\, e^{\int \left(\frac{\frac{1}{2} - \frac{\sqrt{1078+66i\sqrt{11}}}{44}}{x - \frac{i\sqrt{11}}{6} + \frac{1}{6}} + \frac{\frac{1}{2} - \frac{\sqrt{1078-66i\sqrt{11}}}{44}}{x + \frac{i\sqrt{11}}{6} + \frac{1}{6}}\right) dx}$$
$$= (x)\, e^{\frac{\ln(36x^2+12x+12)}{2} - \frac{\sqrt{1078-66i\sqrt{11}}\, \ln(36x^2+12x+12)}{88} + \frac{i\sqrt{1078-66i\sqrt{11}}\, \arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{44} - \frac{\sqrt{1078+66i\sqrt{11}}\, \ln(36x^2+12x+12)}{88} - \frac{i\sqrt{1}}{}}$$
$$= \frac{x\sqrt{2}\, 3^{\frac{3}{4}} e^{\frac{\sqrt{11}\, \arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}}}{6\,(3x^2 + x + 1)^{\frac{1}{4}}}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2+15x}{3x^2+x+1}\,dx} \\ &= z_1 e^{-\frac{5\ln\left(3x^2+x+1\right)}{4}+\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}} \\ &= z_1\left(\frac{\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{22}}}{\left(3x^2+x+1\right)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x\sqrt{2}\,3^{\frac{3}{4}}\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2+15x}{3x^2+x+1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{5\ln\left(3x^2+x+1\right)}{2}+\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{(y_1)^2}\,dx \\ &= y_1\left(\int \frac{2\sqrt{3}\,\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2}dx\right) \end{aligned}$$

Therefore the solution is

$$
y = c_1 y_1 + c_2 y_2
$$

$$
= c_1 \left(\frac{x\sqrt{2}\,3^{\frac{3}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}}\right) + c_2 \left(\frac{x\sqrt{2}\,3^{\frac{3}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}}\left(\int \frac{2\sqrt{3}\,\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2}dx\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{2}\,3^{\frac{3}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}} + \frac{c_2 x\sqrt{2}\,3^{\frac{1}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}\left(\int \frac{\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2}dx\right)}{\left(3x^2+x+1\right)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x\sqrt{2}\,3^{\frac{3}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{6\left(3x^2+x+1\right)^{\frac{3}{2}}} + \frac{c_2 x\sqrt{2}\,3^{\frac{1}{4}}\,\mathrm{e}^{\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}\left(\int \frac{\sqrt{3x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{11}\,\arctan\left(\frac{(1+6x)\sqrt{11}}{11}\right)}{11}}}{x^2}dx\right)}{\left(3x^2+x+1\right)^{\frac{3}{2}}}
$$

Verified OK.

2.491.1 Maple step by step solution

Let's solve

$\left(3x^2+x+1\right) y'' + \left(2+15x\right) y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{12y}{3x^2+x+1} - \frac{(2+15x)y'}{3x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2+15x)y'}{3x^2+x+1} + \frac{12y}{3x^2+x+1} = 0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = \frac{2+15x}{3x^2+x+1}, P_3(x) = \frac{12}{3x^2+x+1}\right]$

- ∘ $\left(x + \frac{I\sqrt{11}}{6} + \frac{1}{6}\right) \cdot P_2(x)$ is analytic at $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(x + \frac{I\sqrt{11}}{6} + \frac{1}{6}\right) \cdot P_2(x)\right)\Bigg|_{x=-\frac{I\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ∘ $\left(x + \frac{I\sqrt{11}}{6} + \frac{1}{6}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

  $\left(\left(x + \frac{I\sqrt{11}}{6} + \frac{1}{6}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=-\frac{I\sqrt{11}}{6}-\frac{1}{6}} = 0$

- ∘ $x = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{I\sqrt{11}}{6} - \frac{1}{6}$

- Multiply by denominators

  $(3x^2 + x + 1)\, y'' + (2 + 15x)\, y' + 12y = 0$

- Change variables using $x = u - \frac{I\sqrt{11}}{6} - \frac{1}{6}$ so that the regular singular point is at $u = 0$

  $(3u^2 - Iu\sqrt{11})\left(\frac{d^2}{du^2}y(u)\right) + \left(-\frac{1}{2} + 15u - \frac{5\,I\sqrt{11}}{2}\right)\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ∘ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

- ∘ Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right) r a_0 u^{-1+r}}{22} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{11}\left(\mathrm{I}\sqrt{11}-22k-55-22r\right)(k+1+r)a_{k+1}}{22} + 3a_k(k+r+2)^2\right)u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$\frac{\mathrm{I}}{22}\sqrt{11}\left(\mathrm{I}\sqrt{11}-33-22r\right) r = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3a_k(k+r+2)^2 - a_{k+1}(k+1+r)\left(\frac{1}{2}+\mathrm{I}\left(k+r+\frac{5}{2}\right)\sqrt{11}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{6a_k\left(k^2+2kr+r^2+4k+4r+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}kr\sqrt{11}+2\,\mathrm{I}\sqrt{11}\,r^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}r\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k+r+1}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Revert the change of variables $u = x+\frac{\mathrm{I}\sqrt{11}}{6}+\frac{1}{6}$

  $$\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{\mathrm{I}\sqrt{11}}{6}+\frac{1}{6}\right)^k, a_{k+1} = \frac{6a_k\left(k^2+4k+4\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}\sqrt{11}+k}\right]$$

- Recursion relation for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

  $$a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+\frac{111\,\mathrm{I}\sqrt{11}}{22}+k-\frac{1}{2}}$$

- Solution for $r = -\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2+\frac{2\,\mathrm{I}\sqrt{11}}{11}\right)}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}\right.$$

- Revert the change of variables $u = x + \frac{\mathrm{I}\sqrt{11}}{6} + \frac{1}{6}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{\mathrm{I}\sqrt{11}}{6} + \frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}, a_{k+1} = \frac{6a_k\left(k^2+2k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)+\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+4k-2\right.}{2\,\mathrm{I}\sqrt{11}\,k^2+4\,\mathrm{I}k\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)\sqrt{11}+2\,\mathrm{I}\sqrt{11}\left(-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}\right)^2+7\,\mathrm{I}k\sqrt{11}+7\,\mathrm{I}\left(\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{\mathrm{I}\sqrt{11}}{6} + \frac{1}{6}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{\mathrm{I}\sqrt{11}}{6} + \frac{1}{6}\right)^{k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{11}}{22}}\right), a_{k+1} = \frac{6a_k\,(k^2+4k+4)}{2\,\mathrm{I}\sqrt{11}\,k^2+1+7\,\mathrm{I}k\sqrt{11}+5\,\mathrm{I}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 143

```
dsolve((1+x+3*x^2)*diff(y(x),x$2)+(2+15*x)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \left(\frac{i\sqrt{11}-6x-1}{i\sqrt{11}+6x+1}\right)^{-\frac{i\sqrt{11}}{22}} x}{(3x^2+x+1)^{\frac{3}{2}}} + \frac{c_2 \left(\frac{i\sqrt{11}-6x-1}{i\sqrt{11}+6x+1}\right)^{-\frac{i\sqrt{11}}{22}} x \left(\int \frac{\sqrt{3x^2+x+1}\left(\frac{i\sqrt{11}+6x+1}{i\sqrt{11}-6x-1}\right)^{-\frac{i\sqrt{11}}{22}}}{x^2} dx\right)}{(3x^2+x+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 3.09 (sec). Leaf size: 93

```
DSolve[(1+x+3*x^2)*y''[x]+(2+15*x)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x e^{\frac{\arctan\left(\frac{6x+1}{\sqrt{11}}\right)}{\sqrt{11}}} \left(c_2 \int_1^x \frac{e^{-\frac{\arctan\left(\frac{6K[1]+1}{\sqrt{11}}\right)}{\sqrt{11}}} \sqrt{3K[1]^2+K[1]+1}}{K[1]^2} dK[1] + c_1\right)}{(3x^2+x+1)^{3/2}}$$

2.492 problem 506

2.492.1 Maple step by step solution 4563

Internal problem ID [7982]
Internal file name [`OUTPUT/6915_Sunday_June_05_2022_05_17_19_PM_14336583/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 506.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x+2)\,y''+(1+x)\,y'+3y=0}$$

Writing the ode as

$$(x+2)\,y''+(1+x)\,y'+3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x+2\\B&=1+x\\C&=3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 21}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 10x - 21$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 21}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 930: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{7}{2\,(x+2)} + \frac{3}{4\,(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{7}{2x} - \frac{9}{2x^2} - \frac{97}{2x^3} - \frac{1291}{4x^4} - \frac{11103}{4x^5} - \frac{98061}{4x^6} - \frac{913053}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10x - 21}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-14x - 25}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{-14x - 25}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 4 gives $-\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{2}\right) - (0) \\ &= -\frac{7}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{7}{2}}{\frac{1}{2}} - 0\right) &= \frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 21}{4\,(x+2)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{7}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{3}{2\,(x+2)} + (-)\left(\frac{1}{2}\right) \\ &= \frac{3}{2\,(x+2)} - \frac{1}{2} \\ &= -\frac{x-1}{2\,(x+2)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2\,(x+2)} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{3}{2\,(x+2)^2}\right) + \left(\frac{3}{2\,(x+2)} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 21}{4\,(x+2)^2}\right)\right) = 0$$

$$\frac{(a_1 + 4)\,x + 2a_0 + a_1 + 4}{x+2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - 4x\right) e^{\int \left(\frac{3}{2(x+2)} - \frac{1}{2}\right) dx} \\ &= \left(x^2 - 4x\right) e^{-\frac{x}{2} + \frac{3\ln(x+2)}{2}} \\ &= x(x-4)\,(x+2)^{\frac{3}{2}}\, e^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{\ln(x+2)}{2}} \\
&= z_1 \left(\sqrt{x+2}\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(x-4)\,(x+2)^2\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-x(x-4)\,(x+2)^2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^3-x^2-10x-6)}{48x\,(x-4)\,(x+2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x(x-4)\,(x+2)^2\,\mathrm{e}^{-x}\right) + c_2\Bigg(x(x-4)\,(x \\
&\quad +2)^2\,\mathrm{e}^{-x}\left(\frac{-x(x-4)\,(x+2)^2\,\mathrm{e}^{-2}\,\mathrm{expIntegral}_1\,(-x-2)-\mathrm{e}^{x}(x^3-x^2-10x-6)}{48x\,(x-4)\,(x+2)^2}\right)\Bigg)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x(x-4)(x+2)^2 \mathrm{e}^{-x} + c_2 \left(-\frac{x(x-4)(x+2)^2 \mathrm{e}^{-x-2} \operatorname{expIntegral}_1(-x-2)}{48} - \frac{x^3}{48} + \frac{x^2}{48} + \frac{5x}{24} + \frac{1}{8} \right) \tag{1}$$

Verification of solutions

$$y = c_1 x(x-4)(x+2)^2 \mathrm{e}^{-x} + c_2 \left(-\frac{x(x-4)(x+2)^2 \mathrm{e}^{-x-2} \operatorname{expIntegral}_1(-x-2)}{48} - \frac{x^3}{48} + \frac{x^2}{48} + \frac{5x}{24} + \frac{1}{8} \right)$$

Verified OK.

2.492.1 Maple step by step solution

Let's solve

$(x+2)\, y'' + (1+x)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{x+2} - \frac{(1+x)y'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x+2} + \frac{3y}{x+2} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+x}{x+2}, P_3(x) = \frac{3}{x+2}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = -1$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$

  - $x = -2$is a regular singular point

Check to see if $x_0 = -2$ is a regular singular point

$x_0 = -2$

- Multiply by denominators

$(x+2)\,y'' + (1+x)\,y' + 3y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-1+u)\left(\frac{d}{du}y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

  - Shift index using $k- > k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+1+r)(k+r-1) + a_k(k+r+3) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+3)}{(k+1+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+1)(k-1)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+2)^{k+2}, a_{k+1} = -\frac{a_k(k+5)}{(k+3)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve((2+x)*diff(y(x),x$2)+(1+x)*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x\left(x^3 - 12x - 16\right) - \frac{c_2 (e^{-2}\,\text{expIntegral}_1\,(-2-x)\,x^4 + e^x x^3 - 12\,e^{-2}\,\text{expIntegral}_1\,(-2-x)\,x^2 - e^x x^2 - 16\,e^{-2}\,\text{expIntegral}}{48}$$

✓ Solution by Mathematica

Time used: 0.189 (sec). Leaf size: 99

```
DSolve[(2+x)*y''[x]+(1+x)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x-1}(c_2(x-4)(x+2)^2 x\,\text{ExpIntegralEi}(x+2) + 384c_1x^4 - c_2e^{x+2}x^3 + x^2(c_2e^{x+2} - 4608c_1) + x(10c_2e}{96\sqrt{2}}$$

2.493 problem 507

2.493.1 Maple step by step solution 4574

Internal problem ID [7983]
Internal file name [OUTPUT/6916_Sunday_June_05_2022_05_17_23_PM_63799600/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 507.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4+x)\,y''+(x+2)\,y'+2y=0}$$

Writing the ode as

$$(4+x)\,y''+(x+2)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4+x\\B&=x+2\\C&=2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 24}{4\,(4 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x - 24 \\ t &= 4(4 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x - 24}{4\,(4 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 932: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4+x)^2$. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(4+x)^2} - \frac{3}{4+x}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{x} + \frac{5}{x^2} - \frac{34}{x^3} + \frac{59}{x^4} - \frac{586}{x^5} + \frac{370}{x^6} - \frac{12484}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x - 24}{4x^2 + 32x + 64} \\
&= Q + \frac{R}{4x^2 + 32x + 64} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-12x - 40}{4x^2 + 32x + 64}\right) \\
&= \frac{1}{4} + \frac{-12x - 40}{4x^2 + 32x + 64}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be found.

$$
\begin{aligned}
b &= (-3) - (0) \\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 0\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 0\right) &&= 3
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x - 24}{4\,(4+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-4	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-3	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 3$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{4+x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{2}{4+x} - \frac{1}{2} \\
&= -\frac{x}{2\,(4+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{4+x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{(4+x)^2}\right) + \left(\frac{2}{4+x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x - 24}{4\,(4+x)^2}\right)\right) = 0
$$

$$
\frac{a_0}{4+x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(\frac{2}{4+x} - \frac{1}{2}\right) dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{2} + 2\ln(4+x)} \\
&= x(4+x)^2\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+2}{4+x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(4+x)} \\
&= z_1 \left((4+x)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x(4+x)^3\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+2}{4+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(4+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-x(4+x)^3\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\,(-x-4)-\mathrm{e}^{x}(x^3+9x^2+22x+6)}{24x\,(4+x)^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x(4+x)^3\,\mathrm{e}^{-x}\right) + c_2\Bigg(x(4 \\
&\quad +x)^3\,\mathrm{e}^{-x}\left(\frac{-x(4+x)^3\,\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\,(-x-4)-\mathrm{e}^{x}(x^3+9x^2+22x+6)}{24x\,(4+x)^3}\right)\Bigg)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x(4+x)^3 e^{-x} + c_2 \left(-\frac{x(4+x)^3 e^{-x-4} \operatorname{expIntegral}_1(-x-4)}{24} - \frac{x^3}{24} - \frac{3x^2}{8} - \frac{11x}{12} - \frac{1}{4}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(4+x)^3 e^{-x} + c_2 \left(-\frac{x(4+x)^3 e^{-x-4} \operatorname{expIntegral}_1(-x-4)}{24} - \frac{x^3}{24} - \frac{3x^2}{8} - \frac{11x}{12} - \frac{1}{4}\right)$$

Verified OK.

2.493.1 Maple step by step solution

Let's solve

$(4+x) y'' + (x+2) y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{4+x} - \frac{(x+2)y'}{4+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+2)y'}{4+x} + \frac{2y}{4+x} = 0$

□ Check to see if $x_0 = -4$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{x+2}{4+x}, P_3(x) = \frac{2}{4+x}\right]$

  ◦ $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

    $\left((4+x) \cdot P_2(x)\right)\Big|_{x=-4} = -2$

  ◦ $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

    $\left((4+x)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

  ◦ $x = -4$is a regular singular point

    Check to see if $x_0 = -4$ is a regular singular point

$x_0 = -4$

- Multiply by denominators

  $(4+x)\, y'' + (x+2)\, y' + 2y = 0$

- Change variables using $x = u-4$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (u-2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$

  - Shift index using $k->k+1$

    $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2)\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-2+r) + a_k(k+r+2) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k(k+r+2)}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = 4 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (4 + x)^{k+3}, a_{k+1} = -\frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 108

```
dsolve((4+x)*diff(y(x),x$2)+(2+x)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x(x^3 + 12x^2 + 48x + 64) - \frac{c_2(e^{-4} \operatorname{expIntegral}_1(-x-4) x^4 + 12 e^{-4} \operatorname{expIntegral}_1(-x-4) x^3 + e^x x^3 + 48 e^{-4} \operatorname{expIntegral}_1(-x -}{24}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 97

```
DSolve[(4+x)*y''[x]+(2+x)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{24} e^{-x-4} \left(c_2 x(x+4)^3 \operatorname{ExpIntegralEi}(x+4) + e^4\left(24c_1 x^4 + x^3(288c_1 - c_2 e^x) + 9x^2(128c_1 - c_2 e^x) + 2x(768c_1 - 11c_2 e^x) - 6c_2 e^x\right)\right)$$

2.494 problem 508

2.494.1 Maple step by step solution 4584

Internal problem ID [7984]
Internal file name [`OUTPUT/6917_Sunday_June_05_2022_05_17_26_PM_94046609/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 508.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+3x)\,y''+10(1+x)\,y'+8y=0}$$

Writing the ode as

$$(2x^2+3x)\,y''+(10x+10)\,y'+8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+3x \\ B &= 10x+10 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6x + 10$$
$$t = (2x^2 + 3x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6x + 10}{(2x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 934: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(2x^2+3x\right)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{22}{27x} + \frac{10}{9x^2} - \frac{5}{36\left(x+\frac{3}{2}\right)^2} + \frac{22}{27\left(x+\frac{3}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3}
\end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2+6x+10}{(2x^2+3x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$
$-\frac{3}{2}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{3x} + \frac{1}{6x+9} + (-)(0) \\
&= -\frac{2}{3x} + \frac{1}{6x+9} \\
&= -\frac{x+2}{x(3+2x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{2}{3x} + \frac{1}{6x+9}\right)(1) + \left(\left(\frac{2}{3x^2} - \frac{1}{6\left(x+\frac{3}{2}\right)^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6x+9}\right)^2 - \left(\frac{-x^2+6x+10}{\left(2x^2+3x\right)^2}\right)\right) = 0
$$

$$
\frac{-4+2a_0}{x(3+2x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-\frac{2}{3x} + \frac{1}{6x+9}\right) dx} \\
&= (x+2)\, \mathrm{e}^{\frac{\ln(3+2x)}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{(x+2)(3+2x)^{\frac{1}{6}}}{x^{\frac{2}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x+10}{2x^2+3x}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{3}-\frac{5\ln(3+2x)}{6}} \\
&= z_1\left(\frac{1}{x^{\frac{5}{3}}\,(3+2x)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x+10}{2x^2+3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{10\ln(x)}{3}-\frac{5\ln(3+2x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}\,(x+2)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}\right) + c_2\left(\frac{x+2}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}}\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}\,(x+2)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x+2)}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}} + \frac{c_2(x+2)\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}(x+2)^2}dx\right)}{(3+2x)^{\frac{2}{3}}\,x^{\frac{7}{3}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x+2)}{(3+2x)^{\frac{2}{3}} x^{\frac{7}{3}}} + \frac{c_2(x+2)\left(\int \frac{x^{\frac{4}{3}}}{(3+2x)^{\frac{1}{3}}(x+2)^2}dx\right)}{(3+2x)^{\frac{2}{3}} x^{\frac{7}{3}}}$$

Verified OK.

2.494.1 Maple step by step solution

Let's solve

$(2x^2+3x)\, y''+(10x+10)\, y'+8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{8y}{x(3+2x)}-\frac{10(1+x)y'}{x(3+2x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{10(1+x)y'}{x(3+2x)}+\frac{8y}{x(3+2x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{10(1+x)}{x(3+2x)}, P_3(x)=\frac{8}{x(3+2x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=\frac{10}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(3+2x)+(10x+10)\, y'+8y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ○ Shift index using $k- > k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(7+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(7+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{7}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(3k+10+3r) + 2a_k(k+r+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{2a_k(k+r+2)^2}{(k+1+r)(3k+10+3r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}\right]$$

- Recursion relation for $r = -\frac{7}{3}$

$$a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}$$

- Solution for $r = -\frac{7}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{7}{3}}, a_{k+1} = -\frac{2a_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{7}{3}}\right), a_{k+1} = -\frac{2a_k(k+2)^2}{(k+1)(3k+10)}, b_{k+1} = -\frac{2b_k\left(k-\frac{1}{3}\right)^2}{\left(k-\frac{4}{3}\right)(3k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((3*x+2*x^2)*diff(y(x),x$2)+10*(1+x)*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+2)}{x^{\frac{7}{3}} (2x+3)^{\frac{2}{3}}} + \frac{c_2(x+2) \left(\int \frac{x^{\frac{4}{3}}}{(x+2)^2 (2x+3)^{\frac{1}{3}}} dx\right)}{x^{\frac{7}{3}} (2x+3)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 245

```
DSolve[(3*x+2*x^2)*y''[x]+10*(1+x)*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$y(x)$

$$\rightarrow \frac{2\ 2^{2/3}\sqrt{3}c_2(x+2)\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{\sqrt[3]{x}+2^{2/3}\sqrt[3]{2x+3}}\right) + 2^{2/3}c_2 x \log\left(2x^{2/3} + 2^{2/3}\sqrt[3]{2x+3}\sqrt[3]{x} + \sqrt[3]{2}(2x+3)^{2/3}\right.}{}$$

2.495 problem 509

Internal problem ID [7985]
Internal file name [`OUTPUT/6918_Sunday_June_05_2022_05_17_30_PM_99761983/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 509.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - (6-7x)\,y' + 8y = 0}$$

Writing the ode as

$$x^2y'' + (-6+7x)\,y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6+7x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 60x + 36}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 60x + 36 \\ t &= 4x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 60x + 36}{4x^4} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 936: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{15}{x^3} + \frac{9}{x^4} + \frac{3}{4x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{3}{x^2} - \frac{5}{2x} - \frac{11}{12} - \frac{55x}{72} - \frac{671x^2}{864} - \frac{4565x^3}{5184} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{3}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 3$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -15. Therefore

$$
\begin{aligned}
b &= (-15) - (0) \\
&= -15
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{3}{x^2} \\
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a}+v\right) &= \frac{1}{2}\left(\frac{-15}{3}+2\right) &= -\frac{3}{2} \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &= \frac{1}{2}\left(-\frac{-15}{3}+2\right) &= \frac{7}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^2 - 60x + 36}{4x^4}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2 - 60x + 36}{4x^4}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{3}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{3}{x^2} - \frac{3}{2x} + (-)\,(0) \\ &= \frac{3}{x^2} - \frac{3}{2x} \\ &= -\frac{3(x-2)}{2x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{x^2} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{6}{x^3} + \frac{3}{2x^2}\right) + \left(\frac{3}{x^2} - \frac{3}{2x}\right)^2 - \left(\frac{3x^2 - 60x + 36}{4x^4}\right)\right) = 0$$

$$\frac{6 + 3a_0}{x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x-2)\, \mathrm{e}^{\int \left(\frac{3}{x^2} - \frac{3}{2x}\right) dx} \\ &= (x-2)\, \mathrm{e}^{-\frac{3}{x} - \frac{3\ln(x)}{2}} \\ &= \frac{(x-2)\, \mathrm{e}^{-\frac{3}{x}}}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6+7x}{x^2} \, dx} \\ &= z_1 e^{-\frac{3}{x} - \frac{7\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{3}{x}}}{x^{\frac{7}{2}}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-2)\, \mathrm{e}^{-\frac{6}{x}}}{x^5}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6+7x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{6}{x} - 7\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(108x - 216)\, \mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + \mathrm{e}^{\frac{6}{x}} x (x^2 + 12x - 36)}{2x - 4} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5}\right) \\
&\quad + c_2 \left(\frac{(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} \left(\frac{(108x-216)\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + \mathrm{e}^{\frac{6}{x}} x(x^2+12x-36)}{2x-4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} + \frac{c_2\left(108(x-2)\,\mathrm{e}^{-\frac{6}{x}}\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + x^3 + 12x^2 - 36x\right)}{2x^5} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x-2)\,\mathrm{e}^{-\frac{6}{x}}}{x^5} + \frac{c_2\left(108(x-2)\,\mathrm{e}^{-\frac{6}{x}}\,\mathrm{expIntegral}_1\left(-\frac{6}{x}\right) + x^3 + 12x^2 - 36x\right)}{2x^5}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 81

```
dsolve(x^2*diff(y(x),x$2)-(6-7*x)*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{6}{x}}(x-2)}{x^5} + \frac{c_2 \left(x^3 \mathrm{e}^{\frac{6}{x}} + 12x^2 \mathrm{e}^{\frac{6}{x}} + 108 \operatorname{expIntegral}_1\left(-\frac{6}{x}\right) x - 36x\, \mathrm{e}^{\frac{6}{x}} - 216 \operatorname{expIntegral}_1\left(-\frac{6}{x}\right)\right) \mathrm{e}^{-\frac{6}{x}}}{2x^5}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]-(6-7*x)*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6/x}\left(-108c_2(x-2)\operatorname{ExpIntegralEi}\left(\frac{6}{x}\right) + c_2 e^{6/x} x(x^2+12x-36) + 2c_1(x-2)\right)}{2x^5}$$

2.496 problem 510

2.496.1 Maple step by step solution 4602

Internal problem ID [7986]
Internal file name [OUTPUT/6919_Sunday_June_05_2022_05_17_33_PM_37121205/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 510.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0}$$

Writing the ode as

$$(2x^2+x+1)\,y''+(1+7x)\,y'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2+x+1 \\ B &= 1+7x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 2x + 5$$
$$t = 4\left(2x^2 + x + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 937: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+x+1)^2$. There is a pole at $x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ of order 2. There is a pole at $x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{\frac{9i\sqrt{7}}{224} - \frac{29}{224}}{\left(x - \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2} + \frac{-\frac{9i\sqrt{7}}{224} - \frac{29}{224}}{\left(x + \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2} - \frac{8i\sqrt{7}}{49\left(x - \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)} + \frac{8i\sqrt{7}}{49\left(x + \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)}$$

For the pole at $x = \frac{i\sqrt{7}}{4} - \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9i\sqrt{7}}{224} - \frac{29}{224}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42 + 14i\sqrt{7}}}{56} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42 + 14i\sqrt{7}}}{56}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{7}}{4} - \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{9i\sqrt{7}}{224} - \frac{29}{224}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{3\sqrt{42 - 14i\sqrt{7}}}{56} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{3\sqrt{42 - 14i\sqrt{7}}}{56}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 2x + 5}{4\left(2x^2 + x + 1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2-2x+5}{4\left(2x^2+x+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{7}}{4}-\frac{1}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42+14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42+14i\sqrt{7}}}{56}$
$-\frac{i\sqrt{7}}{4}-\frac{1}{4}$	2	0	$\frac{1}{2}+\frac{3\sqrt{42-14i\sqrt{7}}}{56}$	$\frac{1}{2}-\frac{3\sqrt{42-14i\sqrt{7}}}{56}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x - \frac{i\sqrt{7}}{4} + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{i\sqrt{7}}{4} + \frac{1}{4}} + (0) \\
&= \frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x - \frac{i\sqrt{7}}{4} + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{i\sqrt{7}}{4} + \frac{1}{4}} \\
&= \frac{1+x}{4x^2 + 2x + 2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x - \frac{i\sqrt{7}}{4} + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{i\sqrt{7}}{4} + \frac{1}{4}}\right)(1) + \left(\left(-\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{\left(x - \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2} - \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{\left(x + \frac{i\sqrt{7}}{4} + \frac{1}{4}\right)^2}\right) + \left(\frac{\frac{1}{2} - 3\sqrt{}}{x - i}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{\frac{1}{2} - \frac{3\sqrt{42+14i\sqrt{7}}}{56}}{x - \frac{i\sqrt{7}}{4} + \frac{1}{4}} + \frac{\frac{1}{2} - \frac{3\sqrt{42-14i\sqrt{7}}}{56}}{x + \frac{i\sqrt{7}}{4} + \frac{1}{4}}\right) dx} \\
&= (1+x)\, e^{\frac{\ln(16x^2+8x+8)}{2} - \frac{3\sqrt{42-14i\sqrt{7}}\, \ln(16x^2+8x+8)}{112} + \frac{3i\sqrt{42-14i\sqrt{7}}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{56} - \frac{3\sqrt{42+14i\sqrt{7}}\, \ln(16x^2+8x+8)}{112} - \frac{3i\sqrt{42+14i\sqrt{}}}{}} \\
&= 2^{\frac{3}{8}} (1+x) \left(2x^2 + x + 1\right)^{\frac{1}{8}} e^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+7x}{2x^2+x+1}\,dx} \\
&= z_1 e^{-\frac{7\ln(2x^2+x+1)}{8}+\frac{3\sqrt{7}\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{3\sqrt{7}\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{28}}}{(2x^2+x+1)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2^{\frac{3}{8}}(1+x)\,\mathrm{e}^{\frac{3\sqrt{7}\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+7x}{2x^2+x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(2x^2+x+1)}{4}+\frac{3\sqrt{7}\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{1}{4}}\mathrm{e}^{-\frac{3\sqrt{7}\arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{2(1+x)^2(2x^2+x+1)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2^{\frac{3}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \right) \\
&\quad + c_2 \left(\frac{2^{\frac{3}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \left(\int \frac{2^{\frac{1}{4}} \mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{2\,(1+x)^2\,(2x^2+x+1)^{\frac{1}{4}}} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 2^{\frac{3}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \\
&\quad + \frac{c_2 2^{\frac{5}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}} \left(\int \frac{\mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(1+x)^2(2x^2+x+1)^{\frac{1}{4}}} dx \right)}{2\,(2x^2+x+1)^{\frac{3}{4}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 2^{\frac{3}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(2x^2+x+1)^{\frac{3}{4}}} \\
&\quad + \frac{c_2 2^{\frac{5}{8}} (1+x)\, \mathrm{e}^{\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}} \left(\int \frac{\mathrm{e}^{-\frac{3\sqrt{7}\, \arctan\left(\frac{(1+4x)\sqrt{7}}{7}\right)}{14}}}{(1+x)^2(2x^2+x+1)^{\frac{1}{4}}} dx \right)}{2\,(2x^2+x+1)^{\frac{3}{4}}}
\end{aligned}
$$

Verified OK.

2.496.1 Maple step by step solution

Let's solve

$(2x^2+x+1)\, y'' + (1+7x)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{2y}{2x^2+x+1} - \frac{(1+7x)y'}{2x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+7x)y'}{2x^2+x+1} + \frac{2y}{2x^2+x+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1+7x}{2x^2+x+1}, P_3(x) = \frac{2}{2x^2+x+1}\right]$

- ○ $\left(x + \frac{\mathrm{I}\sqrt{7}}{4} + \frac{1}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

  $\left(\left(x + \frac{\mathrm{I}\sqrt{7}}{4} + \frac{1}{4}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}} = 0$

- ○ $\left(x + \frac{\mathrm{I}\sqrt{7}}{4} + \frac{1}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

  $\left(\left(x + \frac{\mathrm{I}\sqrt{7}}{4} + \frac{1}{4}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{\mathrm{I}\sqrt{7}}{4}-\frac{1}{4}} = 0$

- ○ $x = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$

- Multiply by denominators

  $(2x^2 + x + 1)\, y'' + (1 + 7x)\, y' + 2y = 0$

- Change variables using $x = u - \frac{\mathrm{I}\sqrt{7}}{4} - \frac{1}{4}$ so that the regular singular point is at $u = 0$

  $(2u^2 - \mathrm{I}u\sqrt{7})\left(\frac{d^2}{du^2}y(u)\right) + \left(-\frac{3}{4} + 7u - \frac{7\,\mathrm{I}\sqrt{7}}{4}\right)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k + r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{\mathrm{I}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)r a_0 u^{-1+r}}{28} + \left(\sum_{k=0}^{\infty}\left(\frac{\mathrm{I}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-28k-49-28r\right)(k+1+r)a_{k+1}}{28} + a_k(k+r+2)(2k+2r+1)\right)\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$\frac{\mathrm{I}}{28}\sqrt{7}\left(3\,\mathrm{I}\sqrt{7}-21-28r\right)r = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-\mathrm{I}\left(k+r+\frac{7}{4}\right)(k+1+r)\,a_{k+1}\sqrt{7}+\frac{(-3k-3r-3)a_{k+1}}{4}+2a_k\left(k+r+\frac{1}{2}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{4a_k\left(2k^2+4kr+2r^2+5k+5r+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,kr+4\,\mathrm{I}\sqrt{7}\,r^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\,r+7\,\mathrm{I}\sqrt{7}+3k+3r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Revert the change of variables $u = x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}\right)^k, a_{k+1} = \frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}+3k}\right]$$

- Recursion relation for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

$$a_{k+1} = \frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\frac{205\,\mathrm{I}\sqrt{7}}{28}+3k}$$

- Solution for $r = \frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1}=\frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}-\frac{7}{4}\right)}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+\cdots}\right.$$

- Revert the change of variables $u=x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}, a_{k+1}=\frac{4a_k\left(2k^2+4k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+2\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+5k+\frac{15\,\mathrm{I}\sqrt{7}}{28}\cdots\right.}{\frac{3}{4}+4\,\mathrm{I}\sqrt{7}\,k^2+8\,\mathrm{I}\sqrt{7}\,k\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)+4\,\mathrm{I}\sqrt{7}\left(\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}\right)^2+11\,\mathrm{I}\sqrt{7}\,k+11\,\mathrm{I}\sqrt{7}\cdots}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x+\frac{\mathrm{I}\sqrt{7}}{4}+\frac{1}{4}\right)^{k+\frac{3\,\mathrm{I}\sqrt{7}}{28}-\frac{3}{4}}\right), a_{k+1}=\frac{4a_k\left(2k^2+5k+2\right)}{3+4\,\mathrm{I}\sqrt{7}\,k^2+11\,\mathrm{I}\sqrt{7}\,k+7\,\mathrm{I}\sqrt{7}\cdots}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 149

```
dsolve((1+x+2*x^2)*diff(y(x),x$2)+(1+7*x)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \left(\frac{i\sqrt{7}-4x-1}{i\sqrt{7}+4x+1}\right)^{-\frac{3i\sqrt{7}}{28}} (x+1)}{(2x^2+x+1)^{\frac{3}{4}}} + \frac{c_2 \left(\frac{i\sqrt{7}-4x-1}{i\sqrt{7}+4x+1}\right)^{-\frac{3i\sqrt{7}}{28}} (x+1) \left(\int \frac{\left(\frac{i\sqrt{7}+4x+1}{i\sqrt{7}-4x-1}\right)^{-\frac{3i\sqrt{7}}{28}}}{(x+1)^2 (2x^2+x+1)^{\frac{1}{4}}} dx\right)}{(2x^2+x+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 2.132 (sec). Leaf size: 102

```
DSolve[(1+x+2*x^2)*y''[x]+(1+7*x)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x+1)e^{\frac{3\arctan\left(\frac{4x+1}{\sqrt{7}}\right)}{2\sqrt{7}}}\left(c_2\int_1^x \frac{e^{-\frac{3\arctan\left(\frac{4K[1]+1}{\sqrt{7}}\right)}{2\sqrt{7}}}}{(K[1]+1)^2\sqrt[4]{2K[1]^2+K[1]+1}}dK[1]+c_1\right)}{(2x^2+x+1)^{3/4}}$$

2.497 problem 511

2.497.1 Maple step by step solution 4613

Internal problem ID [7987]
Internal file name [OUTPUT/6920_Sunday_June_05_2022_05_17_38_PM_36008088/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 511.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+3)\, y''+(2x+1)\, y'-(-x+2)\, y=0}$$

Writing the ode as

$$(x+3)\, y''+(2x+1)\, y'+(x-2)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+3 \\ B &= 2x+1 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35}{4(x+3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35$$
$$t = 4(x+3)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35}{4(x+3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 939: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+3)^2$. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4\left(x+3\right)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35}{4\left(x+3\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35}{4\left(x+3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{5}{2} - \left(-\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2(x+3)} + (-)(0) \\ &= -\frac{5}{2(x+3)} \\ &= -\frac{5}{2(x+3)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2\left(x+3\right)}\right)(0) + \left(\left(\frac{5}{2\left(x+3\right)^{2}}\right) + \left(-\frac{5}{2\left(x+3\right)}\right)^{2} - \left(\frac{35}{4\left(x+3\right)^{2}}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{5}{2(x+3)} dx} \\ &= \frac{1}{\left(x+3\right)^{\frac{5}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x+1}{x+3} \, dx} \\ &= z_1 e^{-x + \frac{5\ln(x+3)}{2}} \\ &= z_1 \left(\left(x+3\right)^{\frac{5}{2}} \mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x+3} \, dx}}{\left(y_1\right)^2} \, dx \\ &= y_1 \int \frac{e^{-2x+5\ln(x+3)}}{\left(y_1\right)^2} \, dx \\ &= y_1 \left(\frac{x(x+6)\left(x^2+9x+27\right)\left(x^2+3x+9\right)}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{x(x+6)\left(x^2+9x+27\right)\left(x^2+3x+9\right)}{6}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{-x}x(x+6)\left(x^2+9x+27\right)\left(x^2+3x+9\right)}{6} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{-x}x(x+6)\left(x^2+9x+27\right)\left(x^2+3x+9\right)}{6}
$$

Verified OK.

2.497.1 Maple step by step solution

Let's solve

$(x+3)\,y'' + (2x+1)\,y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-2)y}{x+3} - \frac{(2x+1)y'}{x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x+3} + \frac{(x-2)y}{x+3} = 0$

□ Check to see if $x_0 = -3$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{x+3}, P_3(x) = \frac{x-2}{x+3}\right]$

  - $(x+3)\cdot P_2(x)$ is analytic at $x = -3$

    $\left((x+3)\cdot P_2(x)\right)\Big|_{x=-3} = -5$

  - $(x+3)^2\cdot P_3(x)$ is analytic at $x = -3$

$$\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = 0$$

- $x=-3$is a regular singular point

  Check to see if $x_0=-3$ is a regular singular point

  $x_0=-3$

- Multiply by denominators

  $$(x+3)\,y''+(2x+1)\,y'+(x-2)\,y=0$$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

  $$u\left(\frac{d^2}{du^2}y(u)\right)+(2u-5)\left(\frac{d}{du}y(u)\right)+(u-5)\,y(u)=0$$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m=0..1$

  $$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(-6+r)\, u^{-1+r} + (a_1(1+r)\,(-5+r) + a_0(-5+2r))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 6\}$

- Each term must be 0

  $a_1(1+r)\,(-5+r) + a_0(-5+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k-5+r) + 2a_k k + 2a_k r - 5a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k-4+r) + 2a_{k+1}(k+1) + 2ra_{k+1} - 5a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+2ra_{k+1}+a_k-3a_{k+1}}{(k+2+r)(k-4+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k-3a_{k+1}}{(k+2)(k-4)}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}$

- Solution for $r = 6$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

- Revert the change of variables $u = x+3$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+6}, a_{k+2} = -\frac{2ka_{k+1}+a_k+9a_{k+1}}{(k+8)(k+2)}, 7a_1 + 7a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve((3+x)*diff(y(x),x$2)+(1+2*x)*diff(y(x),x)-(2-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2e^{-x}(x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 29

```
DSolve[(3+x)*y''[x]+(1+2*x)*y'[x]-(2-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-x-3}(c_2(x+3)^6 + 6c_1)$$

2.498 problem 512

2.498.1 Maple step by step solution 4623

Internal problem ID [7988]
Internal file name [OUTPUT/6921_Sunday_June_05_2022_05_17_41_PM_76592990/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 512.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 3xy' + (2x^2 + 4)\, y = 0}$$

Writing the ode as

$$y'' + 3xy' + (2x^2 + 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 941: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) = 0 \\ a_1 x + 2a_0 + 2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x}{1} \, dx} \\
&= z_1 e^{-\frac{3x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2 - 1\right) \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2 - 1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 1\right) \mathrm{e}^{-x^2}\right) + c_2 \left(\left(x^2 - 1\right) \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2 - 1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 (x^2 - 1) e^{-x^2} + c_2 (x^2 - 1) e^{-x^2} \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 (x^2 - 1) e^{-x^2} + c_2 (x^2 - 1) e^{-x^2} \left(\int \frac{e^{\frac{x^2}{2}}}{(x^2 - 1)^2} dx \right)$$

Verified OK.

2.498.1 Maple step by step solution

Let's solve

$y'' + 3xy' + (2x^2 + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

- ○ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

- ○ Shift index using $k- > k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + 4a_0 + (6a_3 + 7a_1)x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(3k+4) + 2a_{k-2}\right)x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 + 4a_0 = 0, 6a_3 + 7a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)a_{k+2} + 3a_k k + 4a_k + 2a_{k-2} = 0$

- Shift index using $k- >k+2$

  $\left((k+2)^2 + 3k + 8\right)a_{k+4} + 3a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{3ka_{k+2}+2a_k+10a_{k+2}}{k^2+7k+12}, a_2 = -2a_0, a_3 = -\frac{7a_1}{6}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+3*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 - 1)\,\mathrm{e}^{-x^2} + c_2\mathrm{e}^{-x^2}(x^2 - 1)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x-1)^2(x+1)^2}dx\right)$$

✓ Solution by Mathematica
Time used: 0.151 (sec). Leaf size: 63

```
DSolve[y''[x]+3*x*y'[x]+(4+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x^2}\left(\sqrt{2\pi}c_2(x^2 - 1)\,\mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right) + 4c_1(x^2 - 1) - 2c_2e^{\frac{x^2}{2}}x\right)$$

2.499 problem 513

2.499.1 Maple step by step solution 4632

Internal problem ID [7989]
Internal file name [`OUTPUT/6922_Sunday_June_05_2022_05_17_44_PM_39238004/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 513.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4x+2)\, y''-4y'-(4x+6)\, y=0}$$

Writing the ode as

$$(4x+2)\, y''-4y'+(-4x-6)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x+2 \\ B &= -4 \\ C &= -4x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 8x + 6}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 8x + 6 \\ t &= (2x + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 8x + 6}{(2x + 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 943: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(x+\frac{1}{2}\right)^2} + \frac{1}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} - \frac{1}{4x^3} + \frac{11}{32x^4} - \frac{21}{64x^5} + \frac{15}{64x^6} - \frac{3}{32x^7} - \frac{117}{2048x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 8x + 6}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{4x + 5}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{4x + 5}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{4x^2+8x+6}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{2}$ then

$$\begin{aligned}
d&=\alpha_\infty^- -\left(\alpha_{c_1}^-\right)\\
&=-\frac{1}{2}-\left(-\frac{1}{2}\right)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&=-\frac{1}{2\left(x+\frac{1}{2}\right)}+(-)(1)\\
&=-\frac{1}{2\left(x+\frac{1}{2}\right)}-1\\
&=-\frac{2(1+x)}{2x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right)^2 - \left(\frac{4x^2+8x+6}{(2x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(x+\frac{1}{2}\right)} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{\sqrt{2x+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{4x+2} \, dx} \\
&= z_1 e^{\frac{\ln(2x+1)}{2}} \\
&= z_1\left(\sqrt{2x+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{4x+2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(2x+1)}}{(y_1)^2} \, dx \\ &= y_1 \left(x \, e^{2x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(x \, e^{2x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{x} c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{x} c_2 x$$

Verified OK.

2.499.1 Maple step by step solution

Let's solve

$(4x + 2) y'' - 4y' + (-4x - 6) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3+2x)y}{2x+1} + \frac{2y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{2x+1} - \frac{(3+2x)y}{2x+1} = 0$

☐ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{2}{2x+1}, P_3(x) = -\frac{3+2x}{2x+1}\right]$

- $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $$\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$$

- $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $$\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$$

- $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $$(2x+1)\,y'' - 2y' + (-2x-3)\,y = 0$$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $$2u\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + (-2u-2)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $\frac{d}{du}y(u)$ to series expansion

  $$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$$

- Shift index using $k- >k+1$

  $$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-1}$$

- Shift index using $k->k+1$

$$u\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-1}^{\infty}a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}r(-2+r)\,u^{-1+r}+\left(2a_{1}(1+r)\,(-1+r)-2a_{0}\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(2a_{k+1}(k+1+r)\,(k+r-1)-2a_{k}\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,2\}$$

- Each term must be 0

$$2a_{1}(1+r)\,(-1+r)-2a_{0}=0$$

- Each term in the series must be 0, giving the recursion relation

$$2a_{k+1}(k+1+r)\,(k+r-1)-2a_{k}-2a_{k-1}=0$$

- Shift index using $k->k+1$

$$2a_{k+2}(k+2+r)\,(k+r)-2a_{k+1}-2a_{k}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2+r)(k+r)}$$

- Recursion relation for $r=0$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2)k}$$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+2)k}$$

- Recursion relation for $r=2$

$$a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+4)(k+2)}$$

- Solution for $r=2$

$$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+2},a_{k+2}=\frac{a_{k+1}+a_{k}}{(k+4)(k+2)},6a_{1}-2a_{0}=0\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+2}, a_{k+2}=\frac{a_{k+1}+a_k}{(k+4)(k+2)}, 6a_1-2a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((2+4*x)*diff(y(x),x$2)-4*diff(y(x),x)-(6+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\mathrm{e}^{-x}c_1+\mathrm{e}^{x}c_2x$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 29

```
DSolve[(2+4*x)*y''[x]-4*y'[x]-(6+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-x-\frac{1}{2}}\left(c_2e^{2x+1}x+c_1\right)$$

2.500 problem 514

2.500.1 Maple step by step solution 4642

Internal problem ID [7990]
Internal file name [`OUTPUT/6923_Sunday_June_05_2022_05_17_47_PM_97181362/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 514.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 3xy' + (2x^2 + 5)\, y = 0}$$

Writing the ode as

$$y'' - 3xy' + (2x^2 + 5)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3x \\ C &= 2x^2 + 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 26}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 26 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{13}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 945: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{13}{2x} - \frac{169}{4x^3} - \frac{2197}{4x^5} - \frac{142805}{16x^7} - \frac{2599051}{16x^9} - \frac{101362989}{32x^{11}} - \frac{2070701061}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{x^2-26}{4}\\ &= Q+\frac{R}{4}\\ &= \left(\frac{x^2}{4}-\frac{13}{2}\right)+(0)\\ &= \frac{x^2}{4}-\frac{13}{2}\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{13}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{13}{2}\right)-(0)\\ &= -\frac{13}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{x}{2}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}}-1\right) &&= -7\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}}-1\right) &&= 6\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{x^2}{4}-\frac{13}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-7	6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 6$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 6 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 6$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(30x^4 + 20x^3a_5 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{x}{2}\right)(6x^5 + 5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \Bigg(\bigg(-\frac{1}{2}$$

$$a_5x^5 + 2(15 + a_4)\,x^4 + (3a_3 + 20a_5)\,x^3 + 4(a_2 + 3a_4)\,x^2 + (5$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -15, a_1 = 0, a_2 = 45, a_3 = 0, a_4 = -15, a_5 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^6 - 15x^4 + 45x^2 - 15$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^6 - 15x^4 + 45x^2 - 15\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{1} \, dx} \\
&= z_1 e^{\frac{3x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x^2}{2}} \left(x^6 - 15x^4 + 45x^2 - 15\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6 - 15x^4 + 45x^2 - 15\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\right) \\
&\quad + c_2\left(\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right) \\
&\quad + c_2\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right) \\
&\quad + c_2\mathrm{e}^{\frac{x^2}{2}}\left(x^6-15x^4+45x^2-15\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^6-15x^4+45x^2-15\right)^2}dx\right)
\end{aligned}
$$

Verified OK.

2.500.1 Maple step by step solution

Let's solve

$y'' - 3xy' + (2x^2+5)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 5a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(3k-5) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 5a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 3a_k k + 5a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2}(k+2) + 5a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3k a_{k+2} - 2a_k + a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{5a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-3*x*diff(y(x),x)+(5+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^2}{2}} (x^6 - 15x^4 + 45x^2 - 15) + c_2 e^{\frac{x^2}{2}} (x^6 - 15x^4 + 45x^2 - 15) \left(\int \frac{e^{\frac{x^2}{2}}}{(x^6 - 15x^4 + 45x^2 - 15)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 95

```
DSolve[y''[x]-3*x*y'[x]+(5+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\frac{x^2}{2}}\left(\sqrt{2\pi}c_2(x^6-15x^4+45x^2-15)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right)-2c_2e^{\frac{x^2}{2}}x(x^4-14x^2+33)+1440c_1(x^6-15x^4+45x^2-}{1440}$$

2.501 problem 515

2.501.1 Maple step by step solution 4652

Internal problem ID [7991]
Internal file name [`OUTPUT/6924_Sunday_June_05_2022_05_17_50_PM_59728663/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 515.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2y'' + 5xy' + (2x^2 + 4) y = 0}$$

Writing the ode as

$$2y'' + 5xy' + (2x^2 + 4) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 5x \\ C &= 2x^2 + 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 - 12}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 - 12 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2}{16} - \frac{3}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 947: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{4} - \frac{1}{2x} - \frac{1}{6x^3} - \frac{1}{9x^5} - \frac{5}{54x^7} - \frac{7}{81x^9} - \frac{7}{81x^{11}} - \frac{22}{243x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{3x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{16}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^2 - 12}{16} \\
&= Q + \frac{R}{16} \\
&= \left(\frac{9x^2}{16} - \frac{3}{4}\right) + (0) \\
&= \frac{9x^2}{16} - \frac{3}{4}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{4}\right) - (0) \\
&= -\frac{3}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{4}}{\frac{3}{4}} - 1\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2}{16} - \frac{3}{4}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{3x}{4}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{3x}{4}\right) \\ &= -\frac{3x}{4} \\ &= -\frac{3x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3x}{4}\right)(0) + \left(\left(-\frac{3}{4}\right) + \left(-\frac{3x}{4}\right)^2 - \left(\frac{9x^2}{16} - \frac{3}{4}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{3x}{4} dx} \\ &= e^{-\frac{3x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2}\,dx} \\
&= z_1 e^{-\frac{5x^2}{8}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x^2}{8}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5x^2}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}\left(-\frac{i\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} - \frac{i c_2 \mathrm{e}^{-x^2}\sqrt{\pi}\,\sqrt{3}\,\mathrm{erf}\left(\frac{i\sqrt{3}\,x}{2}\right)}{3} \tag{1}
$$

Verification of solutions

$$y = c_1 e^{-x^2} - \frac{i c_2 e^{-x^2} \sqrt{\pi}\, \sqrt{3}\, \mathrm{erf}\left(\frac{i\sqrt{3}\, x}{2}\right)}{3}$$

Verified OK.

2.501.1 Maple step by step solution

Let's solve

$2y'' + 5xy' + (2x^2 + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x^2 - 2)\, y - \frac{5xy'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5xy'}{2} + (x^2 + 2)\, y = 0$

- Multiply by denominators

  $2y'' + 5xy' + (2x^2 + 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k->k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$4a_2 + 4a_0 + (12a_3 + 9a_1) x + \left(\sum_{k=2}^{\infty} \left(2a_{k+2}(k+2)(k+1) + a_k(5k+4) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[4a_2 + 4a_0 = 0, 12a_3 + 9a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(2k^2 + 6k + 4) a_{k+2} + 5a_k k + 4a_k + 2a_{k-2} = 0$$

- Shift index using $k->k+2$

$$\left(2(k+2)^2 + 6k + 16\right) a_{k+4} + 5a_{k+2}(k+2) + 4a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{5ka_{k+2} + 2a_k + 14a_{k+2}}{2(k^2 + 7k + 12)}, a_2 = -a_0, a_3 = -\frac{3a_1}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(2*diff(y(x),x$2)+5*x*diff(y(x),x)+(4+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} \operatorname{erf}\left(\frac{i\sqrt{3}\, x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 42

```
DSolve[2*y''[x]+5*x*y'[x]+(4+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} e^{-x^2} \left(\sqrt{3\pi} c_2 \operatorname{erfi}\left(\frac{\sqrt{3}x}{2}\right) + 3c_1\right)$$

2.502 problem 516

2.502.1 Maple step by step solution 4658

Internal problem ID [7992]
Internal file name [OUTPUT/6925_Sunday_June_05_2022_05_17_53_PM_99454532/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 516.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 949: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s)\\
&= 0 - -\infty\\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx}\\
&= z_1 e^{-x^2}\\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx\\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx\\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\, \mathrm{e}^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\, \mathrm{e}^{-x^2}
$$

Verified OK.

2.502.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

2.503 problem 517

2.503.1 Maple step by step solution 4664

Internal problem ID [7993]
Internal file name [`OUTPUT/6926_Sunday_June_05_2022_05_17_56_PM_981212/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 517.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 2)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 951: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2}
$$

Verified OK.

2.503.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^{k}\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+4} = -\frac{2(2k a_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} x$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

2.504 problem 518

2.504.1 Maple step by step solution 4673

Internal problem ID [7994]
Internal file name [OUTPUT/6927_Sunday_June_05_2022_05_17_58_PM_68889935/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 518.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x^2+x+1)\,y''+x(11x^2+11x+9)\,y'+(7x^2+10x+6)\,y=0}$$

Writing the ode as

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+2x^3+2x^2 \\ B &= 11x^3+11x^2+9x \\ C &= 7x^2+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^4 + 18x^3 + 27x^2 - 2x - 3$$
$$t = 16\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 953: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b=-\frac{5}{24}-\frac{i\sqrt{3}}{24}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^4+18x^3+27x^2-2x-3}{16\left(x^3+x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2+3x+1}{4x\left(x^2+x+1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) + \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x}+\frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}}+\frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx} \\
&= 2\sqrt{2}\, x^{\frac{1}{4}} \left(x^2+x+1\right)^{\frac{3}{4}} \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}-\frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{\frac{9}{4}} \left(x^2+x+1\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{2}\, \sqrt{x^2+x+1}\, \mathrm{e}^{-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2}-\frac{\ln\left(x^2+x+1\right)}{2}-\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\, \left(x^2+x+1\right)^{\frac{3}{2}}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \right) \\
&\quad + c_2 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx \right)}{4x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx \right)}{4x^2}
$$

Verified OK.

2.504.1 Maple step by step solution

Let's solve

$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(11x^2+11x+9)y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)y}{2x^2(x^2+x+1)} = 0$$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = \frac{9}{2}$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 3$$

- ○ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2(x^2+x+1)\,y'' + x(11x^2+11x+9)\,y' + (7x^2+10x+6)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(3+2r)\, x^r + (a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-2, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(2+r)a_0}{3+r}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r+\frac{3}{2}\right)\left((a_k+a_{k-2}+a_{k-1})\,k + (a_k+a_{k-2}+a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k->k+2$

$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})\,(k+2) + (a_{k+2}+a_k+a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k + \frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{ka_k + ka_{k+1} - \frac{1}{2}a_k + \frac{3}{2}a_{k+1}}{k + \frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{kb_k + kb_{k+1} - \frac{1}{2}b_k +}{k + \frac{5}{2}}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 141

```
dsolve(2*x^2*(1+x+x^2)*diff(y(x),x$2)+x*(9+11*x+11*x^2)*diff(y(x),x)+(6+10*x+7*x^2)*y(x)=0,y
```

$$y(x) = \frac{c_1\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}}{x^2} + \frac{c_2\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}\left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{6}}}{(x^2+x+1)^{\frac{3}{2}}\sqrt{x}}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.77 (sec). Leaf size: 93

```
DSolve[2*x^2*(1+x+x^2)*y''[x]+x*(9+11*x+11*x^2)*y'[x]+(6+10*x+7*x^2)*y[x]==0,y[x],x,IncludeS
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

2.505 problem 519

2.505.1 Maple step by step solution 4685

Internal problem ID [7995]
Internal file name [OUTPUT/6928_Sunday_June_05_2022_05_18_05_PM_61795546/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 519.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{3x^2y'' + 2x\left(-2x^2 + x + 1\right)y' + \left(-8x^2 + 2x\right)y = 0}$$

Writing the ode as

$$3x^2y'' + \left(-4x^3 + 2x^2 + 2x\right)y' + \left(-8x^2 + 2x\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -4x^3 + 2x^2 + 2x \\ C &= -8x^2 + 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 4x^3 + 15x^2 - 4x - 2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 955: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} - \frac{4}{9x} - \frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{2x}{3} - \frac{1}{3} + \frac{7}{6x} + \frac{1}{4x^2} - \frac{17}{16x^3} - \frac{31}{32x^4} + \frac{85}{64x^5} + \frac{353}{128x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{2}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{3} + \frac{2x}{3} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{9} - \frac{4}{9}x + \frac{4}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{9}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2} \\ &= Q + \frac{R}{9x^2} \\ &= \left(\frac{4}{9}x^2 - \frac{4}{9}x + \frac{5}{3}\right) + \left(\frac{-4x - 2}{9x^2}\right) \\ &= \frac{4x^2}{9} - \frac{4x}{9} + \frac{5}{3} + \frac{-4x - 2}{9x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{5}{3}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{3}\right) - \left(\frac{1}{9}\right) \\ &= \frac{14}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= -\frac{1}{3} + \frac{2x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) \quad = \frac{1}{2}\left(\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) \quad = \frac{2}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{14}{9}}{\frac{2}{3}} - 1\right) = -\frac{5}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$-\frac{1}{3} + \frac{2x}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{2}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{2}{3} - \left(\frac{2}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{3x} + \left(-\frac{1}{3} + \frac{2x}{3}\right) \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3} \\
&= \frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)(0) + \left(\left(-\frac{2}{3x^2} + \frac{2}{3}\right) + \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right)^2 - \left(\frac{4x^4 - 4x^3 + 15x^2 - 4x - 2}{9x^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} - \frac{1}{3} + \frac{2x}{3}\right) dx} \\
&= x^{\frac{2}{3}} \mathrm{e}^{\frac{x(x-1)}{3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^3 + 2x^2 + 2x}{3x^2} \, dx} \\
&= z_1 e^{\frac{x^2}{3} - \frac{x}{3} - \frac{\ln(x)}{3}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x(x-1)}{3}}}{x^{\frac{1}{3}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3+2x^2+2x}{3x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{2x^2}{3}-\frac{2x}{3}-\frac{2\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \right) + c_2 \left(x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} + c_2 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} + c_2 x^{\frac{1}{3}} \mathrm{e}^{\frac{2x(x-1)}{3}} \left(\int \frac{\mathrm{e}^{-\frac{2x(x-1)}{3}}}{x^{\frac{4}{3}}} dx \right)$$

Verified OK.

2.505.1 Maple step by step solution

Let's solve

$3x^2y'' + (-4x^3 + 2x^2 + 2x)\,y' + (-8x^2 + 2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(4x-1)y}{3x} + \frac{2(2x^2-x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(2x^2-x-1)y'}{3x} - \frac{2(4x-1)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(2x^2-x-1)}{3x}, P_3(x) = -\frac{2(4x-1)}{3x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{2}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3xy'' + (-4x^2 + 2x + 2)\,y' + (-8x + 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+3r) x^{-1+r} + (a_1(1+r)(2+3r) + 2a_0(1+r)) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(3k+2+3r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(1+r)(2+3r) + 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)(3ka_{k+1} + 3ra_{k+1} + 2a_k - 4a_{k-1} + 2a_{k+1}) = 0$

- Shift index using $k->k+1$

  $(k+r+2)(3(k+1)a_{k+2} + 3ra_{k+2} + 2a_{k+1} - 4a_k + 2a_{k+2}) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5+3r}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+6}, 4a_1 + \frac{8a_0}{3} = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{2(-a_{k+1}+2a_k)}{3k+5}, 2a_1 + 2a_0 = 0, b_{k+2} = \frac{2(-b_{k+1}+2b_k)}{3k+6}, 4b_1 +\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(3*x^2*diff(y(x),x$2)+2*x*(1+x-2*x^2)*diff(y(x),x)+(2*x-8*x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = c_1 x^{\frac{1}{3}} e^{\frac{2}{3}x^2-\frac{2}{3}x} + c_2 x^{\frac{1}{3}} e^{\frac{2}{3}x^2-\frac{2}{3}x} \left(\int \frac{e^{-\frac{2}{3}x^2+\frac{2}{3}x}}{x^{\frac{4}{3}}} dx\right)$$

✓ Solution by Mathematica

Time used: 0.379 (sec). Leaf size: 53

```
DSolve[3*x^2*y''[x]+2*x*(1+x-2*x^2)*y'[x]+(2*x-8*x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{\frac{2}{3}(x-1)x}\sqrt[3]{x}\left(c_2\int_1^x \frac{e^{-\frac{2}{3}(K[1]-1)K[1]}}{K[1]^{4/3}}dK[1]+c_1\right)$$

2.506 problem 520

2.506.1 Maple step by step solution 4697

Internal problem ID [7996]
Internal file name [`OUTPUT/6929_Sunday_June_05_2022_05_18_09_PM_59436778/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 520.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{12x^2(1+x)\,y''+x\left(3x^2+35x+11\right)y'-\left(-5x^2-10x+1\right)y=0}$$

Writing the ode as

$$\left(12x^3+12x^2\right)y''+\left(3x^3+35x^2+11x\right)y'+\left(5x^2+10x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 12x^3+12x^2 \\ B &= 3x^3+35x^2+11x \\ C &= 5x^2+10x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 30x^3 - 197x^2 - 190x - 95 \\ t &= 576\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\left(x^2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 957: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 576(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{64} - \frac{7}{64\,(1+x)^2} - \frac{95}{576x^2} - \frac{1}{12\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{95}{576}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{24} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{24}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{1}{3x} - \frac{29}{24x^2} - \frac{193}{72x^3} - \frac{3017}{216x^4} - \frac{40009}{648x^5} - \frac{642029}{1944x^6} - \frac{10350493}{5832x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \\ &= Q + \frac{R}{576x^4 + 1152x^3 + 576x^2} \\ &= \left(\frac{1}{64}\right) + \left(\frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2}\right) \\ &= \frac{1}{64} + \frac{-48x^3 - 206x^2 - 190x - 95}{576x^4 + 1152x^3 + 576x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -48. Dividing this by leading coefficient in t which is 576 gives $-\frac{1}{12}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{12}\right) - (0) \\
&= -\frac{1}{12}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &= -\frac{1}{3} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{12}}{\frac{1}{8}} - 0\right) &= \frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^4 - 30x^3 - 197x^2 - 190x - 95}{576\,(x^2 + x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{8}$	$\frac{1}{8}$
0	2	0	$\frac{19}{24}$	$\frac{5}{24}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{8}$	$-\frac{1}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x+8} + \frac{5}{24x} + (-)\left(\frac{1}{8}\right) \\
&= \frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8} \\
&= \frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8}\right)(0) + \left(\left(-\frac{1}{8(1+x)^2} - \frac{5}{24x^2}\right) + \left(\frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8}\right)^2 - \left(\frac{9x^4 - 30x^3 -}{576}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{8x+8} + \frac{5}{24x} - \frac{1}{8}\right) dx} \\
&= x^{\frac{5}{24}} (1+x)^{\frac{1}{8}} \, e^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx} \\
&= z_1 e^{-\frac{x}{8}-\frac{11\ln(x)}{24}-\frac{7\ln(1+x)}{8}} \\
&= z_1 \left(\frac{e^{-\frac{x}{8}}}{x^{\frac{11}{24}}(1+x)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+35x^2+11x}{12x^3+12x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{4}-\frac{11\ln(x)}{12}-\frac{7\ln(1+x)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}\right) + c_2 \left(\frac{e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}}\left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 e^{-\frac{x}{4}}}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}} + \frac{c_2 e^{-\frac{x}{4}}\left(\int \frac{e^{\frac{x}{4}}}{x^{\frac{5}{12}}(1+x)^{\frac{1}{4}}}dx\right)}{x^{\frac{1}{4}}(1+x)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{4}}}{x^{\frac{1}{4}} (1+x)^{\frac{3}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{4}} \left(\int \frac{\mathrm{e}^{\frac{x}{4}}}{x^{\frac{5}{12}} (1+x)^{\frac{1}{4}}} dx\right)}{x^{\frac{1}{4}} (1+x)^{\frac{3}{4}}}$$

Verified OK.

2.506.1 Maple step by step solution

Let's solve

$(12x^3 + 12x^2) y'' + (3x^3 + 35x^2 + 11x) y' + (5x^2 + 10x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+10x-1)y}{12x^2(1+x)} - \frac{(3x^2+35x+11)y'}{12x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+35x+11)y'}{12x(1+x)} + \frac{(5x^2+10x-1)y}{12x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3x^2+35x+11}{12x(1+x)}, P_3(x) = \frac{5x^2+10x-1}{12x^2(1+x)}\right]$

- ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x)) \bigg|_{x=-1} = \frac{7}{4}$

- ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x)) \bigg|_{x=-1} = 0$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $12x^2(1+x) y'' + x(3x^2 + 35x + 11) y' + (5x^2 + 10x - 1) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$\left(12u^3-24u^2+12u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(3u^3+26u^2-50u+21\right)\left(\frac{d}{du}y(u)\right)+\left(5u^2-6\right)y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(3+4r)\,u^{-1+r}+\left(3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r)\right)u^r+\left(3a_2(2+r)\,(11+4r)-2\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(3+4r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{4}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(7+4r)-2a_0(3+4r)\,(1+3r)=0,3a_2(2+r)\,(11+4r)-2a_1(7+4r)\,(4+3r)+2a$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{2a_0(12r^2+13r+3)}{3(4r^2+11r+7)}, a_2 = \frac{2a_0(54r^3+135r^2+101r+24)}{9(4r^3+23r^2+41r+22)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$12(-2a_k + a_{k-1} + a_{k+1}) k^2 + (24(-2a_k + a_{k-1} + a_{k+1}) r - 26a_k + 3a_{k-2} - 10a_{k-1} + 33a_{k+1}) k +$$

- Shift index using $k- >k+2$

  $$12(-2a_{k+2} + a_{k+1} + a_{k+3}) (k+2)^2 + (24(-2a_{k+2} + a_{k+1} + a_{k+3}) r - 26a_{k+2} + 3a_k - 10a_{k+1} + 3$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 24kra_{k+1} - 48kra_{k+2} + 12r^2a_{k+1} - 24r^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 3ra_k + 38ra_{k+1} - 122r}{3(4k^2+8kr+4r^2+27k+27r+45)}$$

- Recursion relation for $r = 0$

  $$a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7}, a_2 =\right.$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a_k + 26a_{k+1} - 154a_{k+2}}{3(4k^2+27k+45)}, a_1 = \frac{2a_0}{7},\right.$$

- Recursion relation for $r = -\frac{3}{4}$

  $$a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}$$

- Solution for $r = -\frac{3}{4}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0, a_2\right.$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{3}{4}}, a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 20ka_{k+1} - 86ka_{k+2} + \frac{11}{4}a_k + \frac{17}{4}a_{k+1} - 76a_{k+2}}{3(4k^2+21k+27)}, a_1 = 0,\right.$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{3}{4}}\right), a_{k+3} = -\frac{12k^2a_{k+1} - 24k^2a_{k+2} + 3ka_k + 38ka_{k+1} - 122ka_{k+2} + 5a}{3(4k^2+27k+45)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(12*x^2*(1+x)*diff(y(x),x$2)+x*(11+35*x+3*x^2)*diff(y(x),x)-(1-10*x-5*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{x}{4}}}{(x+1)^{\frac{3}{4}} x^{\frac{1}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{4}} \left(\int \frac{\mathrm{e}^{\frac{x}{4}}}{(x+1)^{\frac{1}{4}} x^{\frac{5}{12}}} dx\right)}{(x+1)^{\frac{3}{4}} x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.418 (sec). Leaf size: 61

```
DSolve[12*x^2*(1+x)*y''[x]+x*(11+35*x+3*x^2)*y'[x]-(1-10*x-5*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to \frac{e^{-x/4}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{4}}}{K[1]^{5/12}\sqrt[4]{K[1]+1}}dK[1]+c_1\right)}{\sqrt[4]{x}(x+1)^{3/4}}$$

2.507 problem 521

2.507.1 Maple step by step solution 4706

Internal problem ID [7997]
Internal file name [OUTPUT/6930_Sunday_June_05_2022_05_18_13_PM_57003609/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 521.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' + 4y = 0}$$

Writing the ode as

$$y'' + 3y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-7}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -7 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -\frac{7z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 959: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{7}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{7}\, x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos\left(\frac{\sqrt{7}\, x}{2}\right) \mathrm{e}^{-\frac{3x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\frac{\sqrt{7}\,x}{2}\right) \mathrm{e}^{-\frac{3x}{2}}\right) + c_2 \left(\cos\left(\frac{\sqrt{7}\,x}{2}\right) \mathrm{e}^{-\frac{3x}{2}} \left(\frac{2\sqrt{7}\,\tan\left(\frac{\sqrt{7}\,x}{2}\right)}{7}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(\frac{\sqrt{7}\,x}{2}\right) \mathrm{e}^{-\frac{3x}{2}} + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,x}{2}\right) \mathrm{e}^{-\frac{3x}{2}} \sqrt{7}}{7} \tag{1}
$$

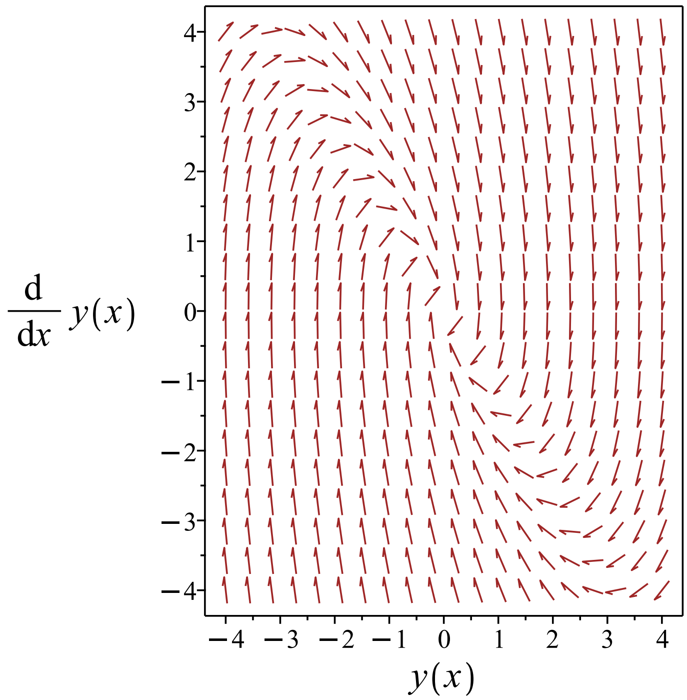

Figure 5: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}} + \frac{2c_2 \sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}} \sqrt{7}}{7}$$

Verified OK.

2.507.1 Maple step by step solution

Let's solve

$y'' + 3y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 3r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-3)\pm(\sqrt{-7})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2}-\frac{I\sqrt{7}}{2}, -\frac{3}{2}+\frac{I\sqrt{7}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \cos\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = \sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}} + c_2 \sin\left(\frac{\sqrt{7}\,x}{2}\right) e^{-\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{7}\,x}{2}\right) + c_2 e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{7}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 42

```
DSolve[y''[x]+3*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/2}\left(c_2 \cos\left(\frac{\sqrt{7}x}{2}\right) + c_1 \sin\left(\frac{\sqrt{7}x}{2}\right)\right)$$

2.508 problem 522

2.508.1 Maple step by step solution 4716

Internal problem ID [7998]
Internal file name [OUTPUT/6931_Sunday_June_05_2022_05_18_15_PM_97034330/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 522.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{18x^2(1+x)\,y''+3x\left(x^2+11x+5\right)y'-\left(-5x^2-2x+1\right)y=0}$$

Writing the ode as

$$\left(18x^3+18x^2\right)y''+\left(3x^3+33x^2+15x\right)y'+\left(5x^2+2x-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 18x^3+18x^2 \\ B &= 3x^3+33x^2+15x \\ C &= 5x^2+2x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 18x^3 - 45x^2 - 18x - 27 \\ t &= 144\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 961: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{4x} - \frac{35}{144\,(1+x)^2} - \frac{3}{16x^2} - \frac{7}{18\,(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} - \frac{5}{6x} - \frac{53}{12x^2} - \frac{523}{12x^3} - \frac{6659}{12x^4} - \frac{94267}{12x^5} - \frac{1432421}{12x^6} - \frac{22802941}{12x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2} \\ &= Q + \frac{R}{144x^4 + 288x^3 + 144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\right) \\ &= \frac{1}{144} + \frac{-20x^3 - 46x^2 - 18x - 27}{144x^4 + 288x^3 + 144x^2}\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -20. Dividing this by leading coefficient in t which is 144 gives $-\frac{5}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{36}\right) - (0) \\ &= -\frac{5}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= -\frac{5}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{36}}{\frac{1}{12}} - 0\right) &= \frac{5}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 18x^3 - 45x^2 - 18x - 27}{144\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{12}$	$\frac{5}{12}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$-\frac{5}{6}$	$\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{5}{6} - \left(\frac{5}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{7}{12\,(1+x)} + \frac{1}{4x} + (-)\left(\frac{1}{12}\right) \\
&= \frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12} \\
&= \frac{7}{12+12x} + \frac{1}{4x} - \frac{1}{12}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)(0) + \left(\left(-\frac{7}{12\,(1+x)^2} - \frac{1}{4x^2}\right) + \left(\frac{7}{12\,(1+x)} + \frac{1}{4x} - \frac{1}{12}\right)^2 - \left(\frac{x^4 - 1\cdots}{\cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{12(1+x)} + \frac{1}{4x} - \frac{1}{12}\right) dx} \\
&= x^{\frac{1}{4}} (1+x)^{\frac{7}{12}} \, \mathrm{e}^{-\frac{x}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx} \\
&= z_1 e^{-\frac{x}{12} - \frac{5\ln(x)}{12} - \frac{5\ln(1+x)}{12}} \\
&= z_1 \left(\frac{e^{-\frac{x}{12}}}{x^{\frac{5}{12}} (1+x)^{\frac{5}{12}}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)^{\frac{1}{6}} e^{-\frac{x}{6}}}{x^{\frac{1}{6}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+33x^2+15x}{18x^3+18x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{6} - \frac{5\ln(x)}{6} - \frac{5\ln(1+x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{\frac{x}{6}}}{\sqrt{x}\, (1+x)^{\frac{7}{6}}} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(1+x)^{\frac{1}{6}} e^{-\frac{x}{6}}}{x^{\frac{1}{6}}} \right) + c_2 \left(\frac{(1+x)^{\frac{1}{6}} e^{-\frac{x}{6}}}{x^{\frac{1}{6}}} \left(\int \frac{e^{\frac{x}{6}}}{\sqrt{x}\, (1+x)^{\frac{7}{6}}} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (1+x)^{\frac{1}{6}} e^{-\frac{x}{6}}}{x^{\frac{1}{6}}} + \frac{c_2 (1+x)^{\frac{1}{6}} e^{-\frac{x}{6}} \left(\int \frac{e^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}} dx \right)}{x^{\frac{1}{6}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+x)^{\frac{1}{6}} \mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}} + \frac{c_2(1+x)^{\frac{1}{6}} \mathrm{e}^{-\frac{x}{6}} \left(\int \frac{\mathrm{e}^{\frac{x}{6}}}{\sqrt{x}\,(1+x)^{\frac{7}{6}}}dx\right)}{x^{\frac{1}{6}}}$$

Verified OK.

2.508.1 Maple step by step solution

Let's solve

$(18x^3+18x^2)\,y''+(3x^3+33x^2+15x)\,y'+(5x^2+2x-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(5x^2+2x-1)y}{18x^2(1+x)}-\frac{(x^2+11x+5)y'}{6x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2+11x+5)y'}{6x(1+x)}+\frac{(5x^2+2x-1)y}{18x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x^2+11x+5}{6x(1+x)}, P_3(x)=\frac{5x^2+2x-1}{18x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=\frac{5}{6}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $18x^2(1+x)\,y''+3x(x^2+11x+5)\,y'+(5x^2+2x-1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$\left(18u^3-36u^2+18u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(3u^3+24u^2-42u+15\right)\left(\frac{d}{du}y(u)\right)+\left(5u^2-8u+2\right)y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$$u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$$

  - Shift index using $k->k-m$

$$u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(-1+6r)\,u^{-1+r}+\left(3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)\right)u^r+\left(3a_2(2+r)\,(11+6r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(-1+6r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{6}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(5+6r)-2a_0(1+3r)\,(-1+6r)=0,3a_2(2+r)\,(11+6r)-2a_1(4+3r)\,(5+6r)+$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{2a_0(18r^2+3r-1)}{3(6r^2+11r+5)}, a_2 = \frac{2a_0(81r^3+126r^2+21r+4)}{9(6r^3+29r^2+45r+22)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$18(-2a_k + a_{k-1} + a_{k+1})k^2 + 3(12(-2a_k + a_{k-1} + a_{k+1})r - 2a_k + a_{k-2} - 10a_{k-1} + 11a_{k+1})k + 1$$

- Shift index using $k- >k+2$

$$18(-2a_{k+2} + a_{k+1} + a_{k+3})(k+2)^2 + 3(12(-2a_{k+2} + a_{k+1} + a_{k+3})r - 2a_{k+2} + a_k - 10a_{k+1} + 11$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 36kra_{k+1} - 72kra_{k+2} + 18r^2a_{k+1} - 36r^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 3ra_k + 42ra_{k+1} - 150r}{3(6k^2+12kr+6r^2+35k+35r+51)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}, a\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a_k + 16a_{k+1} - 154a_{k+2}}{3(6k^2+35k+51)}, a_1 = -\frac{2a_0}{15}\right.$$

- Recursion relation for $r = \frac{1}{6}$

$$a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}$$

- Solution for $r = \frac{1}{6}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}, a_1 = 0, a\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{1}{6}}, a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 48ka_{k+1} - 162ka_{k+2} + \frac{11}{2}a_k + \frac{47}{2}a_{k+1} - 180a_{k+2}}{3(6k^2+37k+57)}, a_1 = \right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+\frac{1}{6}}\right), a_{k+3} = -\frac{18k^2a_{k+1} - 36k^2a_{k+2} + 3ka_k + 42ka_{k+1} - 150ka_{k+2} + 5a}{3(6k^2+35k+51)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve(18*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x+x^2)*diff(y(x),x)-(1-2*x-5*x^2)*y(x)=0,y(x),
```

$$y(x) = c_1 e^{-\frac{x}{6}} \left(\frac{x+1}{x}\right)^{\frac{1}{6}} + c_2 e^{-\frac{x}{6}} \left(\frac{x+1}{x}\right)^{\frac{1}{6}} \left(\int \frac{e^{\frac{x}{6}}}{(x+1)^{\frac{7}{6}} \sqrt{x}} dx\right)$$

✓ Solution by Mathematica

Time used: 0.555 (sec). Leaf size: 73

```
DSolve[18*x^2*(1+x)*y''[x]+3*x*(5+11*x+x^2)*y'[x]-(1-2*x-5*x^2)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to \frac{e^{-x/6}\left(c_2 \int_1^x \frac{e^{\frac{K[1]}{6}}\sqrt[3]{\frac{K[1]}{K[1]+1}}}{K[1]^{5/6}(K[1]+1)^{5/6}}dK[1] + c_1\right)}{\sqrt[6]{\frac{x}{x+1}}}$$

2.509 problem 523

2.509.1 Maple step by step solution 4728

Internal problem ID [7999]
Internal file name [`OUTPUT/6932_Sunday_June_05_2022_05_18_19_PM_18536581/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 523.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + x(3+2x)\,y' - (1-x)\,y = 0}$$

Writing the ode as

$$2x^2y'' + \left(2x^2+3x\right)y' + (x-1)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 2x^2+3x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 5 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 963: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{1}{4x^2} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{3}{64x^6} - \frac{1}{128x^7} + \frac{11}{256x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 + 4x + 5}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{4x + 5}{16x^2}\right) \\
&= \frac{1}{4} + \frac{4x + 5}{16x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{1}{4}\right) - (0) \\
&= \frac{1}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 4x + 5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{4x} - \frac{1}{2} \\
&= -\frac{1}{4x} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{4x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2+4x+5}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{4x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{x^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+3x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^{\frac{3}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+3x}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\sqrt{x}\, \mathrm{e}^{x} - \frac{\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x} \left(\sqrt{x}\, \mathrm{e}^{x} - \frac{\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right)}{2}\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2 \left(-\sqrt{\pi}\, \mathrm{e}^{-x}\, \mathrm{erfi}\left(\sqrt{x}\right) + 2\sqrt{x}\right)}{2x} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2 \left(-\sqrt{\pi}\, \mathrm{e}^{-x}\, \mathrm{erfi}\left(\sqrt{x}\right) + 2\sqrt{x}\right)}{2x}$$

Verified OK.

2.509.1 Maple step by step solution

Let's solve

$2x^2y'' + (2x^2 + 3x)\, y' + (x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{2x^2} - \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+2x)y'}{2x} + \frac{(x-1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3+2x}{2x}, P_3(x) = \frac{x-1}{2x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{2}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(3 + 2x)\, y' + (x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r-\frac{1}{2}\right)\left(a_k(k+r+1) + a_{k-1}\right) = 0$$

- Shift index using $k- >k+1$

  $$2\left(k+\frac{1}{2}+r\right)\left(a_{k+1}(k+2+r) + a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{k+\frac{5}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+\frac{5}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+x*(3+2*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2 \mathrm{e}^{-x} \left(\int \sqrt{x}\, \mathrm{e}^{x} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 33

```
DSolve[2*x^2*y''[x]+x*(3+2*x)*y'[x]-(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x} \left(c_2 x^{3/2} L_{-\frac{3}{2}}^{\frac{3}{2}}(x) + c_1\right)}{x}$$

2.510 problem 524

2.510.1 Maple step by step solution 4739

Internal problem ID [8000]
Internal file name [`OUTPUT/6933_Sunday_June_05_2022_05_18_22_PM_91037410/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 524.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + x(x+5)\,y' - (2-3x)\,y = 0}$$

Writing the ode as

$$2x^2y'' + \left(x^2+5x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= x^2+5x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 14x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 14x + 21$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 14x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 965: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{7}{8x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{7}{4x} - \frac{7}{2x^2} - \frac{49}{2x^3} - \frac{196}{x^4} - \frac{1715}{x^5} - \frac{31899}{2x^6} - \frac{309729}{2x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 14x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-14x + 21}{16x^2}\right) \\ &= \frac{1}{16} + \frac{-14x + 21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -14. Dividing this by leading coefficient in t which is 16 gives $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{7}{8}\right) - (0) \\ &= -\frac{7}{8} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{4}} - 0\right) &&= \frac{7}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 14x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{7}{4}$	$\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{7}{4x} + (-)\left(\frac{1}{4}\right) \\
&= \frac{7}{4x} - \frac{1}{4} \\
&= -\frac{x-7}{4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{7}{4x} - \frac{1}{4}\right)(0) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 14x + 21}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{7}{4x} - \frac{1}{4}\right) dx} \\
&= x^{\frac{7}{4}} \mathrm{e}^{-\frac{x}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + 5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{x}{4} - \frac{5 \ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{4}}}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+5x}{2x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}-\frac{5\ln(x)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} + 2\,\mathrm{e}^{\frac{x}{2}} (x^2+x+3)}{15 x^{\frac{5}{2}}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\right) + c_2 \left(\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} + 2\,\mathrm{e}^{\frac{x}{2}} (x^2+x+3)}{15 x^{\frac{5}{2}}} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + \frac{c_2 \left(-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} \mathrm{e}^{-\frac{x}{2}} - 2x^2 - 2x - 6 \right)}{15x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + \frac{c_2 \left(-i\sqrt{\pi}\,\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{x}}{2}\right) x^{\frac{5}{2}} \mathrm{e}^{-\frac{x}{2}} - 2x^2 - 2x - 6 \right)}{15x^2}$$

Verified OK.

2.510.1 Maple step by step solution

Let's solve

$2x^2y'' + (x^2 + 5x)\, y' + (3x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{2x^2} - \frac{(x+5)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+5)y'}{2x} + \frac{(3x-2)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+5}{2x}, P_3(x) = \frac{3x-2}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + x(x + 5)\, y' + (3x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(2k+2r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right)(k+r+2) = 0$

- Shift index using $k- >k+1$

  $2\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right)(k+r+3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2k+1+2r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{a_k}{2k-3}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k}{2k-3}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2k+2}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k-3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(2*x^2*diff(y(x),x$2)+x*(5+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(\int \frac{\mathrm{e}^{\frac{x}{2}}}{x^{\frac{7}{2}}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 70

```
DSolve[2*x^2*y''[x]+x*(5+x)*y'[x]-(2-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{15}\left(-\frac{2c_2(x^2+x+3)}{x^2} + 15c_1e^{-x/2}\sqrt{x} + \sqrt{2}c_2e^{-x/2}\sqrt{-x}\Gamma\left(\frac{1}{2}, -\frac{x}{2}\right)\right)$$

2.511 problem 525

2.511.1 Maple step by step solution 4750

Internal problem ID [8001]
Internal file name [`OUTPUT/6934_Sunday_June_05_2022_05_18_26_PM_80660180/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 525.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' + x(1+x)\,y' - y = 0}$$

Writing the ode as

$$3x^2y'' + \left(x^2 + x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2 + x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 967: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{1}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{6x} + \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{7}{4x^5} - \frac{7}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{2x + 7}{36x^2}\right) \\ &= \frac{1}{36} + \frac{2x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{18}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{18}\right) - (0) \\ &= \frac{1}{18} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= \frac{1}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{1}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{6}\right) \\
&= -\frac{1}{6x} - \frac{1}{6} \\
&= -\frac{1 + x}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{6}\right)^2 - \left(\frac{x^2 + 2x + 7}{36x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6x} - \frac{1}{6}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{3x^2} \, dx} \\
&= z_1 e^{-\frac{x}{6} - \frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.511.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{3}$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{3}$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(1 + x)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(\left(k+r+\frac{1}{3}\right) a_k + \frac{a_{k-1}}{3}\right)(k+r-1) = 0$

- Shift index using $k- >k+1$

  $3\left(\left(k+\frac{4}{3}+r\right) a_{k+1} + \frac{a_k}{3}\right)(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{3k+4+3r}$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{3k+7}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{3k+7}\right]$$

- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+1} = -\frac{a_k}{3k+3}$

- Solution for $r = -\frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k}{3k+3}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), a_{k+1} = -\frac{a_k}{3k+7}, b_{k+1} = -\frac{b_k}{3k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-\frac{x}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 e^{-\frac{x}{3}} \left(\int x^{\frac{1}{3}} e^{\frac{x}{3}} dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 50

```
DSolve[3*x^2*y''[x]+x*(1+x)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x/3}\left(c_2x^{2/3} - 3\sqrt[3]{3}c_1(-x)^{2/3}\Gamma\left(\frac{4}{3}, -\frac{x}{3}\right)\right)}{x}$$

2.512 problem 526

2.512.1 Maple step by step solution 4759

Internal problem ID [8002]
Internal file name [`OUTPUT/6935_Sunday_June_05_2022_05_18_29_PM_3131615/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 526.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' - xy' + (1-2x)\, y = 0}$$

Writing the ode as

$$2x^2y'' - xy' + (1-2x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 969: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\ &= \frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{2x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4}} \\ &= z_1 \left(x^{\frac{1}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2\sqrt{x}} \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{2\sqrt{x}} \sqrt{x}\right) + c_2 \left(\mathrm{e}^{2\sqrt{x}} \sqrt{x} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2\sqrt{x}}\sqrt{x} - \frac{c_2 e^{-2\sqrt{x}}\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2\sqrt{x}}\sqrt{x} - \frac{c_2 e^{-2\sqrt{x}}\sqrt{x}}{2}$$

Verified OK.

2.512.1 Maple step by step solution

Let's solve

$2x^2y'' - xy' + (1-2x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} - \frac{(2x-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{2x-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$2x^2y'' - xy' + (1-2x)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) - 2a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k - 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$2\left(k+\frac{1}{2}+r\right)(k+r)\, a_{k+1} - 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{(2k+1+2r)(k+r)}$$

- Recursion relation for $r = 1$

$a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{2a_k}{(2k+3)(k+1)}, b_{k+1} = \frac{2b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\sinh\left(2\sqrt{x}\right) + c_2\sqrt{x}\,\cosh\left(2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 41

```
DSolve[2*x^2*y''[x]-x*y'[x]+(1-2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2\sqrt{x}}\sqrt{x}\left(2c_1e^{4\sqrt{x}} - c_2\right)$$

2.513 problem 527

2.513.1 Maple step by step solution 4770

Internal problem ID [8003]
Internal file name [`OUTPUT/6936_Sunday_June_05_2022_05_18_32_PM_30130792/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 527.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2y'' + x(1+x)\,y' - (3x+1)\,y = 0}$$

Writing the ode as

$$3x^2y'' + \left(x^2+x\right)y' + (-3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= x^2 + x \\ C &= -3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 38x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 38x + 7 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 38x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 971: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{19}{18x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{19}{6x} - \frac{59}{2x^2} + \frac{1121}{2x^3} - \frac{53041}{4x^4} + \frac{1404613}{4x^5} - \frac{39845827}{4x^6} + \frac{1184064097}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{6}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+38x+7}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{1}{36}\right) + \left(\frac{38x+7}{36x^2}\right) \\
&= \frac{1}{36} + \frac{38x+7}{36x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 38. Dividing this by leading coefficient in t which is 36 gives $\frac{19}{18}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{19}{18}\right) - (0) \\
&= \frac{19}{18}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= \frac{19}{6} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{19}{18}}{\frac{1}{6}} - 0\right) &&= -\frac{19}{6}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 38x + 7}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{19}{6}$	$-\frac{19}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{19}{6}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{19}{6} - \left(\frac{7}{6}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{7}{6x} + \left(\frac{1}{6}\right) \\
&= \frac{7}{6x} + \frac{1}{6} \\
&= \frac{7 + x}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{7}{6x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(-\frac{7}{6x^2}\right) + \left(\frac{7}{6x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 38x + 7}{36x^2}\right)\right) = 0
$$

$$
\frac{(-a_1 + 20)\, x - 2a_0 + 7a_1}{3x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 70, a_1 = 20\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 20x + 70
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 20x + 70\right) e^{\int \left(\frac{7}{6x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 + 20x + 70\right) e^{\frac{x}{6} + \frac{7 \ln(x)}{6}} \\
&= \left(x^2 + 20x + 70\right) x^{\frac{7}{6}} e^{\frac{x}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{3x^2}\,dx} \\
&= z_1 e^{-\frac{x}{6}-\frac{\ln(x)}{6}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{6}}}{x^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+20x+70\right)x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x}{3}-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+20x+70\right)x\right) + c_2\left(\left(x^2+20x+70\right)x\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+20x+70\right)x + c_2\left(x^2+20x+70\right)x\left(\int \frac{\mathrm{e}^{-\frac{x}{3}}}{x^{\frac{7}{3}}\left(x^2+20x+70\right)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1(x^2+20x+70)\,x + c_2(x^2+20x+70)\,x\left(\int \frac{e^{-\frac{x}{3}}}{x^{\frac{7}{3}}(x^2+20x+70)^2}dx\right)$$

Verified OK.

2.513.1 Maple step by step solution

Let's solve

$3x^2y'' + (x^2+x)\,y' + (-3x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{3x^2} - \frac{(1+x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{3x} - \frac{(3x+1)y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1+x}{3x}, P_3(x) = -\frac{3x+1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(1+x)\,y' + (-3x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+3r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r+1)\,(k+r-1) + a_{k-1}(k-4+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+3r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, -\frac{1}{3}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $3(k+r-1)\left(k+r+\frac{1}{3}\right) a_k + a_{k-1}(k-4+r) = 0$
- Shift index using $k->k+1$

  $3(k+r)\left(k+\frac{4}{3}+r\right) a_{k+1} + a_k(k+r-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+r)(3k+4+3r)}$
- Recursion relation for $r = 1$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(3k+7)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{2a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = \frac{a_1}{20}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{70}$
- Terminating series solution of the ODE for $r = 1$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right)$
- Recursion relation for $r = -\frac{1}{3}$

  $a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}$
- Solution for $r = -\frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+1} = -\frac{a_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + \frac{2}{7}x + \frac{1}{70}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{3}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{10}{3}\right)}{\left(k-\frac{1}{3}\right)(3k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(3*x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x(x^2 + 20x + 70) + c_2 x(x^2 + 20x + 70) \left(\int \frac{e^{-\frac{x}{3}}}{x^{\frac{7}{3}} (x^2 + 20x + 70)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.156 (sec). Leaf size: 78

```
DSolve[3*x^2*y''[x]+x*(1+x)*y'[x]-(1+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x(x^2 + 20x + 70) - \frac{c_2 x(x^2 + 20x + 70)\,\Gamma\left(\frac{2}{3}, \frac{x}{3}\right)}{1680\sqrt[3]{3}} + \frac{c_2 e^{-x/3}(x^3 + 19x^2 + 54x - 18)}{1680\sqrt[3]{x}}$$

2.514 problem 528

2.514.1 Maple step by step solution 4780

Internal problem ID [8004]
Internal file name [`OUTPUT/6937_Sunday_June_05_2022_05_18_35_PM_30844535/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 528.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x+3)\, y''+x(1+5x)\, y'+(1+x)\, y=0}$$

Writing the ode as

$$\left(2x^3+6x^2\right) y''+\left(5x^2+x\right) y'+(1+x)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+6x^2 \\ B &= 5x^2+x \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 - 30x - 35 \\ t &= 16\left(x^2 + 3x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 973: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{108x} + \frac{5}{108\,(x+3)} - \frac{35}{144x^2} + \frac{7}{36\,(x+3)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 30x - 35}{16\,(x^2+3x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 30x - 35}{16\left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{6\left(x+3\right)} + \frac{5}{12x} + (-)\left(0\right) \\ &= -\frac{1}{6\left(x+3\right)} + \frac{5}{12x} \\ &= \frac{x+5}{4x\left(x+3\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(x+3)} + \frac{5}{12x}\right)(0) + \left(\left(\frac{1}{6(x+3)^2} - \frac{5}{12x^2}\right) + \left(-\frac{1}{6(x+3)} + \frac{5}{12x}\right)^2 - \left(\frac{-3x^2 - 30x - 3}{16(x^2+3x)^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{6(x+3)} + \frac{5}{12x}\right) dx} \\ &= \frac{x^{\frac{5}{12}}}{(x+3)^{\frac{1}{6}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2+x}{2x^3+6x^2}\, dx} \\ &= z_1 e^{-\frac{7\ln(x+3)}{6} - \frac{\ln(x)}{12}} \\ &= z_1 \left(\frac{1}{(x+3)^{\frac{7}{6}}\, x^{\frac{1}{12}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2+x}{2x^3+6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x+3)}{3}-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}}\left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}} dx\right)}{(x+3)^{\frac{4}{3}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}} dx\right)}{(x+3)^{\frac{4}{3}}}
$$

Verified OK.

2.514.1 Maple step by step solution

Let's solve

$(2x^3 + 6x^2)\, y'' + (5x^2 + x)\, y' + (1 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(1+x)y}{2x^2(x+3)} - \frac{(1+5x)y'}{2x(x+3)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(1+5x)y'}{2x(x+3)} + \frac{(1+x)y}{2x^2(x+3)} = 0$$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{1+5x}{2x(x+3)}, P_3(x) = \frac{1+x}{2x^2(x+3)}\right]$$

- ◦ $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

$$\left((x+3) \cdot P_2(x)\right)\Bigg|_{x=-3} = \frac{7}{3}$$

- ◦ $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

$$\left((x+3)^2 \cdot P_3(x)\right)\Bigg|_{x=-3} = 0$$

- ◦ $x = -3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -3$

- Multiply by denominators

$$2x^2(x+3)\, y'' + x(1+5x)\, y' + (1+x)\, y = 0$$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

$$(2u^3 - 12u^2 + 18u)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 29u + 42)\left(\frac{d}{du}y(u)\right) + (-2+u)\, y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$6a_0r(4+3r)\, u^{-1+r} + \left(6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2)\right) u^r + \left(\sum_{k=1}^{\infty} (6a_{k+1}(k+r+1)\,(3k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$6r(4+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{4}{3}\right\}$$

- Each term must be 0

$$6a_1(1+r)\,(7+3r) - a_0(12r^2+17r+2) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (4(-6a_k + a_{k-1} + 9a_{k+1})\, r - 17a_k - a_{k-1} + 60a_{k+1})\, k + 2(-6a_k +$$

- Shift index using $k->k+1$

$$2(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (4(-6a_{k+1} + a_k + 9a_{k+2})\, r - 17a_{k+1} - a_k + 60a_{k+2})\,(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 4kra_k - 24kra_{k+1} + 2r^2a_k - 12r^2a_{k+1} + 3ka_k - 41ka_{k+1} + 3ra_k - 41ra_{k+1} + a_k - 31a_{k+1}}{6(3k^2+6kr+3r^2+16k+16r+20)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 12k^2a_{k+1} + 3ka_k - 41ka_{k+1} + a_k - 31a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31 a_{k+1}}{6(3k^2+16k+20)}, 42a_1 - 2a_0 = 0\right]$$

- Recursion relation for $r = -\frac{4}{3}$

$$a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}$$

- Solution for $r = -\frac{4}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Revert the change of variables $u = x+3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{4}{3}}, a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} - \frac{7}{3}k a_k - 9k a_{k+1} + \frac{5}{9}a_k + \frac{7}{3}a_{k+1}}{6(3k^2+8k+4)}, -6a_1 - \frac{2a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{4}{3}}\right), a_{k+2} = -\frac{2k^2 a_k - 12k^2 a_{k+1} + 3k a_k - 41k a_{k+1} + a_k - 31 a_{k+1}}{6(3k^2+16k+20)}, 42a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(2*x^2*(3+x)*diff(y(x),x$2)+x*(1+5*x)*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{(x+3)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{(x+3)^{\frac{1}{3}}}{x^{\frac{5}{6}}}dx\right)}{(x+3)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 50

```
DSolve[2*x^2*(3+x)*y''[x]+x*(1+5*x)*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6\sqrt[3]{3}c_2\sqrt[6]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{3},\frac{1}{6},\frac{7}{6},-\frac{x}{3}\right)+c_1\right)}{(x+3)^{4/3}}$$

2.515 problem 529

2.515.1 Maple step by step solution 4791

Internal problem ID [8005]
Internal file name [`OUTPUT/6938_Sunday_June_05_2022_05_18_38_PM_5219707/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 529.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(4+x)\, y'' - x(-3x+1)\, y' + y = 0}$$

Writing the ode as

$$x^2(4+x)\, y'' + \left(3x^2 - x\right) y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(4+x) \\ B &= 3x^2 - x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 6x - 7$$
$$t = 4\left(x^2 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 975: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -4$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{128x} + \frac{65}{64(4+x)^2} + \frac{5}{128(4+x)} - \frac{7}{64x^2}$$

For the pole at $x = -4$ let b be the coefficient of $\frac{1}{(4+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 6x - 7}{4(x^2 + 4x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 6x - 7}{4\left(x^2 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-4	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{8\left(4 + x\right)} + \frac{1}{8x} + (-)\left(0\right) \\
&= -\frac{5}{8\left(4 + x\right)} + \frac{1}{8x} \\
&= -\frac{x - 1}{2x\left(4 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right)(0) + \left(\left(\frac{5}{8(4+x)^2} - \frac{1}{8x^2}\right) + \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right)^2 - \left(\frac{3x^2 - 6x - 7}{4(x^2+4x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{5}{8(4+x)} + \frac{1}{8x}\right) dx} \\ &= \frac{x^{\frac{1}{8}}}{(4+x)^{\frac{5}{8}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x^2 - x}{x^2(4+x)} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \frac{13\ln(4+x)}{8}} \\ &= z_1 \left(\frac{x^{\frac{1}{8}}}{(4+x)^{\frac{13}{8}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-x}{x^2(4+x)}dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{4}-\frac{13\ln(4+x)}{4}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} \right) + c_2 \left(\frac{x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)}{(4+x)^{\frac{9}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{(4+x)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(4+x)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx \right)}{(4+x)^{\frac{9}{4}}}$$

Verified OK.

2.515.1 Maple step by step solution

Let's solve

$x^2(4+x)\, y'' + (3x^2 - x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x^2(4+x)} - \frac{(3x-1)y'}{x(4+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-1)y'}{x(4+x)} + \frac{y}{x^2(4+x)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x-1}{x(4+x)}, P_3(x) = \frac{1}{x^2(4+x)}\right]$

  - $(4+x) \cdot P_2(x)$ is analytic at $x = -4$

    $\left((4+x) \cdot P_2(x)\right)\Big|_{x=-4} = \frac{13}{4}$

  - $(4+x)^2 \cdot P_3(x)$ is analytic at $x = -4$

    $\left((4+x)^2 \cdot P_3(x)\right)\Big|_{x=-4} = 0$

  - $x = -4$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -4$

- Multiply by denominators

  $x^2(4+x)\, y'' + x(3x-1)\, y' + y = 0$

- Change variables using $x = u-4$ so that the regular singular point is at $u = 0$

  $(u^3 - 8u^2 + 16u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 25u + 52)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$

- Shift index using $k->k +2-m$

  $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

  Rewrite ODE with series expansions

  $4 a_{0} r \left(9+4 r \right) u^{-1+r}+\left(4 a_{1} \left(1+r \right) \left(13+4 r \right)-a_{0} \left(8 r^{2}+17 r -1\right)\right) u^{r}+\left(\sum_{k =1}^{\infty}\left(4 a_{k +1} \left(k +1+r \right) \left(4k\right.\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $4 r \left(9+4 r \right)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{9}{4}\right\}$

- Each term must be 0

  $4 a_{1} \left(1+r \right) \left(13+4 r \right)-a_{0} \left(8 r^{2}+17 r -1\right)=0$

- Each term in the series must be 0, giving the recursion relation

  $\left(-8 a_{k}+a_{k -1}+16 a_{k +1}\right) k^{2}+\left(2 \left(-8 a_{k}+a_{k -1}+16 a_{k +1}\right) r -17 a_{k}+68 a_{k +1}\right) k +\left(-8 a_{k}+a_{k -1}+\right.$

- Shift index using $k->k +1$

  $\left(-8 a_{k +1}+a_{k}+16 a_{k +2}\right) \left(k +1\right)^{2}+\left(2 \left(-8 a_{k +1}+a_{k}+16 a_{k +2}\right) r -17 a_{k +1}+68 a_{k +2}\right) \left(k +1\right)+\left(-\right.$

- Recursion relation that defines series solution to ODE

  $a_{k +2}=-\frac{k^{2} a_{k}-8 k^{2} a_{k +1}+2 k r a_{k}-16 k r a_{k +1}+r^{2} a_{k}-8 r^{2} a_{k +1}+2 k a_{k}-33 k a_{k +1}+2 r a_{k}-33 r a_{k +1}-24 a_{k +1}}{4 \left(4 k^{2}+8 k r +4 r^{2}+25 k +25 r +34\right)}$

- Recursion relation for $r =0$

  $a_{k +2}=-\frac{k^{2} a_{k}-8 k^{2} a_{k +1}+2 k a_{k}-33 k a_{k +1}-24 a_{k +1}}{4 \left(4 k^{2}+25 k +34\right)}$

- Solution for $r =0$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +2}=-\frac{k^{2} a_{k}-8 k^{2} a_{k +1}+2 k a_{k}-33 k a_{k +1}-24 a_{k +1}}{4 \left(4 k^{2}+25 k +34\right)}, 52 a_{1}+a_{0}=0\right]$

- Revert the change of variables $u =4+x$

  $\left[y =\sum_{k =0}^{\infty}a_{k} \left(4+x \right)^{k}, a_{k +2}=-\frac{k^{2} a_{k}-8 k^{2} a_{k +1}+2 k a_{k}-33 k a_{k +1}-24 a_{k +1}}{4 \left(4 k^{2}+25 k +34\right)}, 52 a_{1}+a_{0}=0\right]$

- Recursion relation for $r =-\frac{9}{4}$

  $a_{k +2}=-\frac{k^{2} a_{k}-8 k^{2} a_{k +1}-\frac{5}{2} k a_{k}+3 k a_{k +1}+\frac{9}{16} a_{k}+\frac{39}{4} a_{k +1}}{4 \left(4 k^{2}+7 k -2\right)}$

- Solution for $r =-\frac{9}{4}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}-\frac{5}{2}k a_k+3k a_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Revert the change of variables $u=4+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(4+x)^{k-\frac{9}{4}}, a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}-\frac{5}{2}k a_k+3k a_{k+1}+\frac{9}{16}a_k+\frac{39}{4}a_{k+1}}{4(4k^2+7k-2)}, -20a_1-\frac{5a_0}{4}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k(4+x)^k\right)+\left(\sum_{k=0}^{\infty} b_k(4+x)^{k-\frac{9}{4}}\right), a_{k+2}=-\frac{k^2 a_k-8k^2 a_{k+1}+2k a_k-33k a_{k+1}-24a_{k+1}}{4(4k^2+25k+34)}, 52a_1+$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x^2*(4+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{(x+4)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{(x+4)^{\frac{5}{4}}}{x^{\frac{1}{4}}} dx\right)}{(x+4)^{\frac{9}{4}}}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 89

```
DSolve[x^2*(4+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[4]{x}\left(-10c_2 \arctan\left(\sqrt[4]{\frac{x}{x+4}}\right) + 10c_2 \operatorname{arctanh}\left(\sqrt[4]{\frac{x}{x+4}}\right) + c_2\sqrt[4]{x+4}x^{7/4} + 9c_2\sqrt[4]{x+4}x^{3/4} + 2c_1\right)}{2(x+4)^{9/4}}$$

2.516 problem 530

2.516.1 Maple step by step solution 4801

Internal problem ID [8006]
Internal file name [`OUTPUT/6939_Sunday_June_05_2022_05_18_41_PM_75890810/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 530.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' + 5xy' + (1 + x)\, y = 0}$$

Writing the ode as

$$2x^2y'' + 5xy' + (1 + x)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 5x \\ C &= 1 + x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 - 8x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 - 8x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 977: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right)\\ &= \frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{8x+1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1+2\sqrt{2}\sqrt{-x}}{4x} dx} \\
&= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\sqrt{-x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{2x^2} \, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{\sqrt{2}\sqrt{-x}}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{2x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\sqrt{2}\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{2}\sqrt{-x}}\right)}{2\sqrt{x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{2}\sqrt{-x}}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{2}\sqrt{-x}}}{x} \left(\frac{\sqrt{2}\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{2}\sqrt{-x}}\right)}{2\sqrt{x}}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 e^{\sqrt{2}\,\sqrt{-x}}}{x} - \frac{c_2\sqrt{2}\,\sqrt{-x}\left(e^{\sqrt{2}\,\sqrt{-x}} - e^{-\sqrt{2}\,\sqrt{-x}}\right)}{2x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\sqrt{2}\,\sqrt{-x}}}{x} - \frac{c_2\sqrt{2}\,\sqrt{-x}\left(e^{\sqrt{2}\,\sqrt{-x}} - e^{-\sqrt{2}\,\sqrt{-x}}\right)}{2x^{\frac{3}{2}}}$$

Verified OK.

2.516.1 Maple step by step solution

Let's solve

$2x^2y'' + 5xy' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{2x} - \frac{(1+x)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2x} + \frac{(1+x)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5}{2x}, P_3(x) = \frac{1+x}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 5xy' + (1+x)\,y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(1+2r)\,x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+1)\,(2k+2r+1) + a_{k-1})\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-1, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+r+1)\left(k+r+\frac{1}{2}\right) a_k + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $2(k+2+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(2k+3+2r)}$
- Recursion relation for $r = -1$

$a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}$

- Solution for $r = -1$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = -\frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\sqrt{x}\,\sqrt{2}\right)}{x} + \frac{c_2 \cos\left(\sqrt{x}\,\sqrt{2}\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 60

```
DSolve[2*x^2*y''[x]+5*x*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{i\sqrt{2}\sqrt{x}} + i\sqrt{2}c_2 e^{-i\sqrt{2}\sqrt{x}}}{2x}$$

2.517 problem 531

2.517.1 Maple step by step solution 4812

Internal problem ID [8007]
Internal file name [`OUTPUT/6940_Sunday_June_05_2022_05_18_46_PM_35543723/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 531.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{6x^2y'' + x(10 - x)\, y' - (x + 2)\, y = 0}$$

Writing the ode as

$$6x^2y'' + \left(-x^2 + 10x\right) y' + (-x - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= -x^2 + 10x \\ C &= -x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 28}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 28$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 28}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 979: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{144} + \frac{1}{36x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{12} + \frac{1}{6x} + \frac{1}{x^2} - \frac{2}{x^3} - \frac{2}{x^4} + \frac{28}{x^5} - \frac{56}{x^6} - \frac{272}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{12}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{12} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{144}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 28}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{1}{144}\right) + \left(\frac{4x + 28}{144x^2}\right) \\ &= \frac{1}{144} + \frac{4x + 28}{144x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 144 gives $\frac{1}{36}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{36}\right) - (0) \\ &= \frac{1}{36} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{12} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= \frac{1}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{36}}{\frac{1}{12}} - 0\right) &= -\frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 28}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{6} - \left(-\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + (-)\left(\frac{1}{12}\right) \\
&= -\frac{1}{6x} - \frac{1}{12} \\
&= -\frac{x+2}{12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{6x} - \frac{1}{12}\right)(0) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} - \frac{1}{12}\right)^2 - \left(\frac{x^2+4x+28}{144x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{6x} - \frac{1}{12}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{12}}}{x^{\frac{1}{6}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+10x}{6x^2} \, dx} \\
&= z_1 e^{\frac{x}{12} - \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{12}}}{x^{\frac{5}{6}}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+10x}{6x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x}{6}-\frac{5\ln(x)}{3}}}{(y_1)^2}\, dx \\ &= y_1 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right)}{x}$$

Verified OK.

2.517.1 Maple step by step solution

Let's solve

$6x^2y'' + (-x^2 + 10x)\, y' + (-x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x+2)y}{6x^2} + \frac{(x-10)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-10)y'}{6x} - \frac{(x+2)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-10}{6x}, P_3(x) = -\frac{x+2}{6x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{3}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' - x(x - 10)\, y' + (-x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

$$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

$$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_{0}(1+r)\,(-1+3r)\, x^{r}+\left(\sum_{k=1}^{\infty}\left(2a_{k}(k+r+1)\,(3k+3r-1)-a_{k-1}(k+r)\right) x^{k+r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$2(1+r)\,(-1+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{-1,\frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6(k+r+1)\left(k+r-\frac{1}{3}\right) a_{k}-a_{k-1}(k+r)=0$$

- Shift index using $k->k+1$

$$6(k+2+r)\left(k+\frac{2}{3}+r\right) a_{k+1}-a_{k}(k+r+1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_{k}(k+r+1)}{2(k+2+r)(3k+2+3r)}$$

- Recursion relation for $r=-1$

$$a_{k+1}=\frac{a_{k}k}{2(k+1)(3k-1)}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k-1}, a_{k+1}=\frac{a_{k}k}{2(k+1)(3k-1)}\right]$$

- Recursion relation for $r=\frac{1}{3}$

$$a_{k+1}=\frac{a_{k}\left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)(3k+3)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)\left(3k+3\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = \frac{a_k k}{2(k+1)\left(3k-1\right)}, b_{k+1} = \frac{b_k \left(k+\frac{4}{3}\right)}{2\left(k+\frac{7}{3}\right)\left(3k+3\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(6*x^2*diff(y(x),x$2)+x*(10-x)*diff(y(x),x)-(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + \frac{c_2\left(\int x^{\frac{1}{3}} \mathrm{e}^{\frac{x}{6}} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 38

```
DSolve[6*x^2*y''[x]+x*(10-x)*y'[x]-(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sqrt[3]{x} L_{-\frac{4}{3}}^{\frac{4}{3}}\left(\frac{x}{6}\right) + \frac{6\sqrt[3]{6}c_1}{x}$$

2.518 problem 532

2.518.1 Maple step by step solution 4822

Internal problem ID [8008]
Internal file name [`OUTPUT/6941_Sunday_June_05_2022_05_18_49_PM_11242400/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 532.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(3+4x)\,y''+x(11+4x)\,y'-(3+4x)\,y=0}$$

Writing the ode as

$$\left(4x^3+3x^2\right)y''+\left(4x^2+11x\right)y'+(-3-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+3x^2 \\ B &= 4x^2+11x \\ C &= -3-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x^2 + 8x + 91}{4\,(4x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 48x^2 + 8x + 91 \\ t &= 4\big(4x^2 + 3x\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x^2 + 8x + 91}{4\,(4x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 981: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(4x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{176}{27x} + \frac{91}{36x^2} + \frac{28}{9\left(\frac{3}{4} + x\right)^2} + \frac{176}{27\left(\frac{3}{4} + x\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{91}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{13}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{6}
\end{aligned}$$

For the pole at $x = -\frac{3}{4}$ let b be the coefficient of $\frac{1}{\left(\frac{3}{4}+x\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{28}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{4}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{48x^2 + 8x + 91}{4\left(4x^2 + 3x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x^2+8x+91}{4\left(4x^2+3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{13}{6}$	$-\frac{7}{6}$
$-\frac{3}{4}$	2	0	$\frac{7}{3}$	$-\frac{4}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)} + (-)\,(0) \\
&= -\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)} \\
&= \frac{-7-20x}{8x^2+6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right)(2x+a_1) + \left(\left(\frac{7}{6x^2} + \frac{4}{3\left(\frac{3}{4}+x\right)^2}\right) + \left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right)^2 - \left(\frac{48x^2+8x}{4\,(4x^2+3}\right.\right.
$$

$$
\frac{12a_1x - 8x + 32a_0 -}{x\,(3+4x)}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{7}{48}, a_1 = \frac{2}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{2}{3}x + \frac{7}{48}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{\int \left(-\frac{7}{6x} - \frac{4}{3\left(\frac{3}{4}+x\right)}\right) dx} \\
&= \left(x^2 + \frac{2}{3}x + \frac{7}{48}\right) e^{-\frac{7\ln(x)}{6} - \frac{4\ln(3+4x)}{3}} \\
&= \frac{x^2 + \frac{2}{3}x + \frac{7}{48}}{x^{\frac{7}{6}}\,(3+4x)^{\frac{4}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2+11x}{4x^3+3x^2}\,dx} \\
&= z_1 e^{-\frac{11\ln(x)}{6}+\frac{4\ln(3+4x)}{3}} \\
&= z_1\left(\frac{(3+4x)^{\frac{4}{3}}}{x^{\frac{11}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+11x}{4x^3+3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{11\ln(x)}{3}+\frac{8\ln(3+4x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{2304x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\right) + c_2\left(\frac{x^2+\frac{2}{3}x+\frac{7}{48}}{x^3}\left(\int \frac{2304x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+\frac{2}{3}x+\frac{7}{48}\right)}{x^3} + \frac{c_2(2304x^2+1536x+336)\left(\int \frac{x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)}{x^3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1\left(x^2 + \frac{2}{3}x + \frac{7}{48}\right)}{x^3} + \frac{c_2(2304x^2 + 1536x + 336)\left(\int \frac{x^{\frac{7}{3}}(3+4x)^{\frac{8}{3}}}{(48x^2+32x+7)^2}dx\right)}{x^3}$$

Verified OK.

2.518.1 Maple step by step solution

Let's solve

$(4x^3 + 3x^2)\, y'' + (4x^2 + 11x)\, y' + (-3 - 4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2} - \frac{(11+4x)y'}{x(3+4x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(11+4x)y'}{x(3+4x)} - \frac{y}{x^2} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{11+4x}{x(3+4x)}, P_3(x) = -\frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{11}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(3 + 4x)\, y'' + x(11 + 4x)\, y' + (-3 - 4x)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\, (k + 1 - m + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3 + r)\,(-1 + 3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k + r + 3)\,(3k + 3r - 1) + 4a_{k-1}(k + r)\,(k - 2 + r)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3 + r)\,(-1 + 3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-3, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k + r - \frac{1}{3}\right)(k + r + 3)\, a_k + 4a_{k-1}(k + r)\,(k - 2 + r) = 0$$

- Shift index using $k- >k + 1$

$$3\left(k + \frac{2}{3} + r\right)(k + 4 + r)\, a_{k+1} + 4a_k(k + r + 1)\,(k + r - 1) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k(k+r+1)(k+r-1)}{(3k+2+3r)(k+4+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{4a_k(k-2)(k-4)}{(3k-7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{32a_0}{7}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{3a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{48a_0}{7}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{4a_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(\frac{48}{7}x^2 + \frac{32}{7}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{4b_k\left(k+\frac{4}{3}\right)\left(k-\frac{2}{3}\right)}{(3k+3)\left(k+\frac{13}{3}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 60

```
dsolve(x^2*(3+4*x)*diff(y(x),x$2)+x*(11+4*x)*diff(y(x),x)-(3+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(48x^2+32x+7)}{x^3} + \frac{c_2(48x^2+32x+7)\left(\int \frac{(4x+3)^{\frac{8}{3}} x^{\frac{7}{3}}}{(48x^2+32x+7)^2} dx\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.313 (sec). Leaf size: 339

```
DSolve[x^2*(3+4*x)*y''[x]+x*(11+4*x)*y'[x]-(3+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$y(x)$

$$\rightarrow \frac{-12\sqrt[3]{2}\sqrt{3}c_2(48x^2+32x+7)\arctan\left(\frac{\sqrt{3}\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{8x+6}}\right)+384c_2(4x+3)^{2/3}x^{10/3}+576c_2(4x+3)^{2/3}x^{7/3}}{}$$

2.519 problem 533

2.519.1 Maple step by step solution 4832

Internal problem ID [8009]
Internal file name [`OUTPUT/6942_Sunday_June_05_2022_05_18_52_PM_39165776/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 533.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(3x+2)\,y''+x(4+11x)\,y'-(1-x)\,y=0}$$

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(11x^2+4x\right)y'+(x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 11x^2+4x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{16\left(3x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -35$$
$$t = 16(3x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{16\left(3x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 983: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(3x+2)^2$. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144\left(x + \frac{2}{3}\right)^2}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{16\,(3x+2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{16 (3x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{2}{3}$	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12\left(x+\frac{2}{3}\right)} + (-)(0) \\
&= \frac{5}{12\left(x+\frac{2}{3}\right)} \\
&= \frac{5}{12x+8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{5}{12\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{5}{12\left(x+\frac{2}{3}\right)}\right)^2 - \left(-\frac{35}{16\left(3x+2\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{5}{12\left(x+\frac{2}{3}\right)} dx} \\
&= (3x+2)^{\frac{5}{12}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^2+4x}{6x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(3x+2)}{12}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,(3x+2)^{\frac{5}{12}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^2+4x}{6x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(3x+2)}{6}}}{(y_1)^2}\,dx \\
&= y_1 \left(2(3x+2)^{\frac{1}{6}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x}}\right) + c_2\left(\frac{1}{\sqrt{x}}\left(2(3x+2)^{\frac{1}{6}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x}} + \frac{2c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{\sqrt{x}} + \frac{2c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}}
$$

Verified OK.

2.519.1 Maple step by step solution

Let's solve

$(6x^3+4x^2)\,y'' + (11x^2+4x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{2x^2(3x+2)} - \frac{(4+11x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(4+11x)y'}{2x(3x+2)} + \frac{(x-1)y}{2x^2(3x+2)} = 0$$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{4+11x}{2x(3x+2)}, P_3(x) = \frac{x-1}{2x^2(3x+2)}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left.(x \cdot P_2(x))\right|_{x=0} = 1$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2(3x+2)\, y'' + x(4+11x)\, y' + (x-1)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)(2k+2r-1) + a_{k-1}(2k+2r-1)(3k-2+3r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{3r}{2}-1\right)a_{k-1} + a_k\left(k+r+\frac{1}{2}\right)\right) = 0$$

- Shift index using $k- >k+1$

$$4\left(k+r+\frac{1}{2}\right)\left(\left(\frac{3k}{2}+\frac{1}{2}+\frac{3r}{2}\right)a_k + a_{k+1}\left(k+\frac{3}{2}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{(3k+3r+1)a_k}{2k+3+2r}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{\left(3k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = -\frac{\left(3k+\frac{5}{2}\right)b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+11*x)*diff(y(x),x)-(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2(3x+2)^{\frac{1}{6}}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 32

```
DSolve[2*x^2*(2+3*x)*y''[x]+x*(4+11*x)*y'[x]-(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{c_2\sqrt[6]{6x+4} + 2^{5/6}c_1}{\sqrt{x}}$$

2.520 problem 534

2.520.1 Maple step by step solution 4842

Internal problem ID [8010]
Internal file name [OUTPUT/6943_Sunday_June_05_2022_05_18_55_PM_28263145/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 534.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+2)\,y''+5x(1-x)\,y'-(-8x+2)\,y=0}$$

Writing the ode as

$$x^2(x+2)\,y''+\left(-5x^2+5x\right)y'+(8x-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x+2) \\ B &= -5x^2+5x \\ C &= 8x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 126x + 21$$
$$t = 4\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 985: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{147}{16x} + \frac{147}{16(x+2)} + \frac{285}{16(x+2)^2} + \frac{21}{16x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{285}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{19}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{15}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 - 126x + 21}{4(x^2+2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 126x + 21}{4\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{19}{4}$	$-\frac{15}{4}$
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{9}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{15}{4\left(x+2\right)} - \frac{3}{4x} + (-)\left(0\right) \\
&= -\frac{15}{4\left(x+2\right)} - \frac{3}{4x} \\
&= -\frac{3(3x+1)}{2x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6x a_3 + 2a_2\right) + 2\left(-\frac{15}{4\left(x+2\right)} - \frac{3}{4x}\right)\left(4x^3 + 3x^2 a_3 + 2a_2 x + a_1\right) + \left(\left(\frac{15}{4\left(x+2\right)^2} + \frac{3}{4x^2}\right) + \left(-\frac{}{4}\right.\right.$$

$$\frac{3(4 + a_3)\,x^3 + (8a_2 + }{}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{40}, a_1 = \frac{1}{5}, a_2 = \frac{3}{2}, a_3 = -4\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{\int \left(-\frac{15}{4(x+2)} - \frac{3}{4x}\right) dx} \\
&= \left(x^4 - 4x^3 + \frac{3}{2}x^2 + \frac{1}{5}x + \frac{1}{40}\right) e^{-\frac{3\ln(x)}{4} - \frac{15\ln(x+2)}{4}} \\
&= \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^{\frac{3}{4}}\left(x+2\right)^{\frac{15}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x^2 + 5x}{x^2(x+2)} \, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{4} + \frac{15\ln(x+2)}{4}} \\
&= z_1 \left(\frac{\left(x+2\right)^{\frac{15}{4}}}{x^{\frac{5}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x^2+5x}{x^2(x+2)}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{2}+\frac{15\ln(x+2)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{80\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right) + \left(525x^4 - 2100x^3 + \frac{1575}{2}x^2 + 105\right.\right.}{\sqrt{x\,(x+2)}\,\left(40x^4 - 160x^3 + 60x^2 + 8x + 1\right.}\right.
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2} \right) \\
&\quad + c_2 \left(\frac{40x^4 - 160x^3 + 60x^2 + 8x + 1}{40x^2} \left(\frac{80\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right)\right.}{\sqrt{x\,(x+\right.}\right.\right.
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1(40x^4 - 160x^3 + 60x^2 + 8x + 1)}{40x^2} \\
&\quad + \frac{2c_2\sqrt{x}\left(\left(-525x^4 + 2100x^3 - \frac{1575}{2}x^2 - 105x - \frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right) + \left(525x^4 - 2100x^3 + \frac{1575}{2}x^2 + 10\right.\right.}{\sqrt{x\,(x+2)}\,\left(x + \sqrt{x\,(x+2)}\right)}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(40x^4-160x^3+60x^2+8x+1)}{40x^2} + \frac{2c_2\sqrt{x}\left(\left(-525x^4+2100x^3-\frac{1575}{2}x^2-105x-\frac{105}{8}\right)\ln\left(\frac{\sqrt{x(x+2)}-x}{x}\right)+\left(525x^4-2100x^3+\frac{1575}{2}x^2+10\right.\right.}{\sqrt{x(x+2)}\left(x+\sqrt{x(x+2)}\right)}$$

Verified OK.

2.520.1 Maple step by step solution

Let's solve

$x^2(x+2)\,y''+(-5x^2+5x)\,y'+(8x-2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(4x-1)y}{x^2(x+2)}+\frac{5(x-1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{5(x-1)y'}{x(x+2)}+\frac{2(4x-1)y}{x^2(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{5(x-1)}{x(x+2)}, P_3(x)=\frac{2(4x-1)}{x^2(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-\frac{15}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $x^2(x+2)\,y''-5x(x-1)\,y'+(8x-2)\,y=0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $$(u^3 - 4u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (-5u^2 + 25u - 30)\left(\frac{d}{du}y(u)\right) + (8u - 18)\,y(u) = 0$$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0 r(-17+2r)\, u^{-1+r} + \left(2a_1(1+r)(-15+2r) - a_0(4r^2 - 29r + 18)\right) u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+1+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(-17+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{17}{2}\right\}$$

- Each term must be 0

$$2a_1(1+r)(-15+2r)-a_0(4r^2-29r+18)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k+a_{k-1}+4a_{k+1})k^2+((-8a_k+2a_{k-1}+8a_{k+1})r+29a_k-8a_{k-1}-26a_{k+1})k+(-4a_k+a$$

- Shift index using $k->k+1$

$$(-4a_{k+1}+a_k+4a_{k+2})(k+1)^2+((-8a_{k+1}+2a_k+8a_{k+2})r+29a_{k+1}-8a_k-26a_{k+2})(k+1)-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+2kra_k-8kra_{k+1}+r^2a_k-4r^2a_{k+1}-6ka_k+21ka_{k+1}-6ra_k+21ra_{k+1}+8a_k+7a_{k+1}}{2(2k^2+4kr+2r^2-9k-9r-26)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30a_1-18a_0=0\right]$$

- Revert the change of variables $u=x+2$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+2)^k, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30a_1-18a_0=0\right]$$

- Recursion relation for $r=\frac{17}{2}$

$$a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}$$

- Solution for $r=\frac{17}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+\frac{17}{2}}, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1-\frac{121a_0}{2}=0\right]$$

- Revert the change of variables $u=x+2$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+2)^{k+\frac{17}{2}}, a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}+11ka_k-47ka_{k+1}+\frac{117}{4}a_k-\frac{207}{2}a_{k+1}}{2(2k^2+25k+42)}, 38a_1-\frac{121a_0}{2}=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(x+2)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x+2)^{k+\frac{17}{2}}\right), a_{k+2}=-\frac{k^2a_k-4k^2a_{k+1}-6ka_k+21ka_{k+1}+8a_k+7a_{k+1}}{2(2k^2-9k-26)}, -30\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 88

```
dsolve(x^2*(2+x)*diff(y(x),x$2)+5*x*(1-x)*diff(y(x),x)-(2-8*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(40x^4-160x^3+60x^2+8x+1)}{x^2} + \frac{c_2(40x^4-160x^3+60x^2+8x+1)\left(\int \frac{x^{\frac{3}{2}}(x+2)^{\frac{15}{2}}}{(40x^4-160x^3+60x^2+8x+1)^2}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 47.99 (sec). Leaf size: 1347

```
DSolve[x^2*(2+x)*y''[x]+5*x*(1-x)*y'[x]-(2-8*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

Too large to display

2.521 problem 535

2.521.1 Maple step by step solution 4853

Internal problem ID [8011]
Internal file name [`OUTPUT/6944_Sunday_June_05_2022_05_21_02_PM_98081824/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 535.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^2(1-x^2)\,y''+2x(-13x^2+1)\,y'+(-9x^2+1)\,y=0}$$

Writing the ode as

$$\left(-8x^4+8x^2\right)y''+\left(-26x^3+2x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-8x^4+8x^2\\B&=-26x^3+2x\\C&=-9x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7x^4 - 26x^2 - 15$$
$$t = 64\left(x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 987: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(1+x)^2} - \frac{3}{16(x-1)^2} - \frac{15}{64x^2} + \frac{1}{4x-4} - \frac{1}{4(1+x)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-7x^4 - 26x^2 - 15}{64\left(x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{8}$	$\frac{3}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{7}{8} - \left(\frac{7}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} + (0) \\
&= \frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4} \\
&= \frac{7x^2-3}{8x^3-8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right)(0) + \left(\left(-\frac{3}{8x^2} - \frac{1}{4(x-1)^2} - \frac{1}{4(1+x)^2}\right) + \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8x} + \frac{1}{4x-4} + \frac{1}{4x+4}\right) dx} \\
&= x^{\frac{3}{8}} (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-26x^3+2x}{-8x^4+8x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{8} - \frac{3\ln(1+x)}{4} - \frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{8}} (1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-26x^3+2x}{-8x^4+8x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{4}-\frac{3\ln(1+x)}{2}-\frac{3\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}\right) + c_2 \left(\frac{(x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}}}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}} + \frac{c_2 (x^2-1)^{\frac{1}{4}} x^{\frac{1}{4}} \left(\int \frac{1}{x^{\frac{3}{4}}\sqrt{x^2-1}} dx\right)}{(1+x)^{\frac{3}{4}} (x-1)^{\frac{3}{4}}}$$

Verified OK.

2.521.1 Maple step by step solution

Let's solve

$(-8x^4+8x^2)\,y''+(-26x^3+2x)\,y'+(-9x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x^2-1)y}{8x^2(x^2-1)}-\frac{(13x^2-1)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(13x^2-1)y'}{4x(x^2-1)}+\frac{(9x^2-1)y}{8x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{13x^2-1}{4x(x^2-1)},P_3(x)=\frac{9x^2-1}{8x^2(x^2-1)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=\frac{3}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $8y''x^2(x^2-1)+2x(13x^2-1)\,y'+y(9x^2-1)=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(8u^4-32u^3+40u^2-16u)\left(\frac{d^2}{du^2}y(u)\right)+(26u^3-78u^2+76u-24)\left(\frac{d}{du}y(u)\right)+(9u^2-18u+8)\,y$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-8a_0 r(1+2r)\, u^{-1+r} + (-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r))\, u^r + (-8a_2(2+r)\,(5+2r)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-8r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

  $$[-8a_1(1+r)\,(3+2r) + 4a_0(1+2r)\,(2+5r) = 0, -8a_2(2+r)\,(5+2r) + 4a_1(3+2r)\,(7+5r) -$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{a_0(10r^2+9r+2)}{2(2r^2+5r+3)}, a_2 = \frac{a_0(34r^3+76r^2+41r+5)}{4(2r^3+11r^2+19r+10)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, k^2 + 2(8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})\, r + 18a_k - 7a_{k-2} + 9a_{k-1}$$

- Shift index using $k- >k+2$

  $8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})\left(k+2\right)^2+2(8(5a_{k+2}+a_k-4a_{k+1}-2a_{k+3})\,r+18a_{k+2}-7a_k+9$
- Recursion relation that defines series solution to ODE

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+16kra_k-64kra_{k+1}+80kra_{k+2}+8r^2a_k-32r^2a_{k+1}+40r^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+}}{8(2k^2+4kr+2r^2+13k+13r+21)}$
- Recursion relation for $r=0$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\frac{a_0}{3}\right.$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka_{k+2}+9a_k-96a_{k+1}+240a_{k+2}}{8(2k^2+13k+21)}, a_1=\right.$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}$
- Solution for $r=-\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1=\right.$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-\frac{1}{2}}, a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+10ka_k-78ka_{k+1}+156ka_{k+2}+2a_k-49a_{k+1}+152a_{k+2}}{8(2k^2+11k+15)}, a_1\right.$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k-\frac{1}{2}}\right), a_{k+3}=\frac{8k^2a_k-32k^2a_{k+1}+40k^2a_{k+2}+18ka_k-110ka_{k+1}+196ka}{8(2k^2+13k+21)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 60

```
dsolve(8*x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-13*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x), sings
```

$$y(x) = c_1\sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}} + c_2\sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}} \left(\int \frac{\sqrt{\frac{1}{(x-1)(x+1)}}}{x^{\frac{3}{4}}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 47

```
DSolve[8*x^2*(1-x^2)*y''[x]+2*x*(1-13*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{\sqrt[4]{x}\left(4c_2\sqrt[4]{x}\,\text{Hypergeometric2F1}\left(\frac{1}{8},\frac{1}{2},\frac{9}{8},x^2\right)+c_1\right)}{\sqrt{1-x^2}}$$

2.522 problem 536

2.522.1 Maple step by step solution 4864

Internal problem ID [8012]
Internal file name [`OUTPUT/6945_Sunday_June_05_2022_05_21_05_PM_68453778/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 536.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^{2}\left(x^{2}+1\right) y''-2x\left(-x^{2}+2\right) y'+4y=0}$$

Writing the ode as

$$\left(x^{4}+x^{2}\right) y''+\left(2x^{3}-4x\right) y'+4y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{4}+x^{2} \\ B &= 2x^{3}-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2+2}{(x^3+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2+2$$
$$t = (x^3+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2+2}{(x^3+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 989: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{3}{4 (x - i)^2} + \frac{3}{4 (x + i)^2} + \frac{7i}{4 (x - i)} - \frac{7i}{4 (x + i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_{\infty} &= 0 \\
\alpha_{\infty}^{+} &= 0 \\
\alpha_{\infty}^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+2}{(x^3+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= \frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= \frac{x^2+2}{x^3+x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{x^2}{\sqrt{x^2+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3-4x}{x^4+x^2} \, dx} \\
&= z_1 e^{2\ln(x) - \frac{3\ln(x^2+1)}{2}} \\
&= z_1 \left(\frac{x^2}{(x^2+1)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^4}{\left(x^2+1\right)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-4x}{x^4+x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)-3\ln(x^2+1)}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\frac{-3x^2-1}{3x^3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x^4}{\left(x^2+1\right)^2}\right) + c_2 \left(\frac{x^4}{\left(x^2+1\right)^2}\left(\frac{-3x^2-1}{3x^3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^4}{\left(x^2+1\right)^2} + \frac{c_2(-3x^3-x)}{3\left(x^2+1\right)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^4}{\left(x^2+1\right)^2} + \frac{c_2(-3x^3-x)}{3\left(x^2+1\right)^2}$$

Verified OK.

2.522.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (2x^3 - 4x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{2(x^2-2)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(x^2-2)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(x^2-2)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1) y'' + 2x(x^2 - 2) y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-4+r)\, x^r + a_1 r(-3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-4) + a_{k-2}(k-2+r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 4\}$

- Each term must be 0

  $a_1 r(-3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r-4) + a_{k-2}(k-2+r)) = 0$

- Shift index using $k->k+2$

  $(k+r+1)\,(a_{k+2}(k-2+r) + a_k(k+r)) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r)}{k-2+r}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+1)}{k-1}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0\right]$

- Recursion relation for $r = 4$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 4$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+2} = -\frac{a_k(k+1)}{k-1}, a_1 = 0, b_{k+2} = -\frac{b_k(k+4)}{k+2}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-2*x*(2-x^2)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x(3x^2 + 1)}{(x^2 + 1)^2} + \frac{x^4 c_2}{(x^2 + 1)^2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 35

```
DSolve[x^2*(1+x^2)*y''[x]-2*x*(2-x^2)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{-3c_1x^4 + 3c_2x^3 + c_2x}{3\left(x^2+1\right)^2}$$

2.523 problem 537

2.523.1 Maple step by step solution 4874

Internal problem ID [8013]
Internal file name [OUTPUT/6946_Sunday_June_05_2022_05_21_09_PM_8018023/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 537.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+3)\,y''+(-x^2+2)\,y'-8yx=0}$$

Writing the ode as

$$(x^3+3x)\,y''+(-x^2+2)\,y'-8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^3+3x\\ B&=-x^2+2\\ C&=-8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 74x^2 - 8$$
$$t = 4\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 991: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2} + \frac{85}{144\left(x - i\sqrt{3}\right)^2} + \frac{85}{144\left(x + i\sqrt{3}\right)^2} - \frac{187i\sqrt{3}}{144\left(x - i\sqrt{3}\right)} + \frac{187i\sqrt{3}}{144\left(x + i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{3}
\end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{17}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{12}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{85}{144}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{12}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 74x^2 - 8}{4\left(x^3 + 3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$
$i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$
$-i\sqrt{3}$	2	0	$\frac{17}{12}$	$-\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = \frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= \frac{7}{2} - \left(\frac{7}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} + (0) \\
&= \frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)} \\
&= \frac{2}{3x} + \frac{17x}{6x^2+18}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{3x} + \frac{17}{12\left(x-i\sqrt{3}\right)} + \frac{17}{12\left(x+i\sqrt{3}\right)}\right)(0) + \left(\left(-\frac{2}{3x^2} - \frac{17}{12\left(x-i\sqrt{3}\right)^2} - \frac{17}{12\left(x+i\sqrt{3}\right)^2}\right) + \left(\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{3x} + \frac{17}{12\left(x - i\sqrt{3}\right)} + \frac{17}{12\left(x + i\sqrt{3}\right)}\right) dx} \\
&= x^{\frac{2}{3}} \left(x^2 + 3\right)^{\frac{17}{12}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+2}{x^3+3x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{3} + \frac{5 \ln\left(x^2+3\right)}{12}} \\
&= z_1 \left(\frac{\left(x^2+3\right)^{\frac{5}{12}}}{x^{\frac{1}{3}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{1}{3}} \left(x^2 + 3\right)^{\frac{11}{6}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^3+3x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{2 \ln(x)}{3} + \frac{5 \ln\left(x^2+3\right)}{6}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-8x^4 - 44x^2 - 55}{55 x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}}\right) + c_2 \left(x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}} \left(\frac{-8x^4-44x^2-55}{55x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}} + c_2 \left(-\frac{8}{55}x^4 - \frac{4}{5}x^2 - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{1}{3}} \left(x^2+3\right)^{\frac{11}{6}} + c_2 \left(-\frac{8}{55}x^4 - \frac{4}{5}x^2 - 1\right)
$$

Verified OK.

2.523.1 Maple step by step solution

Let's solve

$(x^3+3x)\,y'' + (-x^2+2)\,y' - 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-2)y'}{x(x^2+3)} + \frac{8y}{x^2+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-2)y'}{x(x^2+3)} - \frac{8y}{x^2+3} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-2}{x(x^2+3)}, P_3(x) = -\frac{8}{x^2+3}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{2}{3}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+3)\,y'' + (-x^2+2)\,y' - 8yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- > k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+3r)\, x^{-1+r} + a_1(1+r)\,(2+3r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(3k+2+3r) + a_{k-1}(k+r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-1+3r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{3}\right\}$

- Each term must be 0

  $a_{1}(1+r)\,(2+3r)=0$

- Each term in the series must be 0, giving the recursion relation

  $\left(a_{k-1}(k-5+r)+3a_{k+1}\left(k+\frac{2}{3}+r\right)\right)(k+r+1)=0$

- Shift index using $k->k+1$

  $\left(a_{k}(k+r-4)+3a_{k+2}\left(k+\frac{5}{3}+r\right)\right)(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r-4)}{3k+5+3r}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+2}=-\frac{a_{k}(k-4)}{3k+5}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k},a_{k+2}=-\frac{a_{k}(k-4)}{3k+5},2a_{1}=0\right]$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{a_{k}\left(k-\frac{11}{3}\right)}{3k+6}$

- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+\frac{1}{3}},a_{k+2}=-\frac{a_{k}\left(k-\frac{11}{3}\right)}{3k+6},4a_{1}=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_{k}x^{k}\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k+\frac{1}{3}}\right),a_{k+2}=-\frac{a_{k}(k-4)}{3k+5},2a_{1}=0,b_{k+2}=-\frac{b_{k}\left(k-\frac{11}{3}\right)}{3k+6},4b_{1}=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*(3+x^2)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^4 + \frac{11}{2}x^2 + \frac{55}{8}\right) + c_2\left(x^2+3\right)^{\frac{11}{6}} x^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 41

```
DSolve[x*(3+x^2)*y''[x]+(2-x^2)*y'[x]-8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt[3]{x}\left(x^2+3\right)^{11/6} - \frac{1}{55}c_2\left(8x^4+44x^2+55\right)$$

2.524 problem 538

2.524.1 Maple step by step solution 4884

Internal problem ID [8014]
Internal file name [OUTPUT/6947_Sunday_June_05_2022_05_21_12_PM_56614714/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 538.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1-x^2)\,y''+x(-19x^2+7)\,y'-(14x^2+1)\,y=0}$$

Writing the ode as

$$(-4x^4+4x^2)\,y''+(-19x^3+7x)\,y'+(-14x^2-1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^4+4x^2 \\ B &= -19x^3+7x \\ C &= -14x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15x^4 - 42x^2 + 9}{64 (x^3 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -15x^4 - 42x^2 + 9$$
$$t = 64(x^3 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-15x^4 - 42x^2 + 9}{64 (x^3 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 993: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(1+x)^2} - \frac{3}{16(x-1)^2} + \frac{9}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-15x^4 - 42x^2 + 9}{64\left(x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{8}$	$\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{8} - \left(\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} + (-)(0) \\
&= -\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \\
&= \frac{3x^2 + 1}{8x^3 - 8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \right)(0) + \left(\left(\frac{1}{8x^2} - \frac{1}{4(x - 1)^2} - \frac{1}{4(1 + x)^2} \right) + \left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x +} \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{8x} + \frac{1}{4x - 4} + \frac{1}{4x + 4} \right) dx} \\
&= \frac{(1 + x)^{\frac{1}{4}} (x - 1)^{\frac{1}{4}}}{x^{\frac{1}{8}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-19x^3+7x}{-4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln(x)}{8}-\frac{3\ln(1+x)}{4}-\frac{3\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{7}{8}}\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-19x^3+7x}{-4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln(x)}{4}-\frac{3\ln(1+x)}{2}-\frac{3\ln(x-1)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2-1\right)^{\frac{1}{4}}}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}} + \frac{c_2\left(x^2-1\right)^{\frac{1}{4}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)}{x\left(1+x\right)^{\frac{3}{4}}\left(x-1\right)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2-1)^{\frac{1}{4}}}{x\,(1+x)^{\frac{3}{4}}\,(x-1)^{\frac{3}{4}}} + \frac{c_2(x^2-1)^{\frac{1}{4}}\left(\int \frac{x^{\frac{1}{4}}}{\sqrt{x^2-1}}dx\right)}{x\,(1+x)^{\frac{3}{4}}\,(x-1)^{\frac{3}{4}}}$$

Verified OK.

2.524.1 Maple step by step solution

Let's solve

$(-4x^4+4x^2)\,y''+(-19x^3+7x)\,y'+(-14x^2-1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(14x^2+1)y}{4x^2(x^2-1)}-\frac{(19x^2-7)y'}{4x(x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(19x^2-7)y'}{4x(x^2-1)}+\frac{(14x^2+1)y}{4x^2(x^2-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{19x^2-7}{4x(x^2-1)}, P_3(x)=\frac{14x^2+1}{4x^2(x^2-1)}\right]$

- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=\frac{3}{2}$

- ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $4y''x^2(x^2-1)+x(19x^2-7)\,y'+(14x^2+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$(4u^4 - 16u^3 + 20u^2 - 8u)\left(\frac{d^2}{du^2}y(u)\right) + (19u^3 - 57u^2 + 50u - 12)\left(\frac{d}{du}y(u)\right) + (14u^2 - 28u + 15)$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  - Shift index using $k- >k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..4$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

  - Shift index using $k- >k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0 r(1+2r)\, u^{-1+r} + (-4a_1(1+r)(3+2r) + 5a_0(4r^2+6r+3))\, u^r + (-4a_2(2+r)(5+2r) +$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- The coefficients of each power of u must be 0

$$[-4a_1(1+r)(3+2r) + 5a_0(4r^2+6r+3) = 0, -4a_2(2+r)(5+2r) + 5a_1(4r^2+14r+13) - a_0$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{5a_0(4r^2+6r+3)}{4(2r^2+5r+3)}, a_2 = \frac{a_0(272r^4+1352r^3+2464r^2+1948r+639)}{16(4r^4+28r^3+71r^2+77r+30)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})k^2 + (8(5a_k + a_{k-2} - 4a_{k-1} - 2a_{k+1})r + 30a_k - a_{k-2} - 9a_{k-1} - 2$$

- Shift index using $k- >k+2$

$$4(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})(k+2)^2 + (8(5a_{k+2} + a_k - 4a_{k+1} - 2a_{k+3})r + 30a_{k+2} - a_k - 9a_k$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+8kra_k-32kra_{k+1}+40kra_{k+2}+4r^2a_k-16r^2a_{k+1}+20r^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}}{4(2k^2+4kr+2r^2+13k+13r+21)}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1 = \frac{5a}{4}\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_{k+2}+14a_k-85a_{k+1}+155a_{k+2}}{4(2k^2+13k+21)}, a_1 =\right.$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a_1 =\right.$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{1}{2}}, a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+11ka_k-57ka_{k+1}+90ka_{k+2}+\frac{15}{2}a_k-\frac{105}{2}a_{k+1}+105a_{k+2}}{4(2k^2+11k+15)}, a\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{1}{2}}\right), a_{k+3} = \frac{4k^2a_k-16k^2a_{k+1}+20k^2a_{k+2}+15ka_k-73ka_{k+1}+110ka_k}{4(2k^2+13k+21)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(4*x^2*(1-x^2)*diff(y(x),x$2)+x*(7-19*x^2)*diff(y(x),x)-(1+14*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1\sqrt{\frac{1}{(x-1)(x+1)}}}{x} + \frac{c_2\sqrt{\frac{1}{(x-1)(x+1)}}\left(\int \sqrt{\frac{1}{(x-1)(x+1)}}\, x^{\frac{1}{4}}dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 50

```
DSolve[4*x^2*(1-x^2)*y''[x]+x*(7-19*x^2)*y'[x]-(1+14*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{4c_2x^{5/4}\,\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{5}{8},\frac{13}{8},x^2\right)+5c_1}{5x\sqrt{1-x^2}}$$

2.525 problem 539

2.525.1 Maple step by step solution 4895

Internal problem ID [8015]
Internal file name [`OUTPUT/6948_Sunday_June_05_2022_05_21_15_PM_51910123/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 539.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2(-x^2+2)\,y''+x(-11x^2+1)\,y'+(-5x^2+1)\,y=0}$$

Writing the ode as

$$(-3x^4+6x^2)\,y''+(-11x^3+x)\,y'+(-5x^2+1)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -3x^4+6x^2 \\ B &= -11x^3+x \\ C &= -5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 - 4x^2 - 35}{36 \left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 - 4x^2 - 35$$
$$t = 36\left(x^3 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 - 4x^2 - 35}{36 \left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 995: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2} - \frac{7}{64\left(x-\sqrt{2}\right)^2} - \frac{7}{64\left(x+\sqrt{2}\right)^2} + \frac{31\sqrt{2}}{384\left(x-\sqrt{2}\right)} - \frac{31\sqrt{2}}{384\left(x+\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 - 4x^2 - 35}{36\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$
$\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} + (0) \\
&= \frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}} \\
&= \frac{5x^2-7}{6x^3-12x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{12x^2} - \frac{1}{8\left(x-\sqrt{2}\right)^2} - \frac{1}{8\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{7}{12x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{12x} + \frac{1}{8x-8\sqrt{2}} + \frac{1}{8x+8\sqrt{2}}\right) dx} \\
&= x^{\frac{7}{12}} \left(x-\sqrt{2}\right)^{\frac{1}{8}} \left(x+\sqrt{2}\right)^{\frac{1}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^3+x}{-3x^4+6x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{12}-\frac{7\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{12}}\left(x^2-2\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^3+x}{-3x^4+6x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{6}-\frac{7\ln\left(x^2-2\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x^{\frac{7}{6}}\left(x^2-2\right)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}\left(\int \frac{1}{x^{\frac{7}{6}}\left(x^2-2\right)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}+\frac{c_2 \sqrt{x}\, \left(\int \frac{1}{x^{\frac{7}{6}} \left(x^2-2\right)^{\frac{1}{4}}}d x \right)}{\left(x^2-2\right)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \sqrt{x}}{\left(x^2-2\right)^{\frac{3}{4}}}+\frac{c_2 \sqrt{x}\, \left(\int \frac{1}{x^{\frac{7}{6}} \left(x^2-2\right)^{\frac{1}{4}}}d x \right)}{\left(x^2-2\right)^{\frac{3}{4}}}
$$

Verified OK.

2.525.1 Maple step by step solution

Let's solve

$\left(-3 x^{4}+6 x^{2}\right) y^{\prime \prime}+\left(-11 x^{3}+x \right) y^{\prime}+\left(-5 x^{2}+1\right) y =0$

- Highest derivative means the order of the ODE is 2

  $y^{\prime \prime}$

- Isolate 2nd derivative

  $y^{\prime \prime}=-\frac{\left(5 x^{2}-1\right) y}{3 x^{2} \left(x^{2}-2\right)}-\frac{\left(11 x^{2}-1\right) y^{\prime}}{3 x \left(x^{2}-2\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y^{\prime \prime}+\frac{\left(11 x^{2}-1\right) y^{\prime}}{3 x \left(x^{2}-2\right)}+\frac{\left(5 x^{2}-1\right) y}{3 x^{2} \left(x^{2}-2\right)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_{2}\left(x \right)=\frac{11 x^{2}-1}{3 x \left(x^{2}-2\right)}, P_{3}\left(x \right)=\frac{5 x^{2}-1}{3 x^{2} \left(x^{2}-2\right)}\right]$

  - $x \cdot P_{2}\left(x \right)$ is analytic at $x =0$

    $\left(x \cdot P_{2}\left(x \right)\right)\bigg|_{x =0}=\frac{1}{6}$

  - $x^{2}\cdot P_{3}\left(x \right)$ is analytic at $x =0$

    $\left(x^{2}\cdot P_{3}\left(x \right)\right)\bigg|_{x =0}=\frac{1}{6}$

  - $x =0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3y''x^2(x^2-2) + x(11x^2-1)\,y' + (5x^2-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-a_0(-1+3r)\,(-1+2r)\,x^r - a_1(2+3r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-a_k(3k+3r-1)\,(2k+2r-1) + \right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(-1+3r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$

- Each term must be 0

  $-a_1(2+3r)(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $-6\left(\frac{(-k-r+1)a_{k-2}}{2}+a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{1}{3}\right)=0$

- Shift index using $k->k+2$

  $-6\left(\frac{(-k-1-r)a_k}{2}+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)\left(k+\frac{5}{3}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{(k+r+1)a_k}{2k+3+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=\frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}$

- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+2}=\frac{\left(k+\frac{4}{3}\right)a_k}{2k+\frac{11}{3}}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=\frac{\left(k+\frac{3}{2}\right)a_k}{2k+4}, a_1=0, b_{k+2}=\frac{\left(k+\frac{4}{3}\right)b_k}{2k+\frac{11}{3}}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(3*x^2*(2-x^2)*diff(y(x),x$2)+x*(1-11*x^2)*diff(y(x),x)+(1-5*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2-2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{(x^2-2)^{\frac{1}{4}} x^{\frac{7}{6}}}dx\right)}{(x^2-2)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 57

```
DSolve[3*x^2*(2-x^2)*y''[x]+x*(1-11*x^2)*y'[x]+(1-5*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{c_1\sqrt{x} - 3\ 2^{3/4} c_2 \sqrt[3]{x}\,\text{Hypergeometric2F1}\left(-\frac{1}{12}, \frac{1}{4}, \frac{11}{12}, \frac{x^2}{2}\right)}{(2-x^2)^{3/4}}$$

2.526 problem 540

2.526.1 Maple step by step solution 4906

Internal problem ID [8016]
Internal file name [`OUTPUT/6949_Sunday_June_05_2022_05_21_18_PM_70020463/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 540.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+2)\,y''-x(-7x^2+12)\,y'+(3x^2+7)\,y=0}$$

Writing the ode as

$$(2x^4+4x^2)\,y''+(7x^3-12x)\,y'+(3x^2+7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3-12x \\ C &= 3x^2+7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 72x^2 + 128}{16\,(x^3 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^4 - 72x^2 + 128 \\ t &= 16\big(x^3 + 2x\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 72x^2 + 128}{16\,(x^3 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 997: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3+2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} + \frac{65}{64\left(x-i\sqrt{2}\right)^2} + \frac{65}{64\left(x+i\sqrt{2}\right)^2} + \frac{135i\sqrt{2}}{128\left(x-i\sqrt{2}\right)} - \frac{135i\sqrt{2}}{128\left(x+i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{65}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 72x^2 + 128}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
$i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$
$-i\sqrt{2}$	2	0	$\frac{13}{8}$	$-\frac{5}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{2}{x} - \frac{5x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{5}{8\left(x - i\sqrt{2}\right)^2} + \frac{5}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{5}{8\left(\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{2}{x} - \frac{5}{8\left(x - i\sqrt{2}\right)} - \frac{5}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \frac{x^2}{\left(x^2 + 2\right)^{\frac{5}{8}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3-12x}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{13\ln\left(x^2+2\right)}{8}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{x^{\frac{3}{2}}}{\left(x^2+2\right)^{\frac{13}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3-12x}{2x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{13\ln\left(x^2+2\right)}{4}+3\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\left(x^2+2\right)^{\frac{5}{4}}}{x^4}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}\right) + c_2 \left(\frac{x^{\frac{7}{2}}}{\left(x^2+2\right)^{\frac{9}{4}}}\left(\int \frac{\left(x^2+2\right)^{\frac{5}{4}}}{x^4}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}}$$

Verified OK.

2.526.1 Maple step by step solution

Let's solve

$(2x^4+4x^2)\, y''+(7x^3-12x)\, y'+(3x^2+7)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+7)y}{2x^2(x^2+2)}-\frac{(7x^2-12)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2-12)y'}{2x(x^2+2)}+\frac{(3x^2+7)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x^2-12}{2(x^2+2)x}, P_3(x)=\frac{3x^2+7}{2x^2(x^2+2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=-3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=\frac{7}{4}$

  - $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2x^2(x^2+2)\,y'' + x(7x^2-12)\,y' + (3x^2+7)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)\,(-7+2r)\,x^r + a_1(1+2r)\,(-5+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)\,(2k+2r-7)+a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-7+2r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{2}, \frac{7}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)(-5+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2}+a_k\left(k+r-\frac{7}{2}\right)\right)\left(k+r-\frac{1}{2}\right)=0$

- Shift index using $k->k+2$

  $4\left(\frac{a_k(k+r+1)}{2}+a_{k+2}\left(k-\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r+1)}{2k-3+2r}$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0\right]$

- Recursion relation for $r=\frac{7}{2}$

  $a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}$

- Solution for $r=\frac{7}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{7}{2}}, a_{k+2}=-\frac{a_k\left(k+\frac{9}{2}\right)}{2k+4}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{7}{2}}\right), a_{k+2}=-\frac{a_k\left(k+\frac{3}{2}\right)}{2k-2}, a_1=0, b_{k+2}=-\frac{b_k\left(k+\frac{9}{2}\right)}{2k+4}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)-x*(12-7*x^2)*diff(y(x),x)+(7+3*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^{\frac{7}{2}}}{(x^2+2)^{\frac{9}{4}}} + \frac{c_2 x^{\frac{7}{2}} \left(\int \frac{(x^2+2)^{\frac{5}{4}}}{x^4} dx\right)}{(x^2+2)^{\frac{9}{4}}}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 57

```
DSolve[2*x^2*(2+x^2)*y''[x]-x*(12-7*x^2)*y'[x]+(7+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{\sqrt{x}\left(3c_1x^3 - 2\sqrt[4]{2}c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}, -\frac{x^2}{2}\right)\right)}{3\left(x^2+2\right)^{9/4}}$$

2.527 problem 541

2.527.1 Maple step by step solution 4917

Internal problem ID [8017]
Internal file name [`OUTPUT/6950_Sunday_June_05_2022_05_21_22_PM_84809605/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 541.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^{2}\left(x^{2}+2\right)y''+x\left(7x^{2}+4\right)y'-\left(-3x^{2}+1\right)y=0}$$

Writing the ode as

$$\left(2x^{4}+4x^{2}\right)y''+\left(7x^{3}+4x\right)y'+\left(3x^{2}-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^{4}+4x^{2}\\ B &= 7x^{3}+4x\\ C &= 3x^{2}-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 24}{16 (x^2 + 2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 + 24 \\ t &= 16(x^2 + 2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 24}{16 (x^2 + 2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 999: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{15}{64\left(x - i\sqrt{2}\right)^2} - \frac{15}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 24}{16\left(x^2+2\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-3x^2+24}{16\left(x^2+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i\sqrt{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{3}{4}-\left(\frac{3}{4}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} + (0) \\
&= \frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)} \\
&= \frac{3x}{4x^2 + 8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{3}{8\left(x - i\sqrt{2}\right)^2} - \frac{3}{8\left(x + i\sqrt{2}\right)^2}\right) + \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{}{8(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{8\left(x - i\sqrt{2}\right)} + \frac{3}{8\left(x + i\sqrt{2}\right)}\right) dx} \\
&= \left(x^2 + 2\right)^{\frac{3}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{7x^3 + 4x}{2x^4 + 4x^2} \, dx} \\
&= z_1 e^{-\frac{5 \ln\left(x^2 + 2\right)}{8} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\left(x^2 + 2\right)^{\frac{5}{8}} \sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3+4x}{2x^4+4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(x^2+2\right)}{4}-\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{1}{\left(x^2+2\right)^{\frac{3}{4}}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}}\right) + c_2\left(\frac{1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}}\left(\int \frac{1}{\left(x^2+2\right)^{\frac{3}{4}}}dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}} + \frac{c_2\left(\int \frac{1}{\left(x^2+2\right)^{\frac{3}{4}}}dx\right)}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}} + \frac{c_2\left(\int \frac{1}{\left(x^2+2\right)^{\frac{3}{4}}}dx\right)}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}}$$

Verified OK.

2.527.1 Maple step by step solution

Let's solve

$\left(2x^4+4x^2\right)y''+\left(7x^3+4x\right)y'+\left(3x^2-1\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2-1)y}{2x^2(x^2+2)}-\frac{(7x^2+4)y'}{2x(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x^2+4)y'}{2x(x^2+2)}+\frac{(3x^2-1)y}{2x^2(x^2+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{7x^2+4}{2(x^2+2)x}, P_3(x)=\frac{3x^2-1}{2x^2(x^2+2)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+2)\,y''+x(7x^2+4)\,y'+\left(3x^2-1\right)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+2r)\,(-1+2r)\,x^{r}+a_{1}(3+2r)\,(1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}(a_{k}(2k+2r+1)\,(2k+2r-1)+a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$

- Each term must be 0

  $a_{1}(3+2r)\,(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2}+a_{k}\left(k+r+\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right)=0$

- Shift index using $k->k+2$

  $4\left(\frac{a_{k}(k+r+1)}{2}+a_{k+2}\left(k+\frac{5}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r+1)}{2k+5+2r}$

- Recursion relation for $r = -\frac{1}{2}$

$a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}$

- Solution for $r = \frac{1}{2}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+6}, a_1 = 0\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0, b_{k+2} = -\frac{b_k\left(k+\frac{3}{2}\right)}{2k+6}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+x*(4+7*x^2)*diff(y(x),x)-(1-3*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}} + \frac{c_2 \left(\int \frac{1}{\left(x^2+2\right)^{\frac{3}{4}}}dx\right)}{\left(x^2+2\right)^{\frac{1}{4}} \sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 68

```
DSolve[2*x^2*(2+x^2)*y''[x]+x*(4+7*x^2)*y'[x]-(1-3*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{c_2 \sqrt[8]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right) Q_{-\frac{1}{4}}^{\frac{1}{4}}\left(\frac{ix}{\sqrt{2}}\right) + 2^{3/8} c_1}{\sqrt{x}\sqrt[4]{x^2+2}\,\text{Gamma}\left(\frac{3}{4}\right)}$$

2.528 problem 542

2.528.1 Maple step by step solution 4928

Internal problem ID [8018]
Internal file name [OUTPUT/6951_Sunday_June_05_2022_05_21_27_PM_98346424/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 542.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(2x^2+1)\,y''+5x(6x^2+1)\,y'-(-40x^2+2)\,y=0}$$

Writing the ode as

$$(4x^4+2x^2)\,y''+(30x^3+5x)\,y'+(40x^2-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+2x^2 \\ B &= 30x^3+5x \\ C &= 40x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 20x^4 + 12x^2 + 21$$
$$t = 16\left(2x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1001: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} + \frac{5}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} + \frac{5}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} + \frac{13i\sqrt{2}}{16\left(x - \frac{i\sqrt{2}}{2}\right)} - \frac{13i\sqrt{2}}{16\left(x + \frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20x^4 + 12x^2 + 21}{16\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{4} - \left(\frac{5}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} + (0) \\ &= \frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)} \\ &= \frac{10x^2+7}{8x^3+4x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)(0) + \left(\left(-\frac{7}{4x^2} + \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} + \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{4x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(\frac{7}{4x} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)}\right)dx} \\ &= \frac{x^{\frac{7}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{30x^3+5x}{4x^4+2x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(x\left(2x^2+1\right)\right)}{4}} \\
&= z_1 \left(\frac{1}{\left(2x^3+x\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{30x^3+5x}{4x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(2x^3+x\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\sqrt{2}\,\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}}\right) + c_2 \left(\frac{x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}}\left(2x^3+x\right)^{\frac{1}{4}}} \left(\int \frac{\sqrt{2}\,\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}} \left(2x^3+x\right)^{\frac{1}{4}}} + \frac{c_2 x^{\frac{3}{4}} 2^{\frac{1}{4}} \left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}} dx\right)}{\left(2x^2+1\right)^{\frac{5}{4}} \left(2x^3+x\right)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{3}{4}} 2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{5}{4}} \left(2x^3+x\right)^{\frac{1}{4}}} + \frac{c_2 x^{\frac{3}{4}} 2^{\frac{1}{4}} \left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}} dx\right)}{\left(2x^2+1\right)^{\frac{5}{4}} \left(2x^3+x\right)^{\frac{1}{4}}}$$

Verified OK.

2.528.1 Maple step by step solution

Let's solve

$\left(4x^4+2x^2\right) y'' + \left(30x^3+5x\right) y' + \left(40x^2-2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(20x^2-1)y}{x^2(2x^2+1)} - \frac{5(6x^2+1)y'}{2x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5(6x^2+1)y'}{2x(2x^2+1)} + \frac{(20x^2-1)y}{x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{5(6x^2+1)}{2x(2x^2+1)}, P_3(x) = \frac{20x^2-1}{x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $2x^2(2x^2+1)\, y'' + 5x(6x^2+1)\, y' + (40x^2-2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k- > k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+2r)\, x^r + a_1(3+r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+2r-1) + 2a_{k-2}(k -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-2, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3+r)(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r+2)\left(a_{k-2}(2k+1+2r)+a_k\left(k+r-\frac{1}{2}\right)\right)=0$

- Shift index using $k->k+2$

  $2(k+r+4)\left(a_k(2k+2r+5)+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_k(2k+2r+5)}{2k+3+2r}$

- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k-2}, a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}}, a_{k+2}=-\frac{2a_k(2k+6)}{2k+4}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-2}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{2a_k(2k+1)}{2k-1}, a_1=0, b_{k+2}=-\frac{2b_k(2k+6)}{2k+4}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(2*x^2*(1+2*x^2)*diff(y(x),x$2)+5*x*(1+6*x^2)*diff(y(x),x)-(2-40*x^2)*y(x)=0,y(x), sin
```

$$y(x) = \frac{c_1\sqrt{x}}{(2x^2+1)^{\frac{3}{2}}} + \frac{c_2\sqrt{x}\left(\int \frac{\sqrt{2x^2+1}}{x^{\frac{7}{2}}}dx\right)}{(2x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 52

```
DSolve[2*x^2*(1+2*x^2)*y''[x]+5*x*(1+6*x^2)*y'[x]-(2-40*x^2)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \to \frac{5c_1x^{5/2} - 2c_2 \text{ Hypergeometric2F1}\left(-\frac{5}{4}, -\frac{1}{2}, -\frac{1}{4}, -2x^2\right)}{5x^2\left(2x^2+1\right)^{3/2}}$$

2.529 problem 543

2.529.1 Maple step by step solution 4939

Internal problem ID [8019]
Internal file name [`OUTPUT/6952_Sunday_June_05_2022_05_21_31_PM_50918889/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 543.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x\left(x^2+1\right)y''+\left(7x^2+4\right)y'+8yx=0}$$

Writing the ode as

$$\left(x^3+x\right)y''+\left(7x^2+4\right)y'+8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+x \\ B &= 7x^2+4 \\ C &= 8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 + 14x^2 + 8 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1003: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^4 + 14x^2 + 8}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)(0) \\ &= -\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\ &= -\frac{1}{x} + \frac{x}{2x^2+2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(\frac{1}{x^2} - \frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2}\right) + \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)dx} \\ &= \frac{(x^2+1)^{\frac{1}{4}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^2+4}{x^3+x}\,dx} \\
&= z_1 e^{-2\ln(x)-\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{1}{x^2\left(x^2+1\right)^{\frac{3}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x^2+1}\,x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2+4}{x^3+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^3-\operatorname{arcsinh}(x)\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{\sqrt{x^2+1}\,x^3}\right) + c_2\left(\frac{1}{\sqrt{x^2+1}\,x^3}\left(\frac{x^3-\operatorname{arcsinh}(x)\sqrt{x^2+1}+x}{2\sqrt{x^2+1}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(x^3-\operatorname{arcsinh}(x)\sqrt{x^2+1}+x\right)}{2\left(x^2+1\right)x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(x^3 - \operatorname{arcsinh}(x)\sqrt{x^2+1} + x\right)}{2\left(x^2+1\right)x^3}$$

Verified OK.

2.529.1 Maple step by step solution

Let's solve

$(x^3+x)\,y'' + (7x^2+4)\,y' + 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8y}{x^2+1} - \frac{(7x^2+4)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(7x^2+4)y'}{x(x^2+1)} + \frac{8y}{x^2+1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{7x^2+4}{x(x^2+1)}, P_3(x) = \frac{8}{x^2+1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\,y'' + (7x^2+4)\,y' + 8yx = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)\,(k+r+4) + a_{k-1}(k+r+3)\,(k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$a_1(1+r)\,(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+1)\,(a_{k+1}(k+r+4) + a_{k-1}(k+r+3)) = 0$$

- Shift index using $k- >k+1$

$$(k+r+2)\left(a_{k+2}(k+5+r)+a_k(k+r+4)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+4)}{k+5+r}$$

- Recursion relation for $r=-3$

$$a_{k+2}=-\frac{a_k(k+1)}{k+2}$$

- Solution for $r=-3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{a_k(k+4)}{k+5}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{a_k(k+4)}{k+5}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{a_k(k+1)}{k+2}, -2a_1=0, b_{k+2}=-\frac{b_k(k+4)}{k+5}, 4b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x*(1+x^2)*diff(y(x),x$2)+(4+7*x^2)*diff(y(x),x)+8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}\,x^3} + \frac{c_2\left(\frac{x\sqrt{x^2+1}}{2} - \frac{\operatorname{arcsinh}(x)}{2}\right)}{\sqrt{x^2+1}\,x^3}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 56

```
DSolve[x*(1+x^2)*y''[x]+(4+7*x^2)*y'[x]+8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x\sqrt{x^2+1} + c_2 \log\left(\sqrt{x^2+1} - x\right) + 2c_1}{2x^3\sqrt{x^2+1}}$$

2.530 problem 544

2.530.1 Maple step by step solution 4949

Internal problem ID [8020]
Internal file name [`OUTPUT/6953_Sunday_June_05_2022_05_21_34_PM_53918227/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 544.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+1)\,y''+x(8x^2+3)\,y'-(-4x^2+3)\,y=0}$$

Writing the ode as

$$\left(2x^4+2x^2\right)y''+\left(8x^3+3x\right)y'+\left(4x^2-3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^4+2x^2\\ B&=8x^3+3x\\ C&=4x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^2 + 21$$
$$t = 16\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^2 + 21}{16\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1005: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{16x^2} - \frac{15}{64(x-i)^2} - \frac{15}{64(x+i)^2} + \frac{27i}{64(x-i)} - \frac{27i}{64(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{36x^2 + 21}{16\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
i	2	0	$\frac{5}{8}$	$\frac{3}{8}$
$-i$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} + (0) \\
&= -\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)} \\
&= -\frac{3}{4x(x^2+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right)(0) + \left(\left(\frac{3}{4x^2} - \frac{3}{8(x-i)^2} - \frac{3}{8(x+i)^2}\right) + \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{}{8}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{3}{4x} + \frac{3}{8(x-i)} + \frac{3}{8(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{8}}}{x^{\frac{3}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x^3+3x}{2x^4+2x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{5\ln(x^2+1)}{8}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}}(x^2+1)^{\frac{5}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^3+3x}{2x^4+2x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}\right) + c_2 \left(\frac{1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{4}}} dx\right)}{\left(x^2+1\right)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

Verified OK.

2.530.1 Maple step by step solution

Let's solve

$(2x^4+2x^2)y''+(8x^3+3x)y'+(4x^2-3)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x^2-3)y}{2x^2(x^2+1)}-\frac{(8x^2+3)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(8x^2+3)y'}{2x(x^2+1)}+\frac{(4x^2-3)y}{2x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{8x^2+3}{2x(x^2+1)}, P_3(x)=\frac{4x^2-3}{2x^2(x^2+1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{3}{2}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{3}{2}$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+1)y''+x(8x^2+3)y'+(4x^2-3)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2r+3)\,(-1+r)\,x^{r}+a_{1}(5+2r)\,r\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(2k+2r+3)\,(k+r-1)+2a_{k-2}(k+r)\,(k$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2r+3)\,(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{1,-\frac{3}{2}\right\}$

- Each term must be 0

  $a_{1}(5+2r)\,r=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(\left(k+r+\frac{3}{2}\right)a_{k}+a_{k-2}(k+r)\right)(k+r-1)=0$

- Shift index using $k->k+2$

  $2\left(\left(k+\frac{7}{2}+r\right)a_{k+2}+a_{k}(k+r+2)\right)(k+r+1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_{k}(k+r+2)}{2k+7+2r}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{2a_k(k+3)}{2k+9}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0\right]$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}$

- Solution for $r = -\frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{2a_k\left(k+\frac{1}{2}\right)}{2k+4}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k(k+3)}{2k+9}, a_1 = 0, b_{k+2} = -\frac{2b_k\left(k+\frac{1}{2}\right)}{2k+4}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(2*x^2*(1+x^2)*diff(y(x),x$2)+x*(3+8*x^2)*diff(y(x),x)-(3-4*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1}{(x^2+1)^{\frac{1}{4}} x^{\frac{3}{2}}} + \frac{c_2 \left(\int \frac{x^{\frac{3}{2}}}{(x^2+1)^{\frac{3}{4}}} dx\right)}{(x^2+1)^{\frac{1}{4}} x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 60

```
DSolve[2*x^2*(1+x^2)*y''[x]+x*(3+8*x^2)*y'[x]-(3-4*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to -\frac{c_2 \text{Hypergeometric2F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, -x^2\right)}{x\sqrt[4]{x^2+1}} + \frac{c_1}{x^{3/2}\sqrt[4]{x^2+1}} + \frac{c_2}{x}$$

2.531 problem 545

2.531.1 Maple step by step solution 4960

Internal problem ID [8021]
Internal file name [OUTPUT/6954_Sunday_June_05_2022_05_21_37_PM_90003406/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 545.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' + 3x(x^2+3)\,y' - (-5x^2+1)\,y = 0}$$

Writing the ode as

$$9x^2y'' + (3x^3+9x)\,y' + (5x^2-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x^3+9x \\ C &= 5x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 5 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1007: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{2}{3x} - \frac{7}{4x^3} - \frac{7}{x^5} - \frac{595}{16x^7} - \frac{889}{4x^9} - \frac{45647}{32x^{11}} - \frac{76811}{8x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 8x^2 - 5}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{x^2}{36} - \frac{2}{9}\right) + \left(-\frac{5}{36x^2}\right) \\ &= \frac{x^2}{36} - \frac{2}{9} - \frac{5}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{2}{9}\right) - (0) \\ &= -\frac{2}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &= -\frac{7}{6} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{2}{9}}{\frac{1}{6}} - 1\right) &= \frac{1}{6} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{6} - \left(\frac{1}{6}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{6x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{6x} - \frac{x}{6} \\ &= \frac{1}{6x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} - \frac{x}{6}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6}\right) + \left(\frac{1}{6x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 8x^2 - 5}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{6x} - \frac{x}{6}\right) dx} \\ &= x^{\frac{1}{6}} e^{-\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^3+9x}{9x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{12} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-\frac{x^2}{12}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{3x^3+9x}{9x^2} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{x^2}{6}-\ln(x)}}{(y_1)^2} dx$$

$$= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x^2}{6}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{-\frac{x^2}{6}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.531.1 Maple step by step solution

Let's solve

$9x^2 y'' + (3x^3 + 9x)\, y' + (5x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2+3)y'}{3x} - \frac{(5x^2-1)y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x^2+3)y'}{3x} + \frac{(5x^2-1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$\left[P_2(x) = \frac{x^2+3}{3x}, P_3(x) = \frac{5x^2-1}{9x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{9}$

- $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$9x^2y'' + 3x(x^2+3)\,y' + (5x^2-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)\,(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term must be 0

$$a_1(4+3r)\,(2+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(3k+3r-1)\,(3a_k k + 3a_k r + a_k + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(3k+3r+5)\,(3a_{k+2}(k+2) + 3a_{k+2}r + a_{k+2} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{3k+7+3r}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = -\frac{a_k}{3k+6}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{a_k}{3k+8}$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k}{3k+8}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{a_k}{3k+6}, a_1 = 0, b_{k+2} = -\frac{b_k}{3k+8}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(3+x^2)*diff(y(x),x)-(1-5*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-\frac{x^2}{6}}}{x^{\frac{1}{3}}} + \frac{c_2 e^{-\frac{x^2}{6}} \left(\int \frac{e^{\frac{x^2}{6}}}{x^{\frac{1}{3}}} dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 61

```
DSolve[9*x^2*y''[x]+3*x*(3+x^2)*y'[x]-(1-5*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{e^{-\frac{x^2}{6}} \left(2c_1 x^{4/3} + \sqrt[3]{6} c_2 (-x^2)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{x^2}{6}\right)\right)}{2x^{5/3}}$$

2.532 problem 546

2.532.1 Maple step by step solution 4971

Internal problem ID [8022]
Internal file name [`OUTPUT/6955_Sunday_June_05_2022_05_21_40_PM_38869907/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 546.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2y'' + x(6x^2+1)\, y' + (9x^2+1)\, y = 0}$$

Writing the ode as

$$6x^2y'' + (6x^3+x)\, y' + (9x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 6x^3 + x \\ C &= 9x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 36x^4 - 132x^2 - 35$$
$$t = 144x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1009: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{5}{12}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{12x} - \frac{13}{12x^3} - \frac{143}{72x^5} - \frac{130}{27x^7} - \frac{17017}{1296x^9} - \frac{597961}{15552x^{11}} - \frac{11016863}{93312x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{36x^4 - 132x^2 - 35}{144x^2} \\ &= Q + \frac{R}{144x^2} \\ &= \left(\frac{x^2}{4} - \frac{11}{12}\right) + \left(-\frac{35}{144x^2}\right) \\ &= \frac{x^2}{4} - \frac{11}{12} - \frac{35}{144x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{11}{12}\right) - (0) \\ &= -\frac{11}{12} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= -\frac{17}{12} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{12}}{\frac{1}{2}} - 1\right) &&= \frac{5}{12} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{36x^4 - 132x^2 - 35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{17}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{5}{12} - \left(\frac{5}{12}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{5}{12x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{5}{12x} - \frac{x}{2} \\
&= \frac{5}{12x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{5}{12x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{12x^2} - \frac{1}{2}\right) + \left(\frac{5}{12x} - \frac{x}{2}\right)^2 - \left(\frac{36x^4 - 132x^2 - 35}{144x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{5}{12x} - \frac{x}{2}\right) dx} \\ &= x^{\frac{5}{12}} \mathrm{e}^{-\frac{x^2}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6x^3 + x}{6x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} - \frac{\ln(x)}{12}} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^{\frac{1}{12}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}} \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6x^3+x}{6x^2} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{x^2}{2}-\frac{\ln(x)}{6}}}{(y_1)^2} dx$$

$$= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \right) + c_2 \left(x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx \right) \right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx \right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx \right)$$

Verified OK.

2.532.1 Maple step by step solution

Let's solve

$6x^2y'' + (6x^3 + x) y' + (9x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+1)y}{6x^2} - \frac{(6x^2+1)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(6x^2+1)y'}{6x} + \frac{(9x^2+1)y}{6x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{6x^2+1}{6x}, P_3(x) = \frac{9x^2+1}{6x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{6}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{6}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $6x^2y'' + x(6x^2+1)\, y' + (9x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)(-1+2r)\, x^r + a_1(2+3r)(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(2k+2r-1)+3a$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term must be 0

  $$a_1(2+3r)(1+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$6\left(\left(k+r-\frac{1}{3}\right)a_k + a_{k-2}\right)\left(k+r-\frac{1}{2}\right) = 0$$

- Shift index using $k- >k+2$

  $$6\left(\left(k+\frac{5}{3}+r\right)a_{k+2} + a_k\right)\left(k+\frac{3}{2}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{3a_k}{3k+5+3r}$$

- Recursion relation for $r = \frac{1}{2}$

  $$a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}$$

- Solution for $r = \frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+2} = -\frac{3a_k}{3k+6}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{3a_k}{3k+6}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{3a_k}{3k+\frac{13}{2}}, a_1 = 0, b_{k+2} = -\frac{3b_k}{3k+6}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(6*x^2*diff(y(x),x$2)+x*(1+6*x^2)*diff(y(x),x)+(1+9*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} + c_2 x^{\frac{1}{3}} e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^{\frac{5}{6}}} dx\right)$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 61

```
DSolve[6*x^2*y''[x]+x*(1+6*x^2)*y'[x]+(1+9*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{e^{-\frac{x^2}{2}} \left(2c_1 x^{11/6} + \sqrt[12]{2} c_2 (-x^2)^{11/12} \, \Gamma\left(\frac{1}{12}, -\frac{x^2}{2}\right)\right)}{2x^{3/2}}$$

2.533 problem 547

2.533.1 Maple step by step solution 4982

Internal problem ID [8023]
Internal file name [`OUTPUT/6956_Sunday_June_05_2022_05_21_44_PM_89387071/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 547.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(x^2+1)\,y''+3x(13x^2+3)\,y'-(-25x^2+1)\,y=0}$$

Writing the ode as

$$\left(9x^4+9x^2\right)y''+\left(39x^3+9x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^2 \\ B &= 39x^3+9x \\ C &= 25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9x^4 + 6x^2 - 5$$
$$t = 36\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1011: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2} - \frac{5}{36(x-i)^2} - \frac{5}{36(x+i)^2} - \frac{i}{12(x-i)} + \frac{i}{12x+12i}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-9x^4 + 6x^2 - 5}{36\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
i	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + \frac{1}{6x - 6i} + \frac{1}{6x + 6i} + (-)(0) \\
&= \frac{1}{6x} + \frac{1}{6x - 6i} + \frac{1}{6x + 6i} \\
&= \frac{1}{6x} + \frac{x}{3x^2 + 3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{6x} + \frac{1}{6x - 6i} + \frac{1}{6x + 6i}\right)(0) + \left(\left(-\frac{1}{6x^2} - \frac{1}{6(x-i)^2} - \frac{1}{6(x+i)^2}\right) + \left(\frac{1}{6x} + \frac{1}{6x - 6i} + \frac{1}{6x +}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{6x} + \frac{1}{6x - 6i} + \frac{1}{6x + 6i}\right) dx} \\
&= x^{\frac{1}{6}} \left(x^2 + 1\right)^{\frac{1}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{39x^3+9x}{9x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{6}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,\left(x^2+1\right)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{39x^3+9x}{9x^4+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+1\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{1}{3}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}\right) + c_2\left(\frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{2}{3}}}\left(\int \frac{1}{x^{\frac{1}{3}}\left(x^2+1\right)^{\frac{1}{3}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{2}{3}}} + \frac{c_2 \left(\int \frac{1}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{1}{3}}}dx\right)}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{2}{3}}} + \frac{c_2 \left(\int \frac{1}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{1}{3}}}dx\right)}{x^{\frac{1}{3}} \left(x^2+1\right)^{\frac{2}{3}}}$$

Verified OK.

2.533.1 Maple step by step solution

Let's solve

$(9x^4+9x^2)\, y''+(39x^3+9x)\, y'+(25x^2-1)\, y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(25x^2-1)y}{9x^2(x^2+1)}-\frac{(13x^2+3)y'}{3x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(13x^2+3)y'}{3x(x^2+1)}+\frac{(25x^2-1)y}{9x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{13x^2+3}{3x(x^2+1)}, P_3(x)=\frac{25x^2-1}{9x^2(x^2+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0}=1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0}=-\frac{1}{9}$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$9x^2(x^2+1)\, y'' + 3x(13x^2+3)\, y' + (25x^2-1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+3r)\,(-1+3r)\, x^r + a_1(4+3r)\,(2+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1) + a_{k-2}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+3r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(4+3r)(2+3r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(\left(k+r-\frac{1}{3}\right)a_{k-2}+a_k\left(k+r+\frac{1}{3}\right)\right)=0$

- Shift index using $k->k+2$

  $9\left(k+\frac{5}{3}+r\right)\left(\left(k+\frac{5}{3}+r\right)a_k+a_{k+2}\left(k+\frac{7}{3}+r\right)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{(3k+3r+5)a_k}{3k+7+3r}$

- Recursion relation for $r=-\frac{1}{3}$

  $a_{k+2}=-\frac{(3k+4)a_k}{3k+6}$

- Solution for $r=-\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{(3k+4)a_k}{3k+6}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{(3k+6)a_k}{3k+8}$

- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{3}}, a_{k+2}=-\frac{(3k+6)a_k}{3k+8}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{3}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{3}}\right), a_{k+2}=-\frac{(3k+4)a_k}{3k+6}, a_1=0, b_{k+2}=-\frac{(3k+6)b_k}{3k+8}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(9*x^2*(1+x^2)*diff(y(x),x$2)+3*x*(3+13*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1}{(x^2+1)^{\frac{2}{3}}\, x^{\frac{1}{3}}} + \frac{c_2 \left(\int \frac{1}{(x^3+x)^{\frac{1}{3}}}dx\right)}{(x^2+1)^{\frac{2}{3}}\, x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 124

```
DSolve[9*x^2*(1+x^2)*y''[x]+3*x*(3+13*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$y(x)$

$$\rightarrow \frac{2\sqrt{3}c_2 \arctan\left(\frac{\sqrt{3}x^{2/3}}{x^{2/3}+2\sqrt[3]{x^2+1}}\right) - 2c_2 \log\left(\sqrt[3]{x^2+1} - x^{2/3}\right) + c_2 \log\left(x^{4/3} + (x^2+1)^{2/3} + \sqrt[3]{x^2+1}x^{2/3}\right.}{4\sqrt[3]{x}\,(x^2+1)^{2/3}}$$

2.534 problem 548

2.534.1 Maple step by step solution 4993

Internal problem ID [8024]
Internal file name [OUTPUT/6957_Sunday_June_05_2022_05_21_47_PM_37498107/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 548.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+4x(6x^2+1)\,y'-(-25x^2+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+4x^2\right)y''+\left(24x^3+4x\right)y'+\left(25x^2-1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=4x^4+4x^2\\ B&=24x^3+4x\\ C&=25x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4\left(x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 6 \\ t &= 4\left(x^2 + 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4\left(x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1013: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x-i\right)^{2}}+\frac{5}{16\left(x+i\right)^{2}}+\frac{7i}{16\left(x-i\right)}-\frac{7i}{16\left(x+i\right)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2-6}{4\left(x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-6}{4\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\ &= \frac{1}{2}-\left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)} \\
&= -\frac{x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-i\right)^2} + \frac{1}{4\left(x+i\right)^2}\right) + \left(-\frac{1}{4\left(x-i\right)} - \frac{1}{4\left(x+i\right)}\right)^2 - \left(\frac{-}{4(\cdots}\right.\right.
$$

$$
\frac{(x^2+1\cdots}{(-x+i)^2(\cdots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= (x)\, \frac{1}{\left(x^2+1\right)^{\frac{1}{4}}} \\
&= \frac{x}{\left(x^2+1\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{24x^3+4x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}-\frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{1}{\sqrt{x}\,\left(x^2+1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{24x^3+4x}{4x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}}\left(\frac{\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}}{x}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{\left(x^2+1\right)^{\frac{3}{2}}} + \frac{c_2\left(\operatorname{arcsinh}(x)\,x-\sqrt{x^2+1}\right)}{\sqrt{x}\,\left(x^2+1\right)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 \left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2+1}\right)}{\sqrt{x}\, (x^2+1)^{\frac{3}{2}}}$$

Verified OK.

2.534.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\, y'' + (24x^3+4x)\, y' + (25x^2-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2-1)y}{4x^2(x^2+1)} - \frac{(6x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(6x^2+1)y'}{x(x^2+1)} + \frac{(25x^2-1)y}{4x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{6x^2+1}{x(x^2+1)}, P_3(x) = \frac{25x^2-1}{4x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\, y'' + 4x(6x^2+1)\, y' + (25x^2-1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r) (k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1 + 2r) (-1 + 2r)\, x^r + a_1 (3 + 2r) (1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (2k + 2r + 1) (2k + 2r - 1) + a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r) (-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1 (3 + 2r) (1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r+\frac{1}{2}\right)\left(\left(k+r+\frac{1}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{2}\right)\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{5}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right)a_k+a_{k+2}\left(k+\frac{3}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r+5)a_k}{2k+3+2r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+4)a_k}{2k+2}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{(2k+4)a_k}{2k+2}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{(2k+6)a_k}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{(2k+6)a_k}{2k+4}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{(2k+4)a_k}{2k+2}, a_1=0, b_{k+2}=-\frac{(2k+6)b_k}{2k+4}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(1+6*x^2)*diff(y(x),x)-(1-25*x^2)*y(x)=0,y(x), sings
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2\left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2+1}\right)}{\sqrt{x}\,(x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 57

```
DSolve[4*x^2*(1+x^2)*y''[x]+4*x*(1+6*x^2)*y'[x]-(1-25*x^2)*y[x]==0,y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{-c_2\sqrt{x^2+1} - c_2 x \log\left(\sqrt{x^2+1} - x\right) + c_1 x}{\sqrt{x}\,(x^2+1)^{3/2}}$$

2.535 problem 549

2.535.1 Maple step by step solution 5003

Internal problem ID [8025]
Internal file name [`OUTPUT/6958_Sunday_June_05_2022_05_21_50_PM_21549691/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 549.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^{2}\left(2x^{2}+1\right)y''+2x\left(34x^{2}+5\right)y'-\left(-30x^{2}+1\right)y=0}$$

Writing the ode as

$$\left(16x^{4}+8x^{2}\right)y''+\left(68x^{3}+10x\right)y'+\left(30x^{2}-1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^{4}+8x^{2} \\ B &= 68x^{3}+10x \\ C &= 30x^{2}-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 132x^4 + 148x^2 - 7 \\ t &= 64\left(2x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1015: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(2x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2} - \frac{3}{16\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{3}{16\left(x + \frac{i\sqrt{2}}{2}\right)^2} - \frac{i\sqrt{2}}{2\left(x - \frac{i\sqrt{2}}{2}\right)} + \frac{i\sqrt{2}}{2x + i\sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{132x^4 + 148x^2 - 7}{64\left(2x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{11}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{11}{8} - \left(\frac{11}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} + (0) \\
&= \frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}} \\
&= \frac{22x^2+7}{16x^3+8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right)(0) + \left(\left(-\frac{7}{8x^2} - \frac{1}{4\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{4\left(x+\frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{7}{8x} + \frac{}{4x\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{7}{8x} + \frac{1}{4x-2i\sqrt{2}} + \frac{1}{4x+2i\sqrt{2}}\right) dx} \\
&= x^{\frac{7}{8}} 2^{\frac{1}{4}} \left(2x^2+1\right)^{\frac{1}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{68x^3+10x}{16x^4+8x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{8}-\frac{3\ln\left(2x^2+1\right)}{4}} \\
&= z_1\left(\frac{1}{x^{\frac{5}{8}}\left(2x^2+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{68x^3+10x}{16x^4+8x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-\frac{3\ln\left(2x^2+1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\sqrt{2}}{2x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}\right) + c_2\left(\frac{x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}}\left(\int \frac{\sqrt{2}}{2x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}2^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}}2^{\frac{3}{4}}\left(\int \frac{1}{x^{\frac{7}{4}}\sqrt{2x^2+1}}dx\right)}{2\sqrt{2x^2+1}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}} 2^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}} 2^{\frac{3}{4}} \left(\int \frac{1}{x^{\frac{7}{4}}\sqrt{2x^2+1}} dx\right)}{2\sqrt{2x^2+1}}$$

Verified OK.

2.535.1 Maple step by step solution

Let's solve

$(16x^4 + 8x^2)\, y'' + (68x^3 + 10x)\, y' + (30x^2 - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(30x^2-1)y}{8x^2(2x^2+1)} - \frac{(34x^2+5)y'}{4x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(34x^2+5)y'}{4x(2x^2+1)} + \frac{(30x^2-1)y}{8x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{34x^2+5}{4x(2x^2+1)}, P_3(x) = \frac{30x^2-1}{8x^2(2x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{8}$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $8x^2(2x^2+1)\, y'' + 2x(34x^2+5)\, y' + (30x^2-1)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r) x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r) (k + r - 1) x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r) (k + 1 - m + r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1 + 2r) (-1 + 4r) x^r + a_1 (3 + 2r) (3 + 4r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (2k + 2r + 1) (4k + 4r - 1) + 2a_{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1 + 2r) (-1 + 4r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$$

- Each term must be 0

$$a_1 (3 + 2r) (3 + 4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$8\left(k+r+\frac{1}{2}\right)\left(\left(2k+2r-\frac{5}{2}\right)a_{k-2}+a_k\left(k+r-\frac{1}{4}\right)\right)=0$$

- Shift index using $k->k+2$

$$8\left(k+\frac{5}{2}+r\right)\left(\left(2k+\frac{3}{2}+2r\right)a_k+a_{k+2}\left(k+\frac{7}{4}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2(4k+4r+3)a_k}{4k+7+4r}$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{4}$

$$a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}$$

- Solution for $r=\frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{2(4k+4)a_k}{4k+8}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2(4k+1)a_k}{4k+5}, a_1=0, b_{k+2}=-\frac{2(4k+4)b_k}{4k+8}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(8*x^2*(1+2*x^2)*diff(y(x),x$2)+2*x*(5+34*x^2)*diff(y(x),x)-(1-30*x^2)*y(x)=0,y(x), si
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{\sqrt{2x^2+1}} + \frac{c_2 x^{\frac{1}{4}} \left(\int \frac{1}{\sqrt{2x^2+1}\, x^{\frac{7}{4}}}dx\right)}{\sqrt{2x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 54

```
DSolve[8*x^2*(1+2*x^2)*y''[x]+2*x*(5+34*x^2)*y'[x]-(1-30*x^2)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to \frac{3c_1x^{3/4} - 4c_2 \,\text{Hypergeometric2F1}\left(-\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, -2x^2\right)}{3\sqrt{x}\sqrt{2x^2+1}}$$

2.536 problem 550

2.536.1 Maple step by step solution 5013

Internal problem ID [8026]
Internal file name [OUTPUT/6959_Sunday_June_05_2022_05_21_54_PM_77787822/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 550.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(1+x)\,y''-x(-3x+1)\,y'+y=0}$$

Writing the ode as

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+2x^2 \\ B &= 3x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1017: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + (-)\,(0) \\ &= \frac{1}{4x} \\ &= \frac{1}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x}\right)(0) + \left(\left(-\frac{1}{4x^2}\right) + \left(\frac{1}{4x}\right)^2 - \left(-\frac{3}{16x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{4x} dx} \\ &= x^{\frac{1}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2 - x}{2x^3 + 2x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{4} - \ln(1+x)} \\ &= z_1 \left(\frac{x^{\frac{1}{4}}}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2 - x}{2x^3 + 2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{\ln(x)}{2} - 2\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1 \left(2\sqrt{x}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{1+x}\right) + c_2\left(\frac{\sqrt{x}}{1+x}(2\sqrt{x})\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{1+x} + \frac{2c_2 x}{1+x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{1+x} + \frac{2c_2 x}{1+x}
$$

Verified OK.

2.536.1 Maple step by step solution

Let's solve

$(2x^3 + 2x^2) y'' + (3x^2 - x) y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x^2(1+x)} - \frac{(3x-1)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x-1)y'}{2x(1+x)} + \frac{y}{2x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x-1}{2x(1+x)}, P_3(x) = \frac{1}{2x^2(1+x)}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 2$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $2x^2(1+x)\,y'' + x(3x-1)\,y' + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(2u^3 - 4u^2 + 2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2 - 7u + 4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(1+r)\,u^{-1+r} + (2a_1(1+r)(2+r) - a_0(1+r)(-1+4r))\,u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)(k+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$$2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1}) k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1}) r - 3a_k - 3a_{k-1} + 6a_{k+1}) k + (-4a_k + 2a$$

- Shift index using $k- >k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2}) (k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2}) r - 3a_{k+1} - 3a_k + 6a_{k+2}) (k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2+2kr+r^2+5k+5r+6)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}, 4a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2+5k+6)}, 4a_1 + a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right), a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2+3k+2)}, 0 = 0,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(1-3*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x+1} + \frac{c_2\sqrt{x}}{x+1}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 25

```
DSolve[2*x^2*(1+x)*y''[x]-x*(1-3*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1\sqrt{x} + 2c_2 x}{x+1}$$

2.537 problem 551

2.537.1 Maple step by step solution 5022

Internal problem ID [8027]
Internal file name [`OUTPUT/6960_Sunday_June_05_2022_05_21_57_PM_83247645/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 551.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2(2x^2+1)\,y''+x(50x^2+1)\,y'+(30x^2+1)\,y=0}$$

Writing the ode as

$$\left(12x^4+6x^2\right)y''+\left(50x^3+x\right)y'+\left(30x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=12x^4+6x^2\\B&=50x^3+x\\C&=30x^2+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1019: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{35}{144x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{35}{144}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{35}{144x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{12}$	$\frac{5}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{12}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{5}{12} - \left(\frac{5}{12}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{12x} + (-)(0) \\ &= \frac{5}{12x} \\ &= \frac{5}{12x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{12x}\right)(0) + \left(\left(-\frac{5}{12x^2}\right) + \left(\frac{5}{12x}\right)^2 - \left(-\frac{35}{144x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{5}{12x} dx} \\ &= x^{\frac{5}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{50x^3 + x}{12x^4 + 6x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{12} - \ln(2x^2 + 1)} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{12}} (2x^2 + 1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{2x^2 + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{50x^3 + x}{12x^4 + 6x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{6} - 2\ln(2x^2 + 1)}}{(y_1)^2} \, dx \\ &= y_1 \left(6x^{\frac{1}{6}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{2x^2+1}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{2x^2+1}\left(6x^{\frac{1}{6}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{2x^2+1} + \frac{6c_2\sqrt{x}}{2x^2+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{2x^2+1} + \frac{6c_2\sqrt{x}}{2x^2+1}
$$

Verified OK.

2.537.1 Maple step by step solution

Let's solve

$(12x^4 + 6x^2)\, y'' + (50x^3 + x)\, y' + (30x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(30x^2+1)y}{6x^2(2x^2+1)} - \frac{(50x^2+1)y'}{6x(2x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(50x^2+1)y'}{6x(2x^2+1)} + \frac{(30x^2+1)y}{6x^2(2x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{50x^2+1}{6x(2x^2+1)}, P_3(x) = \frac{30x^2+1}{6x^2(2x^2+1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{6}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{6}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2(2x^2+1)\,y'' + x(50x^2+1)\,y' + (30x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+3r)\,(-1+2r)\, x^r + a_1(2+3r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)\,(2k+2r-1) + 2a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$
- Each term must be 0

  $a_1(2+3r)(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(3k+3r-1)(2k+2r-1)(a_k+2a_{k-2}) = 0$
- Shift index using $k->k+2$

  $(3k+3r+5)(2k+2r+3)(a_{k+2}+2a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -2a_k$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -2a_k$
- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -2a_k, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -2a_k, a_1 = 0, b_{k+2} = -2b_k, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(6*x^2*(1+2*x^2)*diff(y(x),x$2)+x*(1+50*x^2)*diff(y(x),x)+(1+30*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1\sqrt{x}}{2x^2+1} + \frac{c_2 x^{\frac{1}{3}}}{2x^2+1}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 32

```
DSolve[6*x^2*(1+2*x^2)*y''[x]+x*(1+50*x^2)*y'[x]+(1+30*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt[3]{x}\left(6c_2\sqrt[6]{x}+c_1\right)}{2x^2+1}$$

2.538 problem 552

2.538.1 Maple step by step solution 5031

Internal problem ID [8028]
Internal file name [OUTPUT/6961_Sunday_June_05_2022_05_22_00_PM_6133668/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 552.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{28x^2(-3x+1)\,y''-7x(5+9x)\,y'+7(2+9x)\,y=0}$$

Writing the ode as

$$\left(-84x^3+28x^2\right)y''+\left(-63x^2-35x\right)y'+(63x+14)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -84x^3+28x^2 \\ B &= -63x^2-35x \\ C &= 63x+14 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 33 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1021: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{8x} + (-)\,(0) \\ &= -\frac{3}{8x} \\ &= -\frac{3}{8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8x}\right)(0) + \left(\left(\frac{3}{8x^2}\right) + \left(-\frac{3}{8x}\right)^2 - \left(\frac{33}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{8x} dx} \\ &= \frac{1}{x^{\frac{3}{8}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-63x^2 - 35x}{-84x^3 + 28x^2} \, dx} \\ &= z_1 e^{-\ln(3x-1) + \frac{5\ln(x)}{8}} \\ &= z_1 \left(\frac{x^{\frac{5}{8}}}{3x-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{3x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-63x^2-35x}{-84x^3+28x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-2\ln(3x-1)+\frac{5\ln(x)}{4}}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{4x^{\frac{7}{4}}}{7}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{\frac{1}{4}}}{3x-1}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{3x-1} \left(\frac{4x^{\frac{7}{4}}}{7}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{3x-1} + \frac{4c_2 x^2}{21x-7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{3x-1} + \frac{4c_2 x^2}{21x-7}$$

Verified OK.

2.538.1 Maple step by step solution

Let's solve

$(-84x^3 + 28x^2) y'' + (-63x^2 - 35x) y' + (63x + 14) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2+9x)y}{4x^2(3x-1)} - \frac{(5+9x)y'}{4x(3x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(5+9x)y'}{4x(3x-1)} - \frac{(2+9x)y}{4x^2(3x-1)} = 0$$

- Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{5+9x}{4x(3x-1)}, P_3(x) = -\frac{2+9x}{4x^2(3x-1)}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$(x \cdot P_2(x))\bigg|_{x=0} = -\frac{5}{4}$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $$4y''x^2(3x-1) + x(5+9x)\, y' + (-9x-2)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+4r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} (-a_k(4k+4r-1)(k+r-2) + 3a_{k-1}(4k+4r-1)(k+r-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+4r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{2, \frac{1}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4(k+r-2)(a_k - 3a_{k-1})\left(k+r-\frac{1}{4}\right) = 0$$

- Shift index using $k- > k+1$

$$-4(k+r-1)(a_{k+1} - 3a_k)\left(k+\frac{3}{4}+r\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = 3a_k$$

- Recursion relation for $r = 2$

$$a_{k+1} = 3a_k$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = 3a_k\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+1} = 3a_k$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = 3a_k\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+1} = 3a_k, b_{k+1} = 3b_k\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(28*x^2*(1-3*x)*diff(y(x),x$2)-7*x*(5+9*x)*diff(y(x),x)+7*(2+9*x)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^2}{3x-1} + \frac{c_2 x^{\frac{1}{4}}}{3x-1}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 30

```
DSolve[28*x^2*(1-3*x)*y''[x]-7*x*(5+9*x)*y'[x]+7*(2+9*x)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \rightarrow \frac{4c_2 x^2 + 7c_1 \sqrt[4]{x}}{7-21x}$$

2.539 problem 553

2.539.1 Maple step by step solution 5040

Internal problem ID [8029]
Internal file name [`OUTPUT/6962_Sunday_June_05_2022_05_22_02_PM_99923510/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 553.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{8x^2(-x^2+2)\,y''+2x(-21x^2+10)\,y'-(35x^2+2)\,y=0}$$

Writing the ode as

$$(-8x^4+16x^2)\,y''+(-42x^3+20x)\,y'+(-35x^2-2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -8x^4+16x^2 \\ B &= -42x^3+20x \\ C &= -35x^2-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -7$$
$$t = 64x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1023: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{8} - \left(\frac{1}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x} + (-)(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{8x} dx} \\ &= x^{\frac{1}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-42x^3+20x}{-8x^4+16x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{8} - \ln(x^2-2)} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{8}} (x^2-2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}\,(x^2-2)}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-42x^3+20x}{-8x^4+16x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x)}{4}-2\ln(x^2-2)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{4x^{\frac{3}{4}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}\,(x^2-2)}\right) + c_2 \left(\frac{1}{\sqrt{x}\,(x^2-2)} \left(\frac{4x^{\frac{3}{4}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x}\,(x^2-2)} + \frac{4c_2 x^{\frac{1}{4}}}{3x^2-6} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{\sqrt{x}\,(x^2-2)} + \frac{4c_2 x^{\frac{1}{4}}}{3x^2-6}
$$

Verified OK.

2.539.1 Maple step by step solution

Let's solve

$(-8x^4+16x^2)\,y'' + (-42x^3+20x)\,y' + (-35x^2-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(35x^2+2)y}{8x^2(x^2-2)} - \frac{(21x^2-10)y'}{4x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(21x^2-10)y'}{4x(x^2-2)}+\frac{(35x^2+2)y}{8x^2(x^2-2)}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{21x^2-10}{4x(x^2-2)}, P_3(x)=\frac{35x^2+2}{8x^2(x^2-2)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{5}{4}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=-\frac{1}{8}$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $8y''x^2(x^2-2)+2x(21x^2-10)\,y'+(35x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..2$

    $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..3$

    $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(1+2r)(-1+4r) x^r - 2a_1(3+2r)(3+4r) x^{1+r} + \left(\sum_{k=2}^{\infty} (-2a_k(2k+2r+1)(4k+4r-1)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(1+2r)(-1+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{4}\right\}$

- Each term must be 0

  $-2a_1(3+2r)(3+4r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $-(2k+2r+1)(4k+4r-1)(2a_k - a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $-(2k+2r+5)(4k+4r+7)(2a_{k+2} - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{2}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k}{2}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = \frac{a_k}{2}$

- Solution for $r = \frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2}=\frac{a_k}{2}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=\frac{a_k}{2}, a_1=0, b_{k+2}=\frac{b_k}{2}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(8*x^2*(2-x^2)*diff(y(x),x$2)+2*x*(10-21*x^2)*diff(y(x),x)-(2+35*x^2)*y(x)=0,y(x), sin
```

$$y(x)=\frac{c_1}{(x^2-2)\sqrt{x}}+\frac{c_2 x^{\frac{1}{4}}}{x^2-2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 34

```
DSolve[8*x^2*(2-x^2)*y''[x]+2*x*(10-21*x^2)*y'[x]-(2+35*x^2)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x)\to\frac{\frac{3c_1}{\sqrt{x}}+4c_2\sqrt[4]{x}}{6-3x^2}$$

2.540 problem 554

2.540.1 Maple step by step solution 5047

Internal problem ID [8030]
Internal file name [OUTPUT/6963_Sunday_June_05_2022_05_22_05_PM_48396130/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 554.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+3x+1)\,y''-4x(-3x^2-3x+1)\,y'+3(x^2-x+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(12x^3+12x^2-4x\right)y'+\left(3x^2-3x+3\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+12x^3+4x^2\\B&=12x^3+12x^2-4x\\C&=3x^2-3x+3\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1025: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx} \\
&= z_1 e^{-\ln(x^2+3x+1)+\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{x^2+3x+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2-4x}{4x^4+12x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+3x+1)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2 \left(\frac{\sqrt{x}}{x^2+3x+1}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1}
$$

Verified OK.

2.540.1 Maple step by step solution

Let's solve

$(4x^4+12x^3+4x^2)\, y'' + (12x^3+12x^2-4x)\, y' + (3x^2-3x+3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} - \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+3x-1)y'}{x(x^2+3x+1)} + \frac{3(x^2-x+1)y}{4x^2(x^2+3x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3x^2+3x-1}{x(x^2+3x+1)}, P_3(x) = \frac{3(x^2-x+1)}{4x^2(x^2+3x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{3}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\, y'' + 4x(3x^2+3x-1)\, y' + (3x^2-3x+3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-3+2r)\, x^r + (a_1(1+2r)\,(-1+2r) + 3a_0(1+2r)\,(-1+2r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)(-1+2r) + 3a_0(1+2r)(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -3a_0$

- Each term in the series must be 0, giving the recursion relation

  $(2k+2r-1)(2k+2r-3)(a_k + 3a_{k-1} + a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $(2k+2r+3)(2k+2r+1)(a_{k+2} + 3a_{k+1} + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -3a_{k+1} - a_k$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -3a_{k+1} - a_k$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0, b_{k+2} = -3b_{k+1} - b_k, b_1 =\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)-4*x*(1-3*x-3*x^2)*diff(y(x),x)+3*(1-x+x^2)*y(x)=0,y(
```

$$y(x) = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2 x^{\frac{3}{2}}}{x^2+3x+1}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 28

```
DSolve[4*x^2*(1+3*x+x^2)*y''[x]-4*x*(1-3*x-3*x^2)*y'[x]+3*(1-x+x^2)*y[x]==0,y[x],x,IncludeSi
```

$$y(x) \to \frac{\sqrt{x}(c_2 x + c_1)}{x^2+3x+1}$$

2.541 problem 555

2.541.1 Maple step by step solution 5056

Internal problem ID [8031]
Internal file name [OUTPUT/6964_Sunday_June_05_2022_05_22_08_PM_50123772/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 555.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2(1+x)^2 y'' - x(-11x^2-10x+1)\, y' + (5x^2+1)\, y = 0}$$

Writing the ode as

$$3x^2(1+x)^2 y'' + (11x^3+10x^2-x)\, y' + (5x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2(1+x)^2 \\ B &= 11x^3+10x^2-x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1027: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{36x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{6} - \left(\frac{1}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{6x} + (-)(0) \\
&= \frac{1}{6x} \\
&= \frac{1}{6x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{6x}\right)(0) + \left(\left(-\frac{1}{6x^2}\right) + \left(\frac{1}{6x}\right)^2 - \left(-\frac{5}{36x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{6x} dx} \\ &= x^{\frac{1}{6}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{6} - 2\ln(1+x)} \\ &= z_1 \left(\frac{x^{\frac{1}{6}}}{(1+x)^2}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{11x^3+10x^2-x}{3x^2(1+x)^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\frac{\ln(x)}{3}-4\ln(1+x)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{3x^{\frac{2}{3}}}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{\frac{1}{3}}}{(1+x)^2}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{(1+x)^2}\left(\frac{3x^{\frac{2}{3}}}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{3}}}{(1+x)^2} + \frac{3c_2 x}{2(1+x)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{3}}}{(1+x)^2} + \frac{3c_2 x}{2(1+x)^2}$$

Verified OK.

2.541.1 Maple step by step solution

Let's solve

$3x^2(1+x)^2 y'' + (11x^3+10x^2-x)\,y' + (5x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+1)y}{3x^2(1+x)^2} - \frac{y'(11x-1)}{3x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'(11x-1)}{3x(1+x)}+\frac{(5x^2+1)y}{3x^2(1+x)^2}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{11x-1}{3x(1+x)}, P_3(x)=\frac{5x^2+1}{3x^2(1+x)^2}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=2$

  ◦ $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $3x^2(1+x)^2 y''+x(11x-1)(1+x)y'+(5x^2+1)y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(3u^4-6u^3+3u^2)\left(\frac{d^2}{du^2}y(u)\right)+(11u^3-23u^2+12u)\left(\frac{d}{du}y(u)\right)+(5u^2-10u+6)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  ◦ Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m}\cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=2..4$

  $u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^{m}\cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $3a_{0}(2+r)\,(1+r)\, u^{r}+\left(3a_{1}(3+r)\,(2+r)-a_{0}(2+r)\,(5+6r)\right) u^{1+r}+\left(\sum_{k=2}^{\infty}\left(3a_{k}(k+r+2)\,(k+\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $3(2+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in \{-2,-1\}$

- Each term must be 0

  $3a_{1}(3+r)\,(2+r)-a_{0}(2+r)\,(5+6r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=\frac{a_{0}(5+6r)}{3(3+r)}$

- Each term in the series must be 0, giving the recursion relation

  $3(a_{k}+a_{k-2}-2a_{k-1})\, k^{2}+\left(6(a_{k}+a_{k-2}-2a_{k-1})\, r+9a_{k}-4a_{k-2}-5a_{k-1}\right) k+3(a_{k}+a_{k-2}-2a_{k}$

- Shift index using $k->k+2$

  $3(a_{k+2}+a_{k}-2a_{k+1})\,(k+2)^{2}+\left(6(a_{k+2}+a_{k}-2a_{k+1})\, r+9a_{k+2}-4a_{k}-5a_{k+1}\right)(k+2)+3(a_{k+}$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}+6kra_{k}-12kra_{k+1}+3r^{2}a_{k}-6r^{2}a_{k+1}+8ka_{k}-29ka_{k+1}+8ra_{k}-29ra_{k+1}+5a_{k}-33a_{k+1}}{3(k^{2}+2kr+r^{2}+7k+7r+12)}$

- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}-4ka_{k}-5ka_{k+1}+a_{k}+a_{k+1}}{3(k^{2}+3k+2)}$

- Solution for $r=-2$

  $\left[y(u)=\sum_{k=0}^{\infty} a_{k}u^{k-2}, a_{k+2}=-\frac{3k^{2}a_{k}-6k^{2}a_{k+1}-4ka_{k}-5ka_{k+1}+a_{k}+a_{k+1}}{3(k^{2}+3k+2)}, a_{1}=-\frac{7a_{0}}{3}\right]$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-2}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} - 4k a_k - 5k a_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 = -\frac{7a_0}{3}\right]$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-1}, a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} + 2k a_k - 17k a_{k+1} - 10a_{k+1}}{3(k^2+5k+6)}, a_1 = -\frac{a_0}{6}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-1}\right), a_{k+2} = -\frac{3k^2 a_k - 6k^2 a_{k+1} - 4k a_k - 5k a_{k+1} + a_k + a_{k+1}}{3(k^2+3k+2)}, a_1 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(3*x^2*(1+x)^2*diff(y(x),x$2)-x*(1-10*x-11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x), si
```

$$y(x) = \frac{c_1 x}{(x+1)^2} + \frac{c_2 x^{\frac{1}{3}}}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 29

```
DSolve[3*x^2*(1+x)^2*y''[x]-x*(1-10*x-11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to \frac{2c_1 \sqrt[3]{x} + 3c_2 x}{2(x+1)^2}$$

2.542 problem 556

2.542.1 Maple step by step solution 5066

Internal problem ID [8032]
Internal file name [`OUTPUT/6965_Sunday_June_05_2022_05_22_11_PM_31866754/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 556.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+2x+3)\,y''-x(-15x^2-14x+3)\,y'+(7x^2+3)\,y=0}$$

Writing the ode as

$$\left(4x^4+8x^3+12x^2\right)y''+\left(15x^3+14x^2-3x\right)y'+\left(7x^2+3\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+8x^3+12x^2 \\ B &= 15x^3+14x^2-3x \\ C &= 7x^2+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-7}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -7 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{7}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1029: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{7}{64x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{7}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{8}$	$\frac{1}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{8} - \left(\frac{1}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{8x} + (-)\,(0) \\
&= \frac{1}{8x} \\
&= \frac{1}{8x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{8x}\right)(0) + \left(\left(-\frac{1}{8x^2}\right) + \left(\frac{1}{8x}\right)^2 - \left(-\frac{7}{64x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{8x} dx} \\ &= x^{\frac{1}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{8} - \ln(x^2+2x+3)} \\ &= z_1 \left(\frac{x^{\frac{1}{8}}}{x^2+2x+3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{x^2+2x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{15x^3+14x^2-3x}{4x^4+8x^3+12x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{\frac{\ln(x)}{4}-2\ln(x^2+2x+3)}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{4x^{\frac{3}{4}}}{3}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{\frac{1}{4}}}{x^2+2x+3}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{x^2+2x+3}\left(\frac{4x^{\frac{3}{4}}}{3}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{x^2+2x+3} + \frac{4c_2 x}{3x^2+6x+9} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{x^2+2x+3} + \frac{4c_2 x}{3x^2+6x+9}$$

Verified OK.

2.542.1 Maple step by step solution

Let's solve

$(4x^4+8x^3+12x^2)\, y'' + (15x^3+14x^2-3x)\, y' + (7x^2+3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+3)y}{4x^2(x^2+2x+3)} - \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(15x^2+14x-3)y'}{4x(x^2+2x+3)} + \frac{(7x^2+3)y}{4x^2(x^2+2x+3)} = 0$$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = \frac{15x^2+14x-3}{4x(x^2+2x+3)}, P_3(x) = \frac{7x^2+3}{4x^2(x^2+2x+3)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{4}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$4x^2(x^2 + 2x + 3)\, y'' + x(15x^2 + 14x - 3)\, y' + (7x^2 + 3)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$3a_0(-1+4r)(-1+r)\,x^r + (3a_1(3+4r)\,r + 2a_0r(3+4r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (3a_k(4k+4r-1)(k+r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3(-1+4r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{4}\right\}$$

- Each term must be 0

$$3a_1(3+4r)\,r + 2a_0r(3+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{2a_0}{3}$$

- Each term in the series must be 0, giving the recursion relation

$$(4k+4r-1)(k+r-1)(3a_k + 2a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(4k+4r+7)(k+r+1)(3a_{k+2} + 2a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}, a_1 = -\frac{2a_0}{3}\right]$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -\frac{2a_{k+1}}{3} - \frac{a_k}{3}$$

- Solution for $r = \frac{1}{4}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}, a_1=-\frac{2a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}}\right), a_{k+2}=-\frac{2a_{k+1}}{3}-\frac{a_k}{3}, a_1=-\frac{2a_0}{3}, b_{k+2}=-\frac{2b_{k+1}}{3}-\frac{b_k}{3}, b_1=-\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(4*x^2*(3+2*x+x^2)*diff(y(x),x$2)-x*(3-14*x-15*x^2)*diff(y(x),x)+(3+7*x^2)*y(x)=0,y(x)
```

$$y(x)=\frac{c_1 x}{x^2+2x+3}+\frac{c_2 x^{\frac{1}{4}}}{x^2+2x+3}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 33

```
DSolve[4*x^2*(3+2*x+x^2)*y''[x]-x*(3-14*x-15*x^2)*y'[x]+(3+7*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x)\to\frac{3c_1\sqrt[4]{x}+4c_2 x}{3x^2+6x+9}$$

2.543 problem 557

2.543.1 Maple step by step solution 5076

Internal problem ID [8033]
Internal file name [`OUTPUT/6966_Sunday_June_05_2022_05_22_14_PM_98927914/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 557.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2-2x+1)\,y''-x(x+3)\,y'+(4+x)\,y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+\left(-x^2-3x\right)y'+(4+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 10x - 1 \\ t &= 4x^2(x - 1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1031: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}
\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\
\alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)
\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

There is pole in r at $x = 1$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2\,(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-2}{2} + 2\right) &&= \frac{1}{2} \\ \alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-2}{2} + 2\right) &&= \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= 0 \\ \alpha_\infty^{-} &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2\,(x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\
&= \frac{2x^2+x+1}{2x(x-1)^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\ &= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\ &= z_1 \left(\frac{x^{\frac{3}{2}}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}}\mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x\,(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x\,(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}}\mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x\,(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}}$$

Verified OK.

2.543.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (-x^2-3x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4+x)y}{x^2(x-1)^2} + \frac{(x+3)y'}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x(x-1)^2} + \frac{(4+x)y}{x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x+3}{x(x-1)^2}, P_3(x) = \frac{4+x}{x^2(x-1)^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1)^2 - x(x+3)\,y' + (4+x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k - \right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 2$

- Each term must be 0

  $a_1(-1+r)^2 - a_0(1+2r)(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0(1+2r)}{-1+r}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1})\, k + (a_k + a_{k-2} - 2a_{k-1})\, r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$

- Shift index using $k->k+2$

  $(k+r)\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1})\, r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(x^2*(1-2*x+x^2)*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 \mathrm{e}^{-\frac{4}{x-1}}}{x-1} + \frac{c_2 x^2 \,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right) \mathrm{e}^{-\frac{4x}{x-1}}}{x-1}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 54

```
DSolve[x^2*(1-2*x+x^2)*y''[x]-x*(3+x)*y'[x]+(4+x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}} \sqrt{1-x} x^2 \left(c_2 \,\mathrm{ExpIntegralEi}\left(\frac{4x}{x-1}\right) + e^4 c_1\right)}{(x-1)^{3/2}}$$

2.544 problem 558

2.544.1 Maple step by step solution 5085

Internal problem ID [8034]
Internal file name [OUTPUT/6967_Sunday_June_05_2022_05_22_17_PM_22921311/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 558.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+2)\,y''+5x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2 \\ B &= 5x^2 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1033: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} + \frac{1}{8x+16} + \frac{5}{16\,(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\,(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\,(x+2)} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{4\,(x+2)} + \frac{1}{2x} \\
&= \frac{4+x}{4x\,(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4(x+2)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16(x^2+2x)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(x+2)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{(x+2)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{x+2} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\right) + c_2\left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\left(2\sqrt{x+2} - 2\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} - \frac{2c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) - \sqrt{x+2}\right)}{(x+2)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} - \frac{2c_2\sqrt{x}\left(\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) - \sqrt{x+2}\right)}{(x+2)^{\frac{3}{2}}}
$$

Verified OK.

2.544.1 Maple step by step solution

Let's solve

$(2x^3 + 4x^2)\,y'' + 5x^2 y' + (1+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(1+x)y}{2x^2(x+2)} - \frac{5y'}{2(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{5y'}{2(x+2)} + \frac{(1+x)y}{2x^2(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{5}{2(x+2)}, P_3(x) = \frac{1+x}{2x^2(x+2)}\right]$

- ○ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

$((x+2) \cdot P_2(x))\bigg|_{x=-2} = \frac{5}{2}$

- ○ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

- ○ $x = -2$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -2$

- Multiply by denominators

$2x^2(x+2)\, y'' + 5x^2 y' + (1+x)\, y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

$(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du}y(u)\right) + (-1+u)\, y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ○ Shift index using $k- > k-m$

$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\, u^{-1+r} + (4a_1(1+r)(5+2r) - a_0(8r^2+12r+1))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)(5+2r) - a_0(8r^2+12r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$$

- Shift index using $k->k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+5*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{c_2\sqrt{2}\left(2\sqrt{2}\,\sqrt{x+2} - 4\ \mathrm{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\sqrt{x}}{2\,(x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 55

```
DSolve[2*x^2*(2+x)*y''[x]+5*x^2*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}\left(-2\sqrt{2}c_2\mathrm{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) + 2c_2\sqrt{x+2} + c_1\right)}{(x+2)^{3/2}}$$

2.545 problem 559

2.545.1 Maple step by step solution 5096

Internal problem ID [8035]
Internal file name [OUTPUT/6968_Sunday_June_05_2022_05_22_20_PM_80230886/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 559.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-x^2+2)\,y''-2x(2x^2+1)\,y'+(-2x^2+2)\,y=0}$$

Writing the ode as

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-2x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-2x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 1}{(x^3 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 - 1 \\ t &= (x^3 - 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 1}{(x^3 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1035: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16\left(x - \sqrt{2}\right)^2} + \frac{5}{16\left(x + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{32\left(x - \sqrt{2}\right)} + \frac{3\sqrt{2}}{32\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^2 - 1}{(x^3 - 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\sqrt{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\ &= 0 - (0) \\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} + (0) \\ &= \frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)} \\ &= -\frac{1}{x^3-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\left(x-\sqrt{2}\right)^2} + \frac{1}{4\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{4\left(x-\sqrt{2}\right)} - \frac{1}{4\left(x+\sqrt{2}\right)}\right)dx} \\ &= \frac{\sqrt{x}}{\left(x-\sqrt{2}\right)^{\frac{1}{4}}\left(x+\sqrt{2}\right)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-2x}{-x^4+2x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{5\ln\left(x^2-2\right)}{4}} \\ &= z_1\left(\frac{\sqrt{x}}{\left(x^2-2\right)^{\frac{5}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x^2-2)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-2x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{5\ln\left(x^2-2\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{(x^2-2)^{\frac{3}{2}}}\right) + c_2\left(\frac{x}{(x^2-2)^{\frac{3}{2}}}\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)}{(x^2-2)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2-2}+\sqrt{2}\,\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right)\right)}{(x^2-2)^{\frac{3}{2}}}$$

Verified OK.

2.545.1 Maple step by step solution

Let's solve

$(-x^4+2x^2)\, y''+(-4x^3-2x)\, y'+(-2x^2+2)\, y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2(x^2-1)y}{x^2(x^2-2)}-\frac{2(2x^2+1)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{2(2x^2+1)y'}{x(x^2-2)}+\frac{2(x^2-1)y}{x^2(x^2-2)}=0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$\left[P_2(x)=\frac{2(2x^2+1)}{x(x^2-2)}, P_3(x)=\frac{2(x^2-1)}{x^2(x^2-2)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-1$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ◦ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$y''x^2(x^2-2)+2x(2x^2+1)\, y'+(2x^2-2)\, y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}(-1+r)^{2} x^{r}-2a_{1}r^{2}x^{1+r}+\left(\sum_{k=2}^{\infty}\left(-2a_{k}(k+r-1)^{2}+a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2(-1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $-2a_{1}r^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k}(k+r-1)^{2}+a_{k-2}(k+r)\,(k+r-1)=0$

- Shift index using $k->k+2$

  $-2a_{k+2}(k+r+1)^{2}+a_{k}(k+r+2)\,(k+r+1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(k+r+2)}{2(k+r+1)}$

- Recursion relation for $r = 1$

  $a_{k+2} = \frac{a_k(k+3)}{2(k+2)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k(k+3)}{2(k+2)}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 55

```
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-2*x*(1+2*x^2)*diff(y(x),x)+(2-2*x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1 x}{(x^2-2)^{\frac{3}{2}}} + \frac{c_2\sqrt{2}\, x \left(2\arctan\left(\frac{\sqrt{2}}{\sqrt{x^2-2}}\right) + \sqrt{2}\,\sqrt{x^2-2}\right)}{2\,(x^2-2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica
Time used: 0.098 (sec). Leaf size: 58

```
DSolve[x^2*(2-x^2)*y''[x]-2*x*(1+2*x^2)*y'[x]+(2-2*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{x\left(-\sqrt{2}c_2\text{arctanh}\left(\sqrt{1-\frac{x^2}{2}}\right) + c_2\sqrt{2-x^2} + c_1\right)}{(2-x^2)^{3/2}}$$

2.546 problem 560

2.546.1 Maple step by step solution 5106

Internal problem ID [8036]
Internal file name [OUTPUT/6969_Sunday_June_05_2022_05_22_23_PM_55986350/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 560.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(5-x)\,y' + (9-4x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2-5x\right)y' + (9-4x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1037: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 6x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{6x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{6x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{3}{2}\right) - (0) \\
&= \frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1+x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (1+x)\, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\ &= (1+x)\, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= (1+x)\sqrt{x}\, e^{\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2-5x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\frac{5\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{5}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3(1+x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{-x} + (-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3(1+x)\right) + c_2\left(x^3(1+x)\left(\frac{\mathrm{e}^{-x} + (-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3(1+x) + c_2 x^3\left(-\mathrm{expIntegral}_1(x)\,x + \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3(1+x) + c_2 x^3\left(-\mathrm{expIntegral}_1(x)\,x + \mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\right)
$$

Verified OK.

2.546.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 - 5x)\,y' + (9 - 4x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x-9)y}{x^2} - \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-5)y'}{x} - \frac{(4x-9)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-5}{x}, P_3(x) = -\frac{4x-9}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 9$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x-5)\,y' + (9 - 4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r-3)^2 + a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-3+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 3$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r-3)^2 + a_{k-1}(k-5+r) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k-2+r)^2 + a_k (k+r-4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k (k+r-4)}{(k-2+r)^2}$

- Recursion relation for $r = 3$; series terminates at $k = 1$

  $a_{k+1} = -\frac{a_k (k-1)}{(k+1)^2}$

- Apply recursion relation for $k = 0$

  $a_1 = a_0$

- Terminating series solution of the ODE for $r = 3$. Use reduction of order to find the second li

  $y = a_0 \cdot (1+x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*diff(y(x),x$2)-x*(5-x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^3 (x + 1) + c_2 x^3 (\mathrm{expIntegral}_1(x)\, x + \mathrm{expIntegral}_1(x) - \mathrm{e}^{-x})$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]-x*(5-x)*y'[x]+(9-4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} x^3 (c_2 e^x (x + 1) \,\mathrm{ExpIntegralEi}(-x) + c_1 e^x (x + 1) + c_2)$$

2.547 problem 561

2.547.1 Maple step by step solution 5115

Internal problem ID [8037]
Internal file name [`OUTPUT/6970_Sunday_June_05_2022_05_22_26_PM_6542084/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 561.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+x+1)\,y''+12x^2(1+x)\,y'+(3x^2+3x+1)\,y=0}$$

Writing the ode as

$$(4x^4+4x^3+4x^2)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4x^4+4x^3+4x^2\\B&=12x^3+12x^2\\C&=3x^2+3x+1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^2 - 4x - 1 \\ t &= 4\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 4x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1039: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} - \frac{1}{4x^2} + \frac{-\frac{3}{8} - \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{3}{8} + \frac{i\sqrt{3}}{8}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{4} - \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{4} + \frac{5i\sqrt{3}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{8} - \frac{i\sqrt{3}}{8}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{-2i\sqrt{3} - 2}}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{-2i\sqrt{3} - 2}}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{3}{8}+\frac{i\sqrt{3}}{8}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+\frac{\sqrt{2i\sqrt{3}-2}}{4} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}
\end{aligned}
$$

Since the order of r at ∞ is $4>2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2x^2-4x-1}{4\left(x^3+x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-2i\sqrt{3}-2}}{4}$	$\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{2i\sqrt{3}-2}}{4}$	$\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 1-(1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (-)(0) \\ &= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\ &= \frac{2x^2+1}{2x(x^2+x+1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}\right) +\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{2i\sqrt{3}-2}}{4}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right) dx} \\ &= \sqrt{2}\,\sqrt{x}\,(x^2+x+1)^{\frac{1}{4}}\, e^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{2}}}{\left(x^2+x+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12x^3+12x^2}{4x^4+4x^3+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x\sqrt{x^2+x+1}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}} \left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{2x\sqrt{x^2+x+1}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}} \\
& + \frac{c_2\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{x\sqrt{x^2+x+1}}dx\right)}{2\sqrt{x^2+x+1}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+1}} + \frac{c_2\sqrt{2}\,\sqrt{x}\,\mathrm{e}^{-\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}\left(\int \frac{\mathrm{e}^{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{x\sqrt{x^2+x+1}}dx\right)}{2\sqrt{x^2+x+1}}
$$

Verified OK.

2.547.1 Maple step by step solution

Let's solve

$(4x^4+4x^3+4x^2)\,y''+(12x^3+12x^2)\,y'+(3x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}-\frac{3(1+x)y'}{x^2+x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3(1+x)y'}{x^2+x+1}+\frac{(3x^2+3x+1)y}{4x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3(1+x)}{x^2+x+1}, P_3(x)=\frac{3x^2+3x+1}{4x^2(x^2+x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$4x^2(x^2+x+1)\, y'' + 12x^2(1+x)\, y' + (3x^2+3x+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+2r)^2\, x^r + \left(a_1(1+2r)^2 + a_0(3+2r)\,(1+2r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-1}(2k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=\frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2+a_0(3+2r)(1+2r)=0$$

- Solve for the dependent coefficient(s)

$$a_1=-\frac{(3+2r)a_0}{1+2r}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{1}{2}\right)\left((a_k+a_{k-2}+a_{k-1})\,k+(a_k+a_{k-2}+a_{k-1})\,r-\frac{a_k}{2}-\frac{3a_{k-2}}{2}+\frac{a_{k-1}}{2}\right)=0$$

- Shift index using $k->k+2$

$$4\left(k+\frac{3}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})\,(k+2)+(a_{k+2}+a_k+a_{k+1})\,r-\frac{a_{k+2}}{2}-\frac{3a_k}{2}+\frac{a_{k+1}}{2}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2ra_k+2ra_{k+1}+a_k+5a_{k+1}}{2k+2r+3}$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+2}=-\frac{2ka_k+2ka_{k+1}+2a_k+6a_{k+1}}{2k+4},a_1=-2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 143

```
dsolve(4*x^2*(1+x+x^2)*diff(y(x),x$2)+12*x^2*(1+x)*diff(y(x),x)+(1+3*x+3*x^2)*y(x)=0,y(x), s
```

$$y(x) = c_1\sqrt{\frac{x}{x^2+x+1}}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{2}} + c_2\sqrt{\frac{x}{x^2+x+1}}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{2}}\left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{2}}}{x\sqrt{x^2+x+1}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.914 (sec). Leaf size: 93

```
DSolve[4*x^2*(1+x+x^2)*y''[x]+12*x^2*(1+x)*y'[x]+(1+3*x+3*x^2)*y[x]==0,y[x],x,IncludeSingula
```

$$y(x) \to \frac{\sqrt{x}e^{-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}\left(c_2\int_1^x \frac{e^{\sqrt{3}\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

2.548 problem 562

2.548.1 Maple step by step solution 5126

Internal problem ID [8038]
Internal file name [OUTPUT/6971_Sunday_June_05_2022_05_22_30_PM_20481450/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 562.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y=0}$$

Writing the ode as

$$x^2(x^2+x+1)\,y''+(2x^3+4x^2-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x^2+x+1) \\ B &= 2x^3+4x^2-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 10x^2 - 8x - 1 \\ t &= 4\left(x^3 + x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 - 8x - 1}{4\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1041: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{2x} - \frac{1}{4x^2} + \frac{-\frac{29}{24} - \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{29}{24} + \frac{7i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{4} - \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{3}{4} + \frac{41i\sqrt{3}}{36}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{29}{24} - \frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{-138-42i\sqrt{3}}}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{-138-42i\sqrt{3}}}{12}
\end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b=-\frac{29}{24}+\frac{7i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12}\\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is $4>2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{10x^2-8x-1}{4\left(x^3+x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138-42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2}+\frac{\sqrt{-138+42i\sqrt{3}}}{12}$	$\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}+\alpha_{c_3}^{-}\right)\\
&= 1-(1)\\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} + (-)(0) \\
&= \frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}} \\
&= \frac{2x^2-2x+1}{2x\left(x^2+x+1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2}-\frac{\sqrt{-138+42i}}{12}}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right.}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{\frac{1}{2}-\frac{\sqrt{-138-42i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}-\frac{\sqrt{-138+42i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx} \\
&= \sqrt{2}\,\sqrt{x}\left(x^2+x+1\right)^{\frac{1}{4}} e^{-\frac{7\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)}\,dx} \\
&= z_1 e^{-\frac{3\ln\left(x^2+x+1\right)}{4}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}+\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\sqrt{x}\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{\left(x^2+x+1\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{2}\,x\,\mathrm{e}^{-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+4x^2-x}{x^2\left(x^2+x+1\right)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+x+1\right)}{2}-\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x\sqrt{x^2+x+1}}\,dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}}\right) \\
&\quad + c_2 \left(\frac{\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{2x\sqrt{x^2+x+1}} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} + \frac{c_2\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x\sqrt{x^2+x+1}} dx\right)}{2\sqrt{x^2+x+1}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x^2+x+1}} + \frac{c_2\sqrt{2}\, x\, \mathrm{e}^{-\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{7\sqrt{3}\, \arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x\sqrt{x^2+x+1}} dx\right)}{2\sqrt{x^2+x+1}}
$$

Verified OK.

2.548.1 Maple step by step solution

Let's solve

$x^2(x^2+x+1)\, y'' + (2x^3+4x^2-x)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2(x^2+x+1)} - \frac{(2x^2+4x-1)y'}{x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(2x^2+4x-1)\,y'}{x(x^2+x+1)} + \frac{y}{x^2(x^2+x+1)} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $$\left[P_2(x) = \frac{2x^2+4x-1}{x(x^2+x+1)}, P_3(x) = \frac{1}{x^2(x^2+x+1)}\right]$$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left.(x \cdot P_2(x))\right|_{x=0} = -1$$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left.(x^2 \cdot P_3(x))\right|_{x=0} = 1$$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $$x^2(x^2+x+1)\,y'' + x(2x^2+4x-1)\,y' + y = 0$$
- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$
  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$
  - Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

    Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (a_1 r^2 + a_0 r(3+r)) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k+r-1)(k+2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term must be 0

  $a_1 r^2 + a_0 r(3+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = -\frac{(3+r)a_0}{r}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((a_k + a_{k-2} + a_{k-1})k + (a_k + a_{k-2} + a_{k-1})r - a_k - 2a_{k-2} + 2a_{k-1}\right) = 0$
- Shift index using $k- >k+2$

  $(k+r+1)\left((a_{k+2} + a_k + a_{k+1})(k+2) + (a_{k+2} + a_k + a_{k+1})r - a_{k+2} - 2a_k + 2a_{k+1}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + 4a_{k+1}}{k+r+1}$
- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{ka_k + ka_{k+1} + a_k + 5a_{k+1}}{k+2}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{ka_k + ka_{k+1} + a_k + 5a_{k+1}}{k+2}, a_1 = -4a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 137

```
dsolve(x^2*(1+x+x^2)*diff(y(x),x$2)-x*(1-4*x-2*x^2)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x \left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{7i\sqrt{3}}{6}}}{\sqrt{x^2+x+1}} + \frac{c_2 x \left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{7i\sqrt{3}}{6}} \left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{7i\sqrt{3}}{6}}}{x\sqrt{x^2+x+1}} dx\right)}{\sqrt{x^2+x+1}}$$

✓ Solution by Mathematica

Time used: 0.914 (sec). Leaf size: 90

```
DSolve[x^2*(1+x+x^2)*y''[x]-x*(1-4*x-2*x^2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{xe^{-\frac{7\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{7\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{K[1]\sqrt{K[1]^2+K[1]+1}}dK[1]+c_1\right)}{\sqrt{x^2+x+1}}$$

2.549 problem 563

2.549.1 Maple step by step solution 5137

Internal problem ID [8039]
Internal file name [OUTPUT/6972_Sunday_June_05_2022_05_22_37_PM_27642812/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 563.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{9x^2y'' + 3x(-2x^2 + 3x + 5)\, y' + (-14x^2 + 12x + 1)\, y = 0}$$

Writing the ode as

$$9x^2y'' + (-6x^3 + 9x^2 + 15x)\, y' + (-14x^2 + 12x + 1)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -6x^3 + 9x^2 + 15x \\ C &= -14x^2 + 12x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 12x^3 + 33x^2 - 18x - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1043: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{x}{3} + \frac{11}{12} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{2} + \frac{1}{x} + \frac{3}{4x^2} - \frac{3}{4x^3} - \frac{27}{8x^4} - \frac{117}{32x^5} + \frac{405}{64x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= -\frac{1}{2}+\frac{x}{3} \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}-\frac{1}{3}x+\frac{1}{9}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q+\frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4-12x^3+33x^2-18x-9}{36x^2} \\ &= Q+\frac{R}{36x^2} \\ &= \left(\frac{1}{9}x^2-\frac{1}{3}x+\frac{11}{12}\right)+\left(\frac{-18x-9}{36x^2}\right) \\ &= \frac{x^2}{9}-\frac{x}{3}+\frac{11}{12}+\frac{-18x-9}{36x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $\frac{11}{12}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{11}{12}\right)-\left(\frac{1}{4}\right) \\ &= \frac{2}{3} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= -\frac{1}{2}+\frac{x}{3} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{\frac{2}{3}}{\frac{1}{3}}-1\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{\frac{2}{3}}{\frac{1}{3}}-1\right) &&= -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$-\frac{1}{2} + \frac{x}{3}$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(-\frac{1}{2} + \frac{x}{3} \right) \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \\ &= \frac{1}{2x} - \frac{1}{2} + \frac{x}{3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 12x^3 + 33x^2 - 18x - 9}{36x^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2} + \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{\frac{x(-3+x)}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6x^3 + 9x^2 + 15x}{9x^2} \, dx} \\ &= z_1 e^{\frac{x^2}{6} - \frac{x}{2} - \frac{5\ln(x)}{6}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x(-3+x)}{6}}}{x^{\frac{5}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-6x^3+9x^2+15x}{9x^2}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\frac{x^2}{3}-x-\frac{5\ln(x)}{3}}}{(y_1)^2}\,dx$$

$$= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x(-3+x)}{3}}}{x}dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}}\right) + c_2 \left(\frac{\mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}}\left(\int \frac{\mathrm{e}^{-\frac{x(-3+x)}{3}}}{x}dx\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{\frac{x(-3+x)}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x(-3+x)}{3}}}{x}dx\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{\frac{x(-3+x)}{3}}}{x^{\frac{1}{3}}} + \frac{c_2 \mathrm{e}^{\frac{x(-3+x)}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x(-3+x)}{3}}}{x}dx\right)}{x^{\frac{1}{3}}}$$

Verified OK.

2.549.1 Maple step by step solution

Let's solve

$9x^2y'' + (-6x^3 + 9x^2 + 15x)\,y' + (-14x^2 + 12x + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = \frac{(14x^2-12x-1)y}{9x^2} + \frac{(2x^2-3x-5)y'}{3x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(2x^2-3x-5)y'}{3x} - \frac{(14x^2-12x-1)y}{9x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = -\frac{2x^2-3x-5}{3x}, P_3(x) = -\frac{14x^2-12x-1}{9x^2}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{9}$$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$9x^2y'' - 3x(2x^2 - 3x - 5)\, y' + (-14x^2 + 12x + 1)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+3r)^2\, x^r + \left(a_1(4+3r)^2 + 3a_0(4+3r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)^2 + 3a_{k-1}(3k+3r+1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{3}$$

- Each term must be 0

$$a_1(4+3r)^2 + 3a_0(4+3r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{3a_0}{4+3r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(3k+3r+1)^2 + (3k+3r+1)\,(-2a_{k-2} + 3a_{k-1}) = 0$$

- Shift index using $k- > k+2$

$$a_{k+2}(3k+3r+7)^2 + (3k+3r+7)\,(-2a_k + 3a_{k+1}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+3r+7}$$

- Recursion relation for $r = -\frac{1}{3}$

$$a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}$$

- Solution for $r = -\frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2} = \frac{2a_k - 3a_{k+1}}{3k+6}, a_1 = -a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(5+3*x-2*x^2)*diff(y(x),x)+(1+12*x-14*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1 e^{\frac{1}{3}x^2 - x}}{x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{1}{3}x^2 - x} \left(\int \frac{e^{-\frac{1}{3}x^2 + x}}{x} dx\right)}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 52

```
DSolve[9*x^2*y''[x]+3*x*(5+3*x-2*x^2)*y'[x]+(1+12*x-14*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{e^{\frac{1}{3}(x-3)x}\left(c_2 \int_1^x \frac{e^{K[1]-\frac{K[1]^2}{3}}}{K[1]} dK[1] + c_1\right)}{\sqrt[3]{x}}$$

2.550 problem 564

2.550.1 Maple step by step solution 5149

Internal problem ID [8040]
Internal file name [`OUTPUT/6973_Sunday_June_05_2022_05_22_41_PM_78197349/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 564.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''+x\left(3x^2+14x+5\right)y'+\left(12x^2+18x+4\right)y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(3x^3+14x^2+5x\right)y'+\left(12x^2+18x+4\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=3x^3+14x^2+5x\\C&=12x^2+18x+4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^4 - 12x^3 - 16x^2 - 4x - 1$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1045: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16} - \frac{1}{4x^2} - \frac{15}{64\left(x+\frac{1}{2}\right)^2} - \frac{21}{16\left(x+\frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{15}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{4} - \frac{7}{8x} - \frac{19}{48x^2} - \frac{151}{288x^3} - \frac{139}{192x^4} - \frac{11383}{10368x^5} - \frac{38729}{20736x^6} - \frac{1212655}{373248x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \\ &= Q + \frac{R}{16x^4 + 16x^3 + 4x^2} \\ &= \left(\frac{9}{16}\right) + \left(\frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2}\right) \\ &= \frac{9}{16} + \frac{-21x^3 - \frac{73}{4}x^2 - 4x - 1}{16x^4 + 16x^3 + 4x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -21. Dividing this by leading coefficient in t which is 16 gives $-\frac{21}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{21}{16}\right) - (0) \\ &= -\frac{21}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{3}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= -\frac{7}{8} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{21}{16}}{\frac{3}{4}} - 0\right) &= \frac{7}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 12x^3 - 16x^2 - 4x - 1}{4\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{8}$	$\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{3}{4}$	$-\frac{7}{8}$	$\frac{7}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{7}{8} - \left(\frac{7}{8}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} + (-)\left(\frac{3}{4}\right) \\
&= \frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4} \\
&= \frac{-3x^2+2x+1}{4x^2+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right)^2 - \left(\frac{9x^4 - 12x}{4}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{3}{8\left(x+\frac{1}{2}\right)} - \frac{3}{4}\right) dx} \\
&= \sqrt{x}\,(2x+1)^{\frac{3}{8}}\, e^{-\frac{3x}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{3x}{4}-\frac{5\ln(x)}{2}-\frac{5\ln(2x+1)}{8}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{4}}}{x^{\frac{5}{2}}\left(2x+1\right)^{\frac{5}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3+14x^2+5x}{2x^3+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3x}{2}-5\ln(x)-\frac{5\ln(2x+1)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x}{2}}}{x\left(2x+1\right)^{\frac{3}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}}\left(\int \frac{\mathrm{e}^{\frac{3x}{2}}}{x\left(2x+1\right)^{\frac{3}{4}}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\mathrm{e}^{-\frac{3x}{2}}}{x^2\left(2x+1\right)^{\frac{1}{4}}} + \frac{c_2\mathrm{e}^{-\frac{3x}{2}}\left(\int \frac{\mathrm{e}^{\frac{3x}{2}}}{x(2x+1)^{\frac{3}{4}}}dx\right)}{x^2\left(2x+1\right)^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 e^{-\frac{3x}{2}}}{x^2 (2x+1)^{\frac{1}{4}}} + \frac{c_2 e^{-\frac{3x}{2}} \left(\int \frac{e^{\frac{3x}{2}}}{x(2x+1)^{\frac{3}{4}}} dx\right)}{x^2 (2x+1)^{\frac{1}{4}}}$$

Verified OK.

2.550.1 Maple step by step solution

Let's solve

$(2x^3 + x^2) y'' + (3x^3 + 14x^2 + 5x) y' + (12x^2 + 18x + 4) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(6x^2+9x+2)y}{x^2(2x+1)} - \frac{(3x^2+14x+5)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x^2+14x+5)y'}{x(2x+1)} + \frac{2(6x^2+9x+2)y}{x^2(2x+1)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x^2+14x+5}{x(2x+1)}, P_3(x) = \frac{2(6x^2+9x+2)}{x^2(2x+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x+1) y'' + x(3x^2 + 14x + 5) y' + (12x^2 + 18x + 4) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)^2 x^r + \left(a_1 (3+r)^2 + 2a_0 (3+r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)^2 + 2a_{k-1} (k+r+2)^2 + 3a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term must be 0

$$a_1 (3+r)^2 + 2a_0 (3+r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -2a_0$$

- Each term in the series must be 0, giving the recursion relation

$$((2k+2r+4)\, a_{k-1}+a_k(k+r+2)+3a_{k-2})\,(k+r+2)=0$$

- Shift index using $k->k+2$

$$((2k+8+2r)\, a_{k+1}+a_{k+2}(k+r+4)+3a_k)\,(k+r+4)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2ka_{k+1}+2ra_{k+1}+3a_k+8a_{k+1}}{k+r+4}$$

- Recursion relation for $r=-2$

$$a_{k+2}=-\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{2ka_{k+1}+3a_k+4a_{k+1}}{k+2}, a_1=-2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+14*x+3*x^2)*diff(y(x),x)+(4+18*x+12*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{c_1 e^{-\frac{3x}{2}}}{(2x+1)^{\frac{1}{4}} x^2} + \frac{c_2 e^{-\frac{3x}{2}} \left(\int \frac{e^{\frac{3x}{2}}}{(2x+1)^{\frac{3}{4}} x} dx\right)}{(2x+1)^{\frac{1}{4}} x^2}$$

✓ Solution by Mathematica

Time used: 0.442 (sec). Leaf size: 61

```
DSolve[x^2*(1+2*x)*y''[x]+x*(5+14*x+3*x^2)*y'[x]+(4+18*x+12*x^2)*y[x]==0,y[x],x,IncludeSingu
```

$$y(x) \to \frac{e^{-3x/2}\left(c_2 \int_1^x \frac{e^{\frac{3K[1]}{2}}}{K[1](2K[1]+1)^{3/4}} dK[1] + c_1\right)}{x^2 \sqrt[4]{2x+1}}$$

2.551 problem 565

2.551.1 Maple step by step solution 5160

Internal problem ID [8041]
Internal file name [`OUTPUT/6974_Sunday_June_05_2022_05_22_45_PM_14665731/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 565.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{16x^2y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0}$$

Writing the ode as

$$16x^2y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 8x^3 + 4x^2 + 24x \\ C &= 18x^2 + 5x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 4x^3 - 31x^2 - 8x - 16 \\ t &= 64x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1047: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} - \frac{1}{8x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} + \frac{1}{8} - \frac{1}{x} + \frac{1}{4x^2} - \frac{21}{8x^3} + \frac{37}{16x^4} - \frac{377}{32x^5} + \frac{1137}{64x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{1}{8} + \frac{x}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64} + \frac{1}{16}x + \frac{1}{16}x^2$$

This shows that the coefficient of 1 in the above is $\frac{1}{64}$. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2} \\ &= Q + \frac{R}{64x^2} \\ &= \left(\frac{1}{16}x^2 + \frac{1}{16}x - \frac{31}{64}\right) + \left(\frac{-8x - 16}{64x^2}\right) \\ &= \frac{x^2}{16} + \frac{x}{16} - \frac{31}{64} + \frac{-8x - 16}{64x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{31}{64}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{31}{64}\right) - \left(\frac{1}{64}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{8} + \frac{x}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{1}{8} + \frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{8} + \frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{1}{8} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right)^2 - \left(\frac{4x^4 + 4x^3 - 31x^2 - 8x - 16}{64x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{8} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x(1+x)}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8x^3+4x^2+24x}{16x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{8} - \frac{x}{8} - \frac{3\ln(x)}{4}} \\ &= z_1 \left(\frac{e^{-\frac{x(1+x)}{8}}}{x^{\frac{3}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{8x^3+4x^2+24x}{16x^2}dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{-\frac{x^2}{4}-\frac{x}{4}-\frac{3\ln(x)}{2}}}{(y_1)^2} dx$$

$$= y_1 \left(\int \frac{\mathrm{e}^{\frac{x(1+x)}{4}}}{x} dx\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\mathrm{e}^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} \left(\int \frac{\mathrm{e}^{\frac{x(1+x)}{4}}}{x} dx\right)\right)$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x(1+x)}{4}} \left(\int \frac{\mathrm{e}^{\frac{x(1+x)}{4}}}{x} dx\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{x(1+x)}{4}}}{x^{\frac{1}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{x(1+x)}{4}} \left(\int \frac{\mathrm{e}^{\frac{x(1+x)}{4}}}{x} dx\right)}{x^{\frac{1}{4}}}$$

Verified OK.

2.551.1 Maple step by step solution

Let's solve

$16x^2y'' + (8x^3 + 4x^2 + 24x)\, y' + (18x^2 + 5x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(18x^2+5x+1)y}{16x^2} - \frac{(2x^2+x+6)y'}{4x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(2x^2+x+6)y'}{4x} + \frac{(18x^2+5x+1)y}{16x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{2x^2+x+6}{4x}, P_3(x) = \frac{18x^2+5x+1}{16x^2}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$16x^2y'' + 4x(2x^2 + x + 6)\, y' + (18x^2 + 5x + 1)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2\, x^r + \left(a_1(5+4r)^2 + a_0(5+4r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r+1)^2 + a_{k-1}(4k+4r+1)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -\frac{1}{4}$$

- Each term must be 0

$$a_1(5+4r)^2 + a_0(5+4r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{a_0}{5+4r}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(4k+4r+1)^2 + (4k+4r+1)\,(2a_{k-2} + a_{k-1}) = 0$$

- Shift index using $k- > k+2$

$$a_{k+2}(4k+4r+9)^2 + (4k+4r+9)\,(2a_k + a_{k+1}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+4r+9}$$

- Recursion relation for $r = -\frac{1}{4}$

$$a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}$$

- Solution for $r = -\frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{4}}, a_{k+2} = -\frac{2a_k + a_{k+1}}{4k+8}, a_1 = -\frac{a_0}{4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunB  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(16*x^2*diff(y(x),x$2)+4*x*(6+x+2*x^2)*diff(y(x),x)+(1+5*x+18*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1 \mathrm{e}^{-\frac{1}{4}x^2-\frac{1}{4}x}}{x^{\frac{1}{4}}} + \frac{c_2 \mathrm{e}^{-\frac{1}{4}x^2-\frac{1}{4}x}\left(\int \frac{\mathrm{e}^{\frac{1}{4}x^2+\frac{1}{4}x}}{x}dx\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.348 (sec). Leaf size: 51

```
DSolve[16*x^2*y''[x]+4*x*(6+x+2*x^2)*y'[x]+(1+5*x+18*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{e^{-\frac{1}{4}x(x+1)}\left(c_2 \int_1^x \frac{e^{\frac{1}{4}K[1](K[1]+1)}}{K[1]}dK[1] + c_1\right)}{\sqrt[4]{x}}$$

2.552 problem 566

2.552.1 Maple step by step solution 5172

Internal problem ID [8042]
Internal file name [`OUTPUT/6975_Sunday_June_05_2022_05_22_49_PM_42044977/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 566.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(1+x)\,y''+3x\left(-x^2+11x+5\right)y'+\left(-7x^2+16x+1\right)y=0}$$

Writing the ode as

$$\left(9x^3+9x^2\right)y''+\left(-3x^3+33x^2+15x\right)y'+\left(-7x^2+16x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+9x^2 \\ B &= -3x^3+33x^2+15x \\ C &= -7x^2+16x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 + 6x^3 + 3x^2 - 18x - 9 \\ t &= 36\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1049: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{7}{36(1+x)^2} - \frac{1}{4x^2} + \frac{1}{9x+9}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{1}{3x} - \frac{5}{6x^2} + \frac{5}{6x^3} - \frac{7}{3x^4} + \frac{41}{6x^5} - \frac{149}{6x^6} + \frac{277}{3x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \\ &= Q + \frac{R}{36x^4 + 72x^3 + 36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2}\right) \\ &= \frac{1}{36} + \frac{4x^3 + 2x^2 - 18x - 9}{36x^4 + 72x^3 + 36x^2} \end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is 4. Dividing this by leading coefficient in t which is 36 gives $\frac{1}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{9}\right) - (0) \\ &= \frac{1}{9} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{6} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &&= \frac{1}{3} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{9}}{\frac{1}{6}} - 0\right) &&= -\frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 + 6x^3 + 3x^2 - 18x - 9}{36\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{6}$	$-\frac{1}{6}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{6(1+x)} + \frac{1}{2x} + \left(\frac{1}{6}\right) \\ &= -\frac{1}{6(1+x)} + \frac{1}{2x} + \frac{1}{6} \\ &= -\frac{1}{6+6x} + \frac{1}{2x} + \frac{1}{6}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)(0) + \left(\left(\frac{1}{6(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{6(1+x)} + \frac{1}{2x} + \frac{1}{6}\right)^2 - \left(\frac{x^4 + 6x^3 +}{36(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{6(1+x)} + \frac{1}{2x} + \frac{1}{6}\right) dx} \\ &= \frac{\sqrt{x}\, e^{\frac{x}{6}}}{(1+x)^{\frac{1}{6}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}-\frac{5\ln(x)}{6}-\frac{7\ln(1+x)}{6}} \\
&= z_1\left(\frac{e^{\frac{x}{6}}}{x^{\frac{5}{6}}(1+x)^{\frac{7}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}(1+x)^{\frac{4}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+33x^2+15x}{9x^3+9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}-\frac{5\ln(x)}{3}-\frac{7\ln(1+x)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{(1+x)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}(1+x)^{\frac{4}{3}}}\right) + c_2\left(\frac{e^{\frac{x}{3}}}{x^{\frac{1}{3}}(1+x)^{\frac{4}{3}}}\left(\int \frac{(1+x)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{\frac{x}{3}}}{x^{\frac{1}{3}}(1+x)^{\frac{4}{3}}} + \frac{c_2 e^{\frac{x}{3}}\left(\int \frac{(1+x)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x}dx\right)}{x^{\frac{1}{3}}(1+x)^{\frac{4}{3}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 e^{\frac{x}{3}}}{x^{\frac{1}{3}} (1+x)^{\frac{4}{3}}} + \frac{c_2 e^{\frac{x}{3}} \left(\int \frac{(1+x)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x} dx\right)}{x^{\frac{1}{3}} (1+x)^{\frac{4}{3}}}$$

Verified OK.

2.552.1 Maple step by step solution

Let's solve

$(9x^3 + 9x^2) y'' + (-3x^3 + 33x^2 + 15x) y' + (-7x^2 + 16x + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(7x^2-16x-1)y}{9x^2(1+x)} + \frac{(x^2-11x-5)y'}{3x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-11x-5)y'}{3x(1+x)} - \frac{(7x^2-16x-1)y}{9x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-11x-5}{3x(1+x)}, P_3(x) = -\frac{7x^2-16x-1}{9x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = \frac{7}{3}$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $9x^2(1+x) y'' - 3x(x^2 - 11x - 5) y' + (-7x^2 + 16x + 1) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(9u^3-18u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(-3u^3+42u^2-60u+21)\left(\frac{d}{du}y(u)\right)+(-7u^2+30u-22)\,y(u)=$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

$$u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..3$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0r(4+3r)\,u^{-1+r}+\left(3a_1(1+r)\,(7+3r)-2a_0(9r^2+21r+11)\right)u^r+\left(3a_2(2+r)\,(10+3r)-2\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$3r(4+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{4}{3}\right\}$$

- The coefficients of each power of u must be 0

$$[3a_1(1+r)\,(7+3r)-2a_0(9r^2+21r+11)=0, 3a_2(2+r)\,(10+3r)-2a_1(9r^2+39r+41)+3a$$

- Solve for the dependent coefficient(s)

$\left\{a_1 = \frac{2a_0(9r^2+21r+11)}{3(3r^2+10r+7)}, a_2 = \frac{a_0(243r^4+1593r^3+3699r^2+3567r+1174)}{9(9r^4+78r^3+241r^2+312r+140)}\right\}$

- Each term in the series must be 0, giving the recursion relation

$9(-2a_k + a_{k-1} + a_{k+1})k^2 + 3(6(-2a_k + a_{k-1} + a_{k+1})r - 14a_k - a_{k-2} + 5a_{k-1} + 10a_{k+1})k + 9(-$

- Shift index using $k- >k+2$

$9(-2a_{k+2} + a_{k+1} + a_{k+3})(k+2)^2 + 3(6(-2a_{k+2} + a_{k+1} + a_{k+3})r - 14a_{k+2} - a_k + 5a_{k+1} + 10a_{k+}$

- Recursion relation that defines series solution to ODE

$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}+18kra_{k+1}-36kra_{k+2}+9r^2a_{k+1}-18r^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-3ra_k+51ra_{k+1}-114ra}{3(3k^2+6kr+3r^2+22k+22r+39)}$

- Recursion relation for $r = 0$

$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}$

- Solution for $r = 0$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21}, a_2 =\right.$

- Revert the change of variables $u = 1 + x$

$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a_k+72a_{k+1}-178a_{k+2}}{3(3k^2+22k+39)}, a_1 = \frac{22a_0}{21},\right.$

- Recursion relation for $r = -\frac{4}{3}$

$a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}$

- Solution for $r = -\frac{4}{3}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3}, a_2 =\right.$

- Revert the change of variables $u = 1 + x$

$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k-\frac{4}{3}}, a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+27ka_{k+1}-66ka_{k+2}-3a_k+20a_{k+1}-58a_{k+2}}{3(3k^2+14k+15)}, a_1 = \frac{2a_0}{3},\right.$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k-\frac{4}{3}}\right), a_{k+3} = -\frac{9k^2a_{k+1}-18k^2a_{k+2}-3ka_k+51ka_{k+1}-114ka_{k+2}-7a}{3(3k^2+22k+39)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(9*x^2*(1+x)*diff(y(x),x$2)+3*x*(5+11*x-x^2)*diff(y(x),x)+(1+16*x-7*x^2)*y(x)=0,y(x),
```

$$y(x) = \frac{c_1 e^{\frac{x}{3}}}{(x+1)^{\frac{4}{3}} x^{\frac{1}{3}}} + \frac{c_2 e^{\frac{x}{3}} \left(\int \frac{(x+1)^{\frac{1}{3}} e^{-\frac{x}{3}}}{x} dx\right)}{(x+1)^{\frac{4}{3}} x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 50

```
DSolve[9*x^2*(1+x)*y''[x]+3*x*(5+11*x-x^2)*y'[x]+(1+16*x-7*x^2)*y[x]==0,y[x],x,IncludeSingul
```

$$y(x) \to \frac{e^{x/3}\left(c_1 - \sqrt[3]{3e}c_2\Gamma\left(\frac{1}{3}, \frac{x+1}{3}\right)\right)}{\sqrt[3]{x}(x+1)^{4/3}}$$

2.553 problem 567

2.553.1 Maple step by step solution 5183

Internal problem ID [8043]
Internal file name [`OUTPUT/6976_Sunday_June_05_2022_05_22_55_PM_64635033/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 567.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{36x^2(1-2x)\, y''+24x(1-9x)\, y'+(1-70x)\, y=0}$$

Writing the ode as

$$\left(-72x^3+36x^2\right) y''+\left(-216x^2+24x\right) y'+(1-70x)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -72x^3+36x^2 \\ B &= -216x^2+24x \\ C &= 1-70x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -32x^2 + 48x - 9$$
$$t = 36\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1051: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{1}{3x} + \frac{7}{36\left(x - \frac{1}{2}\right)^2} - \frac{1}{3\left(x - \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-32x^2 + 48x - 9}{36\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{2}{9}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{2}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{3}-\left(\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} + (-)\,(0) \\ &= \frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)} \\ &= \frac{4x-3}{12x^2-6x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{6\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{-32x^2+48x-9}{36\left(2x^2-x\right)^2}\right)\right.$$

$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{1}{6\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{(2x-1)^{\frac{1}{6}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-216x^2+24x}{-72x^3+36x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{3} - \frac{7\ln(2x-1)}{6}} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{3}}\,(2x-1)^{\frac{7}{6}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-216x^2+24x}{-72x^3+36x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{2\ln(x)}{3}-\frac{7\ln(2x-1)}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left(3(2x-1)^{\frac{1}{3}} + \frac{\ln\left((2x-1)^{\frac{2}{3}} - (2x-1)^{\frac{1}{3}} + 1\right)}{2} \right. \\
&\qquad \left. - \sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - \ln\left((2x-1)^{\frac{1}{3}}+1\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}\right) + c_2\left(\frac{x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}\left(3(2x-1)^{\frac{1}{3}} + \frac{\ln\left((2x-1)^{\frac{2}{3}} - (2x-1)^{\frac{1}{3}} + 1\right)}{2}\right.\right. \\
&\qquad \left.\left. - \sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - \ln\left((2x-1)^{\frac{1}{3}}+1\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} - \frac{c_2\left(\sqrt{3}\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - 3(2x-1)^{\frac{1}{3}} + \ln\left((2x-1)^{\frac{1}{3}}+1\right) - \frac{\ln\left((2x-1)^{\frac{2}{3}}-(2x-1)^{\frac{1}{3}}+1\right)}{2}\right)x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}} - \frac{c_2\left(\sqrt{3}\,\arctan\left(\frac{\left(2(2x-1)^{\frac{1}{3}}-1\right)\sqrt{3}}{3}\right) - 3(2x-1)^{\frac{1}{3}} + \ln\left((2x-1)^{\frac{1}{3}}+1\right) - \frac{\ln\left((2x-1)^{\frac{2}{3}}-(2x-1)^{\frac{1}{3}}+1\right)}{2}\right) x^{\frac{1}{6}}}{(2x-1)^{\frac{4}{3}}}$$

Verified OK.

2.553.1 Maple step by step solution

Let's solve

$(-72x^3+36x^2)\,y''+(-216x^2+24x)\,y'+(1-70x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(70x-1)y}{36x^2(2x-1)}-\frac{2(9x-1)y'}{3x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(9x-1)y'}{3x(2x-1)}+\frac{(70x-1)y}{36x^2(2x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{2(9x-1)}{3x(2x-1)}, P_3(x)=\frac{70x-1}{36x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=\frac{1}{36}$

  ◦ $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

$$36y''x^2(2x-1)+24x(9x-1)\,y'+y(70x-1)=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

$$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k->k-m$

$$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m\cdot y'$ to series expansion for $m=1..2$

$$x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- Convert $x^m\cdot y''$ to series expansion for $m=2..3$

$$x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+6r)^2\,x^r+\left(\sum_{k=1}^{\infty}\left(-a_k(6k+6r-1)^2+2a_{k-1}(6k+1+6r)\,(6k+6r-1)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+6r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=\frac{1}{6}$$

- Each term in the series must be 0, giving the recursion relation

$$-36\left(\left(-2k-2r-\frac{1}{3}\right)a_{k-1}+a_k\left(k+r-\frac{1}{6}\right)\right)\left(k+r-\frac{1}{6}\right)=0$$

- Shift index using $k- >k+1$

$$-36\left(\left(-2k-\frac{7}{3}-2r\right)a_k+a_{k+1}\left(k+\frac{5}{6}+r\right)\right)\left(k+\frac{5}{6}+r\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{2(6k+6r+7)a_k}{6k+6r+5}$$

- Recursion relation for $r=\frac{1}{6}$

$$a_{k+1}=\frac{2(6k+8)a_k}{6k+6}$$

- Solution for $r=\frac{1}{6}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{6}}, a_{k+1}=\frac{2(6k+8)a_k}{6k+6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(36*x^2*(1-2*x)*diff(y(x),x$2)+24*x*(1-9*x)*diff(y(x),x)+(1-70*x)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^{\frac{1}{6}}}{(-1+2x)^{\frac{4}{3}}} + \frac{c_2 x^{\frac{1}{6}} \left(\int \frac{(-1+2x)^{\frac{1}{3}}}{x} dx\right)}{(-1+2x)^{\frac{4}{3}}}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 111

```
DSolve[36*x^2*(1-2*x)*y''[x]+24*x*(1-9*x)*y'[x]+(1-70*x)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{\sqrt[6]{x}\left(-2\sqrt{3}c_2 \arctan\left(\frac{2\sqrt[3]{1-2x}+1}{\sqrt{3}}\right) + 6c_2\sqrt[3]{1-2x} + 2c_2 \log\left(\sqrt[3]{1-2x}-1\right) - c_2 \log\left((1-2x)^{2/3} + \sqrt[3]{1\right.\right.}{2(1-2x)^{4/3}}$$

2.554 problem 568

2.554.1 Maple step by step solution 5194

Internal problem ID [8044]
Internal file name [`OUTPUT/6977_Sunday_June_05_2022_05_22_59_PM_93429649/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 568.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1+x)\,y''-x(3-x)\,y'+4y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(x^2-3x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=x^2-3x\\C&=4\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 - 10x - 1 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1053: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2+x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\ &= -\frac{1}{1+x} + \frac{1}{2x} \\ &= -\frac{x-1}{2x(1+x)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2 - 10x - 1}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x-1)\, e^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\ &= (x-1)\, e^{\frac{\ln(x)}{2} - \ln(1+x)} \\ &= \frac{(x-1)\sqrt{x}}{1+x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-3x}{x^2(1+x)}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} - 2\ln(1+x)} \\ &= z_1 \left(\frac{x^{\frac{3}{2}}}{(1+x)^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-1)\,x^2}{(1+x)^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-4\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x-1)\,x^2}{(1+x)^3}\right) + c_2\left(\frac{(x-1)\,x^2}{(1+x)^3}\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x-1)\,x^2}{(1+x)^3} + \frac{c_2(\ln(x)(x-1)-4)\,x^2}{(1+x)^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x-1)\,x^2}{(1+x)^3} + \frac{c_2(\ln(x)(x-1)-4)\,x^2}{(1+x)^3}$$

Verified OK.

2.554.1 Maple step by step solution

Let's solve

$x^2(1+x)\, y'' + (x^2-3x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(1+x)} - \frac{(-3+x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-3+x)y'}{x(1+x)} + \frac{4y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{-3+x}{x(1+x)}, P_3(x) = \frac{4}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 4$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\, y'' + x(-3+x)\, y' + 4y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2-5u+4)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, u^{-1+r} + \left(a_1(1+r)\,(4+r) - a_0(2r^2+3r-4)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+4+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)\,(4+r) - a_0(2r^2+3r-4) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+4+r) - a_k(2k^2+4kr+2r^2+3k+3r-4) + a_{k-1}(k+r-1)^2 = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) - a_{k+1}\left(2(k+1)^2+4(k+1)\,r+2r^2+3k-1+3r\right) + a_k(k+r)^2 =$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}+2kra_k-4kra_{k+1}+r^2a_k-2r^2a_{k+1}-7ka_{k+1}-7ra_{k+1}-a_{k+1}}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{k^2a_k-2k^2a_{k+1}-6ka_k+5ka_{k+1}+9a_k+2a_{k+1}}{(k-1)(k+2)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 6k a_k + 5k a_{k+1} + 9a_k + 2a_{k+1}}{(k-1)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 7k a_{k+1} - a_{k+1}}{(k+2)(k+5)}, 4a_1 + 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3-x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 (x-1)}{(x+1)^3} + \frac{c_2 x^2 (x \ln(x) - \ln(x) - 4)}{(x+1)^3}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 33

```
DSolve[x^2*(1+x)*y''[x]-x*(3-x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2(c_1(x-1)+c_2(x-1)\log(x)-4c_2)}{(x+1)^3}$$

2.555 problem 569

2.555.1 Maple step by step solution 5203

Internal problem ID [8045]
Internal file name [OUTPUT/6978_Sunday_June_05_2022_05_23_01_PM_52723399/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 569.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-2x)\,y''-x(5-4x)\,y'+(9-4x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(4x^2-5x\right)y'+(9-4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= 4x^2-5x \\ C &= 9-4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 1}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 1 \\ t &= 4\left(2x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 1}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1055: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4x^2} + \frac{3}{4\left(x - \frac{1}{2}\right)^2} - \frac{1}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)} \\
&= -\frac{1}{2x\,(2x-1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{8x-1}{4\left(2x^2-x\right)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \frac{\sqrt{x}}{\sqrt{2x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^2-5x}{-2x^3+x^2} \, dx} \\
&= z_1 e^{\frac{5\ln(x)}{2} - \frac{3\ln(2x-1)}{2}} \\
&= z_1 \left(\frac{x^{\frac{5}{2}}}{(2x-1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^3}{(2x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2-5x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-3\ln(2x-1)}}{(y_1)^2}\,dx \\
&= y_1(2x - \ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^3}{(2x-1)^2}\right) + c_2 \left(\frac{x^3}{(2x-1)^2}(2x - \ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^3}{(2x-1)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(2x-1)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^3}{(2x-1)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(2x-1)^2}
$$

Verified OK.

2.555.1 Maple step by step solution

Let's solve

$(-2x^3 + x^2)\, y'' + (4x^2 - 5x)\, y' + (9 - 4x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-9)y}{x^2(2x-1)} + \frac{(4x-5)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4x-5)y'}{x(2x-1)} + \frac{(4x-9)y}{x^2(2x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{4x-5}{x(2x-1)}, P_3(x) = \frac{4x-9}{x^2(2x-1)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -5$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 9$

- ○ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(2x-1) - x(4x-5)\,y' + (4x-9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- ○ Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 3$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-3)^2 + 2a_{k-1}(k+r-2)(k+r-3) = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+r-2)^2 + 2a_k(k+r-1)(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r-1)}{k+r-2}$$

- Recursion relation for $r=3$

$$a_{k+1} = \frac{2a_k(k+2)}{k+1}$$

- Solution for $r=3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+2)}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5-4*x)*diff(y(x),x)+(9-4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^3}{(-1+2x)^2} + \frac{c_2 x^3 (2x - \ln(x))}{(-1+2x)^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 29

```
DSolve[x^2*(1-2*x)*y''[x]-x*(5-4*x)*y'[x]+(9-4*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{x^3(-2c_2 x + c_2 \log(x) + c_1)}{(1-2x)^2}$$

2.556 problem 570

2.556.1 Maple step by step solution 5212

Internal problem ID [8046]
Internal file name [OUTPUT/6979_Sunday_June_05_2022_05_23_04_PM_99294650/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 570.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2(x+2)\,y''+x^2y'+(1-x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+x^2y'+(1-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+4x^2\\ B &= x^2\\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 8x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1057: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2+2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{8x} - \frac{3}{8\,(x+2)} - \frac{3}{16\,(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+8x-16}{16\,(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 8x - 16}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= \frac{5}{4} - \left(\frac{5}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{3}{4\left(x+2\right)} + \frac{1}{2x} + (0) \\ &= \frac{3}{4\left(x+2\right)} + \frac{1}{2x} \\ &= \frac{5x+4}{4x\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4(x+2)} + \frac{1}{2x}\right)(0) + \left(\left(-\frac{3}{4(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{3}{4(x+2)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2 + 8x - 16}{16(x^2 + 2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{3}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \sqrt{x}\,(x+2)^{\frac{3}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{2x^3 + 4x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{(x+2)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\sqrt{x+2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{2x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x+2)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) \sqrt{x+2} - 2}{2\sqrt{x+2}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\,\sqrt{x+2}\right) + c_2 \left(\sqrt{x}\,\sqrt{x+2} \left(-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right) \sqrt{x+2} - 2}{2\sqrt{x+2}} \right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\sqrt{x+2} - \frac{c_2\sqrt{x}\left(\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2} - 2\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\sqrt{x+2} - \frac{c_2\sqrt{x}\left(\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\sqrt{x+2} - 2\right)}{2}$$

Verified OK.

2.556.1 Maple step by step solution

Let's solve

$(2x^3 + 4x^2)\, y'' + x^2 y' + (1 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = \frac{(x-1)y}{2x^2(x+2)} - \frac{y'}{2(x+2)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{y'}{2(x+2)} - \frac{(x-1)y}{2x^2(x+2)} = 0$$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{1}{2(x+2)}, P_3(x) = -\frac{x-1}{2x^2(x+2)}\right]$$

- ○ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

$$\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2} = \frac{1}{2}$$

- ○ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

$$\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- ○ $x=-2$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = -2$$

- Multiply by denominators

$$2x^2(x+2)\,y'' + x^2y' + (1-x)\,y = 0$$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

$$(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - 4u + 4)\left(\frac{d}{du}y(u)\right) + (3-u)\,y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- ○ Shift index using $k->k-m$

$$u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(-1+2r)\, u^{-1+r} + (4a_1(1+r)\,(1+2r) - a_0(8r^2-4r-3))\, u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+1+r)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(1+2r) - a_0(8r^2-4r-3) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r + 4a_k - 5a_{k-1} + 12a_{k+1})\, k + 2(-4a_k +$$

- Shift index using $k->k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r + 4a_{k+1} - 5a_k + 12a_{k+2})\,(k+1) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k - a_{k+1}}{4(2k^2+4kr+2r^2+7k+7r+6)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} - ka_k - 12ka_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 + 3a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20k a_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20k a_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + k a_k - 20k a_{k+1} - a_k - 9a_{k+1}}{4(2k^2+9k+10)}, 12a_1 + 3a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - k a_k - 12k a_{k+1} - a_k - a_{k+1}}{4(2k^2+7k+6)}, 4a_1 +\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)+x^2*diff(y(x),x)+(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x^2+2x} - \frac{c_2\sqrt{2}\left(\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x - \sqrt{2}\,\sqrt{x+2} + 2\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\sqrt{x\,(x+2)}}{2\,(x+2)}$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 65

```
DSolve[2*x^2*(2+x)*y''[x]+x^2*y'[x]+(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}\left(2\left(c_1\sqrt{x+2}+c_2\right) - \sqrt{2}c_2\sqrt{x+2}\operatorname{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right)\right)}{2\sqrt[4]{2}}$$

2.557 problem 571

2.557.1 Maple step by step solution 5223

Internal problem ID [8047]
Internal file name [OUTPUT/6980_Sunday_June_05_2022_05_23_08_PM_7033910/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 571.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^{2}(1+x)\,y''-x(-x+6)\,y'+(8-x)\,y=0}$$

Writing the ode as

$$\left(2x^{3}+2x^{2}\right)y''+\left(x^{2}-6x\right)y'+(8-x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^{3}+2x^{2} \\ B &= x^{2}-6x \\ C &= 8-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 - 20x - 4}{16(x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 - 20x - 4$$
$$t = 16(x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 - 20x - 4}{16(x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1059: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{21}{16(1+x)^2} - \frac{1}{4x^2} + \frac{3}{4(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 - 20x - 4}{16(x^2+x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 - 20x - 4}{16\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c1}^- + \alpha_{c2}^+\right) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{4\left(1 + x\right)} + \frac{1}{2x} + (-)\left(0\right) \\ &= -\frac{3}{4\left(1 + x\right)} + \frac{1}{2x} \\ &= -\frac{x - 2}{4x\left(1 + x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{3}{4(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{5x^2 - 20x - 4}{16(x^2+x)^2}\right)\right) =$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{3}{4(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(1+x)^{\frac{3}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-6x}{2x^3+2x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} - \frac{7\ln(1+x)}{4}} \\ &= z_1 \left(\frac{x^{\frac{3}{2}}}{(1+x)^{\frac{7}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2}{(1+x)^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-6x}{2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3\ln(x)-\frac{7\ln(1+x)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\ln\left(\sqrt{1+x}-1\right)-\ln\left(\sqrt{1+x}+1\right)+\frac{(2x+8)\sqrt{1+x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{x^2}{(1+x)^{\frac{5}{2}}}\right) \\ &\quad + c_2\left(\frac{x^2}{(1+x)^{\frac{5}{2}}}\left(\ln\left(\sqrt{1+x}-1\right)-\ln\left(\sqrt{1+x}+1\right)+\frac{(2x+8)\sqrt{1+x}}{3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2}{(1+x)^{\frac{5}{2}}} + \frac{c_2 x^2\left(2\sqrt{1+x}\,x+8\sqrt{1+x}+3\ln\left(\sqrt{1+x}-1\right)-3\ln\left(\sqrt{1+x}+1\right)\right)}{3\,(1+x)^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{(1+x)^{\frac{5}{2}}} + \frac{c_2 x^2\left(2\sqrt{1+x}\,x+8\sqrt{1+x}+3\ln\left(\sqrt{1+x}-1\right)-3\ln\left(\sqrt{1+x}+1\right)\right)}{3\,(1+x)^{\frac{5}{2}}}$$

Verified OK.

2.557.1 Maple step by step solution

Let's solve

$(2x^3+2x^2)\,y''+(x^2-6x)\,y'+(8-x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(x-8)y}{2x^2(1+x)}-\frac{(x-6)y'}{2x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x-6)y'}{2x(1+x)}-\frac{(x-8)y}{2x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x-6}{2x(1+x)},P_3(x)=-\frac{x-8}{2x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{7}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $2x^2(1+x)\,y''+x(x-6)\,y'+(8-x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-8u+7)\left(\frac{d}{du}y(u)\right)+(9-u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(5+2r)\, u^{-1+r} + \left(a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+7+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(1+r)\,(7+2r) - a_0(4r^2+4r-9) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-4a_k + 2a_{k-1} + 2a_{k+1})\, k^2 + \left((-8a_k + 4a_{k-1} + 4a_{k+1})\, r - 4a_k - 5a_{k-1} + 9a_{k+1}\right) k + (-4a_k + 2a$$

- Shift index using $k- >k + 1$

  $$(-4a_{k+1} + 2a_k + 2a_{k+2})\,(k+1)^2 + \left((-8a_{k+1} + 4a_k + 4a_{k+2})\, r - 4a_{k+1} - 5a_k + 9a_{k+2}\right)(k+1) +$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2 a_k - 4r^2 a_{k+1} - ka_k - 12ka_{k+1} - ra_k - 12ra_{k+1} - a_k + a_{k+1}}{2k^2 + 4kr + 2r^2 + 13k + 13r + 18}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 + 9a_0 = 0\right]$$

- Recursion relation for $r = -\frac{5}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}$$

- Solution for $r = -\frac{5}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k-\frac{5}{2}}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 11ka_k + 8ka_{k+1} + 14a_k + 6a_{k+1}}{2k^2 + 3k - 2}, -3a_1 - 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k-\frac{5}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - ka_k - 12ka_{k+1} - a_k + a_{k+1}}{2k^2 + 13k + 18}, 7a_1 +\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve(2*x^2*(1+x)*diff(y(x),x$2)-x*(6-x)*diff(y(x),x)+(8-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{(x+1)^{\frac{5}{2}}} + \frac{c_2 x^2 \left(\frac{2\sqrt{x+1}\,x}{3} + \frac{8\sqrt{x+1}}{3} + \ln\left(\sqrt{x+1}-1\right) - \ln\left(\sqrt{x+1}+1\right)\right)}{(x+1)^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 50

```
DSolve[2*x^2*(1+x)*y''[x]-x*(6-x)*y'[x]+(8-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{x^2\left(-6c_2 \operatorname{arctanh}\left(\sqrt{x+1}\right) + 2c_2\sqrt{x+1}(x+4) + 3c_1\right)}{3(x+1)^{5/2}}$$

2.558 problem 572

2.558.1 Maple step by step solution 5233

Internal problem ID [8048]
Internal file name [`OUTPUT/6981_Sunday_June_05_2022_05_23_11_PM_2206266/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 572.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''+x(5+9x)\,y'+(4+3x)\,y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(9x^2+5x\right)y'+(4+3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^3+x^2 \\ B &= 9x^2+5x \\ C &= 4+3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^2 + 6x - 1$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1061: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{2x} + \frac{5}{16\left(x + \frac{1}{2}\right)^2} - \frac{5}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 + 6x - 1}{4\left(2x^2 + x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right) \\
&= \frac{7}{4} - \left(\frac{7}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} + (0) \\
&= \frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)} \\
&= \frac{1+7x}{4x^2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2+6x-1}{4\left(2x^2+x\right)^2}\right)\right)
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \sqrt{x}\,(2x+1)^{\frac{5}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^2+5x}{2x^3+x^2}\, dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2} + \frac{\ln(2x+1)}{4}} \\
&= z_1 \left(\frac{(2x+1)^{\frac{1}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(2x+1)^{\frac{3}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^2+5x}{2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{\ln(2x+1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{2x+1}\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}-1\right) - 2\sqrt{2x+1}\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}+1\right) + 4x + \frac{8}{3}}{(2x+1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(2x+1)^{\frac{3}{2}}}{x^2}\right) \\
&\quad + c_2 \left(\frac{(2x+1)^{\frac{3}{2}}}{x^2}\left(\frac{2\sqrt{2x+1}\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}-1\right) - 2\sqrt{2x+1}\left(x+\frac{1}{2}\right)\ln\left(\sqrt{2x+1}+1\right) + 4x + \frac{8}{3}}{(2x+1)^{\frac{3}{2}}}\right.\right.
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1 (2x+1)^{\frac{3}{2}}}{x^2} \\
&\quad + \frac{c_2\left((6x+3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}-1\right) + (-6x-3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}+1\right) + 12x + 8\right)}{3x^2}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(2x+1)^{\frac{3}{2}}}{x^2} + \frac{c_2\left((6x+3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}-1\right)+(-6x-3)\sqrt{2x+1}\,\ln\left(\sqrt{2x+1}+1\right)+12x+8\right)}{3x^2}$$

Verified OK.

2.558.1 Maple step by step solution

Let's solve

$(2x^3+x^2)\,y''+(9x^2+5x)\,y'+(4+3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4+3x)y}{x^2(2x+1)}-\frac{(5+9x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(5+9x)y'}{x(2x+1)}+\frac{(4+3x)y}{x^2(2x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{5+9x}{x(2x+1)}, P_3(x)=\frac{4+3x}{x^2(2x+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=4$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\,y''+x(5+9x)\,y'+(4+3x)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+2)^2 + a_{k-1}(k+r+2)\,(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r+2)\left(a_k (k+r+2) + a_{k-1}(2k-1+2r)\right) = 0$$

- Shift index using $k- >k + 1$

$$(k+r+3)\left(a_{k+1}(k+r+3) + a_k (2k+2r+1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(2k+2r+1)}{k+r+3}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{a_k(2k-3)}{k+1}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{a_k(2k-3)}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 130

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(5+9*x)*diff(y(x),x)+(4+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x+1)^{\frac{3}{2}}}{x^2} + \frac{c_2\left(-12\ln\left(\sqrt{2x+1}+1\right)x^2 - 12\ln\left(\sqrt{2x+1}+1\right)x + 12\ln\left(\sqrt{2x+1}-1\right)x^2 + 12\ln\left(\sqrt{2x+1}-1\right)\right.}{3x^2\sqrt{2x+1}}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 56

```
DSolve[x^2*(1+2*x)*y''[x]+x*(5+9*x)*y'[x]+(4+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{2c_2\left(-3(2x+1)^{3/2}\operatorname{arctanh}\left(\sqrt{2x+1}\right)+6x+4\right)+3c_1(2x+1)^{3/2}}{3x^2}$$

2.559 problem 573

2.559.1 Maple step by step solution 5243

Internal problem ID [8049]
Internal file name [`OUTPUT/6982_Sunday_June_05_2022_05_23_14_PM_6646513/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 573.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-2x)\,y''-x(4x+5)\,y'+(9+4x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+\left(-4x^2-5x\right)y'+(9+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -4x^2-5x \\ C &= 9+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 32x^2 + 56x - 1$$
$$t = 4\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1063: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{13}{x} - \frac{1}{4x^2} + \frac{35}{4\left(x - \frac{1}{2}\right)^2} - \frac{13}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 56x - 1}{4\left(2x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -1-(-2) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} + (-)(0) \\
&= \frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)} \\
&= \frac{-8x-1}{4x^2-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)(1) + \left(\left(-\frac{1}{2x^2} + \frac{5}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+56x-1}{4\left(2x^2-x\right)^2}\right)\right)$$

$$\frac{-1+8a_0}{x\left(2x-1\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{1}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{1}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{8}\right) e^{\int \left(\frac{1}{2x} - \frac{5}{2\left(x-\frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{1}{8}\right) e^{\frac{\ln(x)}{2} - \frac{5\ln(2x-1)}{2}} \\
&= \frac{\left(x+\frac{1}{8}\right)\sqrt{x}}{\left(2x-1\right)^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2-5x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{7\ln(2x-1)}{2}} \\
&= z_1\left(\frac{x^{\frac{5}{2}}}{(2x-1)^{\frac{7}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x+\frac{1}{8}\right)x^3}{\left(2x-1\right)^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2-5x}{-2x^3+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-7\ln(2x-1)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{32x^3}{3}-44x^2+\frac{203x}{2}-64\ln\left(x\right)-\frac{3125}{128x+16}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x+\frac{1}{8}\right)x^3}{\left(2x-1\right)^6}\right) + c_2\left(\frac{\left(x+\frac{1}{8}\right)x^3}{\left(2x-1\right)^6}\left(\frac{32x^3}{3}-44x^2+\frac{203x}{2}-64\ln\left(x\right)-\frac{3125}{128x+16}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x+\frac{1}{8}\right)x^3}{\left(2x-1\right)^6} + \frac{32c_2x^3\left(\left(-6x-\frac{3}{4}\right)\ln\left(x\right)+x^4-4x^3+9x^2+\frac{609x}{512}-\frac{9375}{4096}\right)}{3\left(2x-1\right)^6} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1\left(x+\frac{1}{8}\right)x^3}{\left(2x-1\right)^6}+\frac{32c_2x^3\left(\left(-6x-\frac{3}{4}\right)\ln\left(x\right)+x^4-4x^3+9x^2+\frac{609x}{512}-\frac{9375}{4096}\right)}{3\left(2x-1\right)^6}$$

Verified OK.

2.559.1 Maple step by step solution

Let's solve

$(-2x^3+x^2)\,y''+(-4x^2-5x)\,y'+(9+4x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(9+4x)y}{x^2(2x-1)}-\frac{(4x+5)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x+5)y'}{x(2x-1)}-\frac{(9+4x)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4x+5}{x(2x-1)},P_3(x)=-\frac{9+4x}{x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-5$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=9$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(4x+5)\,y'+(-4x-9)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- > k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- > k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- > k + 2 - m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0(-3+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(-3+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 3$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_k(k+r-3)^2 + 2a_{k-1}(k+1+r)\,(k-2+r) = 0$$

- Shift index using $k- > k + 1$

  $$-a_{k+1}(k-2+r)^2 + 2a_k(k+r+2)\,(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(k+r+2)(k+r-1)}{(k-2+r)^2}$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{2a_k(k+5)(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)-x*(5+4*x)*diff(y(x),x)+(9+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^3(8x+1)}{(-1+2x)^6} + \frac{c_2 x^3\left(\frac{4x^4}{3} - \frac{16x^3}{3} - 8x\ln(x) + 12x^2 - \ln(x) + \frac{203x}{128} - \frac{3125}{1024}\right)}{(-1+2x)^6}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 63

```
DSolve[x^2*(1-2*x)*y''[x]-x*(5+4*x)*y'[x]+(9+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{x^3(c_2(4096x^4 - 16384x^3 + 36864x^2 + 4872x - 9375) - 48c_1(8x + 1) - 3072c_2(8x + 1)\log(x))}{384(1 - 2x)^6}$$

2.560 problem 574

2.560.1 Maple step by step solution 5253

Internal problem ID [8050]
Internal file name [OUTPUT/6983_Sunday_June_05_2022_05_23_17_PM_17144998/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 574.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)\,y''+x(7+x)\,y'+(9-x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+(9-x)\,y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= x^2+7x \\ C &= 9-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 82x - 1$$
$$t = 4\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 82x - 1}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1065: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x} + \frac{20}{(x-1)^2} - \frac{1}{4x^2} - \frac{20}{x-1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 82x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2+82x-1}{4\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{1}{2} - \left(-\frac{7}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{4}{x-1} + (-)(0) \\ &= \frac{1}{2x} - \frac{4}{x-1} \\ &= -\frac{1+7x}{2x(x-1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{4}{x-1}\right)(4x^3 + 3x^2a_3 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} + \frac{4}{(x-1)^2}\right) + \left(\frac{1}{2x} - \frac{}{x}\right.\right.$$

$$\frac{(a_3 - 16)\,x^3 + (4a_2 - 9a_3)\,x^2 +}{x\,(x -}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 16, a_2 = 36, a_3 = 16\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 16x^3 + 36x^2 + 16x + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{4}{x-1}\right)dx} \\
&= (x^4 + 16x^3 + 36x^2 + 16x + 1)\, \mathrm{e}^{\frac{\ln(x)}{2} - 4\ln(x-1)} \\
&= \frac{(x^4 + 16x^3 + 36x^2 + 16x + 1)\sqrt{x}}{(x-1)^4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+7x}{-x^3+x^2}\, dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2} + 4\ln(x-1)} \\
&= z_1 \left(\frac{(x-1)^4}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+7x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-7\ln(x)+8\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{120x^3 + 450x^2 + 280x + 25}{3x^4 + 48x^3 + 108x^2 + 48x + 3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3}\right) \\
&\quad + c_2 \left(\frac{x^4 + 16x^3 + 36x^2 + 16x + 1}{x^3} \left(\ln(x) + \frac{120x^3 + 450x^2 + 280x + 25}{3x^4 + 48x^3 + 108x^2 + 48x + 3}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} \\
&\quad + \frac{c_2(25 + 3\ln(x)(x^4 + 16x^3 + 36x^2 + 16x + 1) + 120x^3 + 450x^2 + 280x)}{3x^3}
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
y &= \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} \\
&\quad + \frac{c_2(25 + 3\ln(x)(x^4 + 16x^3 + 36x^2 + 16x + 1) + 120x^3 + 450x^2 + 280x)}{3x^3}
\end{aligned}$$

Verified OK.

2.560.1 Maple step by step solution

Let's solve

$\left(-x^3+x^2\right)y''+\left(x^2+7x\right)y'+\left(9-x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-9+x)y}{x^2(x-1)}+\frac{(7+x)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(7+x)y'}{x(x-1)}+\frac{(-9+x)y}{x^2(x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{7+x}{x(x-1)},P_3(x)=\frac{-9+x}{x^2(x-1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=7$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=9$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(x-1)-x(7+x)y'+(-9+x)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(3+r)^{2}\,x^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+3)^{2}+a_{k-1}(k-2+r)^{2}\right)x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-(3+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-3$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r+3)^{2}+a_{k-1}(k-2+r)^{2}=0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+4+r)^{2}+a_{k}(k+r-1)^{2}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+r-1)^{2}}{(k+4+r)^{2}}$

- Recursion relation for $r=-3$; series terminates at $k=4$

  $a_{k+1}=\frac{a_{k}(k-4)^{2}}{(k+1)^{2}}$

- Apply recursion relation for $k=0$

$a_1 = 16a_0$

- Apply recursion relation for $k = 1$

$a_2 = \frac{9a_1}{4}$

- Express in terms of a_0

$a_2 = 36a_0$

- Apply recursion relation for $k = 2$

$a_3 = \frac{4a_2}{9}$

- Express in terms of a_0

$a_3 = 16a_0$

- Apply recursion relation for $k = 3$

$a_4 = \frac{a_3}{16}$

- Express in terms of a_0

$a_4 = a_0$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

$y = a_0 \cdot (x^4 + 16x^3 + 36x^2 + 16x + 1)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 75

```
dsolve(x^2*(1-x)*diff(y(x),x$2)+x*(7+x)*diff(y(x),x)+(9-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4 + 16x^3 + 36x^2 + 16x + 1)}{x^3} + \frac{c_2\left(x^4 \ln(x) + 16x^3 \ln(x) + 36x^2 \ln(x) + 40x^3 + 16x \ln(x) + 150x^2 + \ln(x) + \frac{280x}{3} + \frac{25}{3}\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 78

```
DSolve[x^2*(1-x)*y''[x]+x*(7+x)*y'[x]+(9-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5c_2(24x^3 + 90x^2 + 56x + 5) + 3c_1(x^4 + 16x^3 + 36x^2 + 16x + 1) + 3c_2(x^4 + 16x^3 + 36x^2 + 16x + 1) \log}{3x^3}$$

2.561 problem 575

2.561.1 Maple step by step solution 5263

Internal problem ID [8051]
Internal file name [OUTPUT/6984_Sunday_June_05_2022_05_23_20_PM_39239997/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 575.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(1-x^2)\,y' + (x^2+1)\,y = 0}$$

Writing the ode as

$$\begin{align*} x^2y'' + (x^3-x)\,y' + (x^2+1)\,y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= x^2 \\ B &= x^3 - x \tag{3}\\ C &= x^2+1 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1067: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 4x^2 - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{4} - 1 - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) = -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\
&= \frac{1}{2x} - \frac{x}{2} \\
&= \frac{1}{2x} - \frac{x}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^3 - x}{x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{x^2}{4}} \\ &= z_1 \left(\sqrt{x}\, e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\,\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\,\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}
$$

Verified OK.

2.561.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3 - x)\,y' + (x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-1)\,y' + (x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{k+r+1}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{k+2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 x\,\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{expIntegral}_1\left(-\frac{x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{2}}x\left(c_1\,\mathrm{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.562 problem 576

2.562.1 Maple step by step solution 5273

Internal problem ID [8052]
Internal file name [OUTPUT/6985_Sunday_June_05_2022_05_23_23_PM_35128224/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 576.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-3x(1-x^2)\,y'+4y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(3x^3-3x)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 3x^3-3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 - 10x^2 - 1}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 - 10x^2 - 1 \\ t &= 4(x^3 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 - 10x^2 - 1}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1069: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{i}{4x-4i} - \frac{i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 - 10x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} + (-)\,(0) \\ &= \frac{1}{2x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)} \\ &= \frac{1}{2x} - \frac{x}{x^2+1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2\,(x-i)} - \frac{1}{2\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{2\,(x-i)^2} + \frac{1}{2\,(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{2\,(x-i)} - \frac{}{2(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)dx} \\ &= \frac{\sqrt{x}}{\sqrt{x^2+1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x^3-3x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{x^{\frac{3}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2}{\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{x^2}{2}+\ln\left(x\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{\left(x^2+1\right)^2}\right) + c_2 \left(\frac{x^2}{\left(x^2+1\right)^2}\left(\frac{x^2}{2}+\ln\left(x\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{\left(x^2+1\right)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2}+\ln\left(x\right)\right)}{\left(x^2+1\right)^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x^2}{\left(x^2+1\right)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2} + \ln\left(x\right)\right)}{\left(x^2+1\right)^2}$$

Verified OK.

2.562.1 Maple step by step solution

Let's solve

$(x^4 + x^2)\, y'' + (3x^3 - 3x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x^2+1)} - \frac{3(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(x^2-1)y'}{x(x^2+1)} + \frac{4y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3(x^2-1)}{x(x^2+1)}, P_3(x) = \frac{4}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\, y'' + 3x(x^2-1)\, y' + 4y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\, x^r + a_1(-1+r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 + a_{k-2}(k+r-2)\,(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term must be 0

$$a_1(-1+r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k(k+r-2) + a_{k-2}(k+r)) = 0$$

- Shift index using $k- >k+2$

$$(k+r)\,(a_{k+2}(k+r) + a_k(k+r+2)) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k+r+2)}{k+r}$$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+4)}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{a_k(k+4)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-3*x*(1-x^2)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{(x^2+1)^2} + \frac{c_2 x^2 \left(\frac{x^2}{2} + \ln(x)\right)}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 36

```
DSolve[x^2*(1+x^2)*y''[x]-3*x*(1-x^2)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{x^2(c_2 x^2 + 2c_2 \log(x) + 2c_1)}{2(x^2+1)^2}$$

2.563 problem 577

2.563.1 Maple step by step solution 5282

Internal problem ID [8053]
Internal file name [`OUTPUT/6986_Sunday_June_05_2022_05_23_27_PM_82915174/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 577.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + 2y'x^3 + (3x^2 + 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + 2y'x^3 + (3x^2 + 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 2x^3 \\ C &= 3x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1071: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{1}{x} - \frac{5}{2x^3} - \frac{10}{x^5} - \frac{105}{2x^7} - \frac{310}{x^9} - \frac{1965}{x^{11}} - \frac{13060}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{4}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^2 - 4}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{x^2}{16} - \frac{1}{2}\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{16} - \frac{1}{2} - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= -\frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{4}} - 1\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\
&= \frac{1}{2x} - \frac{x}{4} \\
&= \frac{1}{2x} - \frac{x}{4}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 - 8x^2 - 4}{16x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^3}{4x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{8}} \\ &= z_1 \left(e^{-\frac{x^2}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, e^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x^3}{4x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{4}}}{(y_1)^2} dx$$
$$= y_1 \left(-\frac{\operatorname{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\left(-\frac{\operatorname{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} - \frac{c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} \operatorname{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} - \frac{c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} \operatorname{expIntegral}_1\left(-\frac{x^2}{4}\right)}{2}$$

Verified OK.

2.563.1 Maple step by step solution

Let's solve

$4x^2y'' + 2y'x^3 + (3x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{xy'}{2} - \frac{(3x^2+1)y}{4x^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{xy'}{2} + \frac{(3x^2+1)y}{4x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{x}{2}, P_3(x) = \frac{3x^2+1}{4x^2}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 0$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{4}$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$4x^2y'' + 2y'x^3 + (3x^2+1)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ○ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^3 \cdot y'$ to series expansion

$$x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+2}$$

- ○ Shift index using $k- >k - 2$

$$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + a_1(1+2r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1) = 0$$

- Shift index using $k- > k+2$

$$a_{k+2}(2k+2r+3)^2 + a_k(2k+2r+3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{2k+2r+3}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{a_k}{2k+4}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k}{2k+4}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(4*x^2*diff(y(x),x$2)+2*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\,\mathrm{expIntegral}_1\left(-\frac{x^2}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]+2*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-\frac{x^2}{4}}\sqrt{x}\left(c_2\,\mathrm{ExpIntegralEi}\left(\frac{x^2}{4}\right) + 2c_1\right)$$

2.564 problem 578

2.564.1 Maple step by step solution 5292

Internal problem ID [8054]
Internal file name [`OUTPUT/6987_Sunday_June_05_2022_05_23_30_PM_82190300/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 578.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-2x^2+1)\,y'+y=0}$$

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(2x^3-x\right)y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=2x^3-x\\C&=1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1073: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{3}{16(x-i)^2} - \frac{3}{16(x+i)^2} - \frac{5i}{16(x-i)} + \frac{5i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-i$	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3}\right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} + (-)(0) \\ &= \frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i} \\ &= \frac{1}{2x} + \frac{x}{2x^2+2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{4x-4i} + \frac{1}{4x+4i}\right) dx} \\ &= \sqrt{x}\,(x^2+1)^{\frac{1}{4}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^3-x}{x^4+x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{3\ln(x^2+1)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(x^2+1)^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{\sqrt{x^2+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{3\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{\sqrt{x^2+1}}\right) + c_2 \left(\frac{x}{\sqrt{x^2+1}}\left(-\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 x}{\sqrt{x^2+1}} - \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 x}{\sqrt{x^2+1}} - \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}}$$

Verified OK.

2.564.1 Maple step by step solution

Let's solve

$\left(x^4+x^2\right) y''+\left(2 x^3-x\right) y'+y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{y}{x^2\left(x^2+1\right)}-\frac{\left(2 x^2-1\right) y'}{x\left(x^2+1\right)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{\left(2 x^2-1\right) y'}{x\left(x^2+1\right)}+\frac{y}{x^2\left(x^2+1\right)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=\frac{2 x^2-1}{x\left(x^2+1\right)}, P_3(x)=\frac{1}{x^2\left(x^2+1\right)}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x=0$

$\left.\left(x \cdot P_2(x)\right)\right|_{x=0}=-1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x=0$

$\left.\left(x^2 \cdot P_3(x)\right)\right|_{x=0}=1$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2\left(x^2+1\right) y''+x\left(2 x^2-1\right) y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m=1..3$

$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r) x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k-2+r)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term must be 0

  $$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\,(a_k(k+r-1) + a_{k-2}(k-2+r)) = 0$$

- Shift index using $k->k+2$

  $$(k+r+1)\,(a_{k+2}(k+r+1) + a_k(k+r)) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)}{k+r+1}$$

- Recursion relation for $r = 1$

  $$a_{k+2} = -\frac{a_k(k+1)}{k+2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+1)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-2*x^2)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{\sqrt{x^2+1}} + \frac{c_2 x \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 33

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1-2*x^2)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x\left(c_1 - c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right)}{\sqrt{x^2+1}}$$

2.565 problem 579

2.565.1 Maple step by step solution 5301

Internal problem ID [8055]
Internal file name [`OUTPUT/6988_Sunday_June_05_2022_05_23_33_PM_47531544/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 579.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x^2+2)\,y''+7y'x^3+(3x^2+1)\,y=0}$$

Writing the ode as

$$\left(2x^4+4x^2\right)y''+7y'x^3+\left(3x^2+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^4+4x^2 \\ B &= 7x^3 \\ C &= 3x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^4 - 16 \\ t &= 16\left(x^3 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1075: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3+2x)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{7}{64\left(x - i\sqrt{2}\right)^2} - \frac{7}{64\left(x + i\sqrt{2}\right)^2} - \frac{9i\sqrt{2}}{128\left(x - i\sqrt{2}\right)} + \frac{9i\sqrt{2}}{128\left(x + i\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_{\infty} = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_{\infty} &= 0 \\
\alpha_{\infty}^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_{\infty}^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^4 - 16}{16\left(x^3 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-i\sqrt{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}} + (0) \\
&= \frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}} \\
&= \frac{1}{2x} + \frac{x}{4x^2+8}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{8\left(x-i\sqrt{2}\right)^2} - \frac{1}{8\left(x+i\sqrt{2}\right)^2}\right) + \left(\frac{1}{2x} + \frac{}{8x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{1}{8x-8i\sqrt{2}} + \frac{1}{8x+8i\sqrt{2}}\right) dx} \\
&= \left(x^2+2\right)^{\frac{1}{8}} \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x^3}{2x^4+4x^2}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(x^2+2\right)}{8}} \\
&= z_1\left(\frac{1}{\left(x^2+2\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^3}{2x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(x^2+2\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{1}{x\left(x^2+2\right)^{\frac{1}{4}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}\right) + c_2\left(\frac{\sqrt{x}}{\left(x^2+2\right)^{\frac{3}{4}}}\left(\int \frac{1}{x\left(x^2+2\right)^{\frac{1}{4}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2 \sqrt{x}\left(\int \frac{1}{x(x^2+2)^{\frac{1}{4}}}dx\right)}{(x^2+2)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2 \sqrt{x}\left(\int \frac{1}{x(x^2+2)^{\frac{1}{4}}}dx\right)}{(x^2+2)^{\frac{3}{4}}}$$

Verified OK.

2.565.1 Maple step by step solution

Let's solve

$(2x^4+4x^2)\, y'' + 7y'x^3 + (3x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x^2+1)y}{2x^2(x^2+2)} - \frac{7y'x}{2(x^2+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7y'x}{2(x^2+2)} + \frac{(3x^2+1)y}{2x^2(x^2+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{7x}{2(x^2+2)}, P_3(x) = \frac{3x^2+1}{2x^2(x^2+2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{4}$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$2x^2(x^2+2)\,y'' + 7y'x^3 + (3x^2+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r+2}$

  - Shift index using $k- >k-2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + a_{k-2}(2k+2r-1)\,(k+r-1)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = \frac{1}{2}$

- Each term must be 0

  $a_1(1+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\frac{a_{k-2}(k+r-1)}{2} + a_k\left(k+r-\frac{1}{2}\right)\right)\left(k+r-\frac{1}{2}\right) = 0$

- Shift index using $k- > k+2$

  $4\left(\frac{a_k(k+r+1)}{2} + a_{k+2}\left(k+\frac{3}{2}+r\right)\right)\left(k+\frac{3}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{2k+2r+3}$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{a_k\left(k+\frac{3}{2}\right)}{2k+4}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(2*x^2*(2+x^2)*diff(y(x),x$2)+7*x^3*diff(y(x),x)+(1+3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{(x^2+2)^{\frac{3}{4}}} + \frac{c_2\sqrt{x}\left(\int \frac{1}{(x^2+2)^{\frac{1}{4}}x}dx\right)}{(x^2+2)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 77

```
DSolve[2*x^2*(2+x^2)*y''[x]+7*x^3*y'[x]+(1+3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sqrt{x}\left(2^{3/4}c_2\arctan\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) - 2^{3/4}c_2\text{arctanh}\left(\frac{\sqrt[4]{x^2+2}}{\sqrt[4]{2}}\right) + 2c_1\right)}{2(x^2+2)^{3/4}}$$

2.566 problem 580

2.566.1 Maple step by step solution 5312

Internal problem ID [8056]
Internal file name [OUTPUT/6989_Sunday_June_05_2022_05_23_36_PM_89137668/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 580.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-4x^2+1)\,y'+(2x^2+1)\,y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(4x^3-x)\,y'+(2x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 4x^3-x \\ C &= 2x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -6x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-6x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1077: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{3i}{16(x-i)} - \frac{3i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-6x^2 - 1}{4 \left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)} + (0) \\
&= \frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)} \\
&= \frac{1}{2x^3+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{1}{4\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{1}{4\,(x-i)^2} + \frac{1}{4\,(x+i)^2}\right) + \left(\frac{1}{2x} - \frac{1}{4\,(x-i)} - \frac{}{4\,(}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{4(x-i)} - \frac{1}{4(x+i)}\right) dx} \\
&= \frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x^3 - x}{x^4 + x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - \frac{5\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{\sqrt{x}}{\left(x^2+1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x}{(x^2+1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3-x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)-\frac{5\ln(x^2+1)}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x}{(x^2+1)^{\frac{3}{2}}}\right) + c_2 \left(\frac{x}{(x^2+1)^{\frac{3}{2}}} \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x \left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}}$$

Verified OK.

2.566.1 Maple step by step solution

Let's solve

$(x^4 + x^2)\, y'' + (4x^3 - x)\, y' + (2x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x^2+1)y}{x^2(x^2+1)} - \frac{(4x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2-1)y'}{x(x^2+1)} + \frac{(2x^2+1)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4x^2-1}{x(x^2+1)}, P_3(x) = \frac{2x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2 + 1)\, y'' + x(4x^2 - 1)\, y' + (2x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m} \cdot y'$ to series expansion for $m = 1..3$

  $x^{m} \cdot y' = \sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m} \cdot y''$ to series expansion for $m = 2..4$

  $x^{m} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- > k+2-m$

  $x^{m} \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r} + a_{1} r^{2} x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k}(k+r-1)^{2} + a_{k-2}(k+r)\,(k+r-1)\right) x^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2} = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_{1} r^{2} = 0$

- Solve for the dependent coefficient(s)

  $a_{1} = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k}(k+r-1) + a_{k-2}(k+r)\right) = 0$

- Shift index using $k- > k+2$

  $(k+r+1)\left(a_{k+2}(k+r+1) + a_{k}(k+r+2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_{k}(k+r+2)}{k+r+1}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k(k+3)}{k+2}$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k+3)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1-4*x^2)*diff(y(x),x)+(1+2*x^2)*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1 x}{(x^2+1)^{\frac{3}{2}}} + \frac{c_2 x\left(\sqrt{x^2+1} - \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}{(x^2+1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 45

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1-4*x^2)*y'[x]+(1+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{x\left(-c_2 \operatorname{arctanh}\left(\sqrt{x^2+1}\right) + c_2\sqrt{x^2+1} + c_1\right)}{(x^2+1)^{3/2}}$$

2.567 problem 581

2.567.1 Maple step by step solution 5321

Internal problem ID [8057]
Internal file name [`OUTPUT/6990_Sunday_June_05_2022_05_23_39_PM_9447000/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 581.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(x^2+4)\,y''+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+16x^2\right)y''+\left(9x^3+24x\right)y'+\left(-9x^2+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+16x^2 \\ B &= 9x^3+24x \\ C &= -9x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 153x^4 + 704x^2 - 256$$
$$t = 64\left(x^3 + 4x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1079: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(x^3 + 4x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{39}{256(x-2i)^2} - \frac{39}{256(x+2i)^2} - \frac{377i}{512(x-2i)} + \frac{377i}{512(x+2i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(x-2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{39}{256}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{16} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{3}{16} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{153}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{17}{8} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{153x^4 + 704x^2 - 256}{64\left(x^3 + 4x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$
$-2i$	2	0	$\frac{13}{16}$	$\frac{3}{16}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{17}{8}$	$-\frac{9}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{17}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{17}{8} - \left(\frac{17}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x - c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{13}{16\,(x - 2i)} + \frac{13}{16\,(x + 2i)} + (0) \\
&= \frac{1}{2x} + \frac{13}{16\,(x - 2i)} + \frac{13}{16\,(x + 2i)} \\
&= \frac{1}{2x} + \frac{13x}{8x^2 + 32}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{13}{16\,(x - 2i)} + \frac{13}{16\,(x + 2i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{13}{16\,(x - 2i)^2} - \frac{13}{16\,(x + 2i)^2}\right) + \left(\frac{1}{2x} + \frac{1}{16\,(x\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(\frac{1}{2x} + \frac{13}{16(x-2i)} + \frac{13}{16(x+2i)}\right) dx} \\
&= \sqrt{x}\, \left(x^2 + 4\right)^{\frac{13}{16}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{9x^3+24x}{4x^4+16x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}-\frac{3\ln\left(x^2+4\right)}{16}} \\
&= z_1\left(\frac{1}{x^{\frac{3}{4}}\left(x^2+4\right)^{\frac{3}{16}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{9x^3+24x}{4x^4+16x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{3\ln\left(x^2+4\right)}{8}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{1}{x\left(x^2+4\right)^{\frac{13}{8}}}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}\right) + c_2\left(\frac{\left(x^2+4\right)^{\frac{5}{8}}}{x^{\frac{1}{4}}}\left(\int \frac{1}{x\left(x^2+4\right)^{\frac{13}{8}}}dx\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2(x^2+4)^{\frac{5}{8}} \left(\int \frac{1}{x(x^2+4)^{\frac{13}{8}}}dx\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2(x^2+4)^{\frac{5}{8}} \left(\int \frac{1}{x(x^2+4)^{\frac{13}{8}}}dx\right)}{x^{\frac{1}{4}}}$$

Verified OK.

2.567.1 Maple step by step solution

Let's solve

$(4x^4+16x^2)\, y'' + (9x^3+24x)\, y' + (-9x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(9x^2-1)y}{4x^2(x^2+4)} - \frac{3(3x^2+8)y'}{4x(x^2+4)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3(3x^2+8)y'}{4x(x^2+4)} - \frac{(9x^2-1)y}{4x^2(x^2+4)} = 0$

□ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = \frac{3(3x^2+8)}{4x(x^2+4)}, P_3(x) = -\frac{9x^2-1}{4x^2(x^2+4)}\right]$

- ∘ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{3}{2}$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{16}$

- ∘ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$4x^2(x^2+4)\,y''+3x(3x^2+8)\,y'+(-9x^2+1)\,y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k-m$

$$x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+4r)^2\,x^r+a_1(5+4r)^2\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(4k+4r+1)^2+a_{k-2}(4k+4r+1)\,(k-3+r)\right)x^{k+}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+4r)^2=0$$

- Values of r that satisfy the indicial equation

$$r=-\frac{1}{4}$$

- Each term must be 0

$a_1(5+4r)^2=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $16\left(\frac{a_{k-2}(k-3+r)}{4}+a_k\left(k+r+\frac{1}{4}\right)\right)\left(k+r+\frac{1}{4}\right)=0$

- Shift index using $k->k+2$

  $16\left(\frac{a_k(k+r-1)}{4}+a_{k+2}\left(k+\frac{9}{4}+r\right)\right)\left(k+\frac{9}{4}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k(k+r-1)}{4k+4r+9}$

- Recursion relation for $r=-\frac{1}{4}$

  $a_{k+2}=-\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8}$

- Solution for $r=-\frac{1}{4}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{4}},a_{k+2}=-\frac{a_k\left(k-\frac{5}{4}\right)}{4k+8},a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(4*x^2*(4+x^2)*diff(y(x),x$2)+3*x*(8+3*x^2)*diff(y(x),x)+(1-9*x^2)*y(x)=0,y(x), singso
```

$$y(x) = \frac{c_1(x^2+4)^{\frac{5}{8}}}{x^{\frac{1}{4}}} + \frac{c_2(x^2+4)^{\frac{5}{8}}\left(\int \frac{1}{(x^2+4)^{\frac{13}{8}}x}dx\right)}{x^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 198

```
DSolve[4*x^2*(4+x^2)*y''[x]+3*x*(8+3*x^2)*y'[x]+(1-9*x^2)*y[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{c_2\left(5\ 2^{3/4}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt[8]{x^2+4}}{\sqrt[4]{2}}\right)+5\sqrt[4]{2}(x^2+4)^{5/8}\arctan\left(\frac{\sqrt{2}-\sqrt[4]{x^2+4}}{2^{3/4}\sqrt[8]{x^2+4}}\right)-5\ 2^{3/4}(x^2+4)^{5/8}\ a\right.}{80\sqrt[4]{x}}$$

2.568 problem 582

2.568.1 Maple step by step solution 5332

Internal problem ID [8058]
Internal file name [OUTPUT/6991_Sunday_June_05_2022_05_23_42_PM_59599598/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 582.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2(x^2+3)\,y''+x(11x^2+3)\,y'+(5x^2+1)\,y=0}$$

Writing the ode as

$$(3x^4+9x^2)\,y''+(11x^3+3x)\,y'+(5x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^4+9x^2 \\ B &= 11x^3+3x \\ C &= 5x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^4 + 18x^2 - 81$$
$$t = 36\left(x^3 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1081: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^3+3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i\sqrt{3}$ of order 2. There is a pole at $x = -i\sqrt{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} - \frac{5}{36\left(x-i\sqrt{3}\right)^2} - \frac{5}{36\left(x+i\sqrt{3}\right)^2} - \frac{7i\sqrt{3}}{108\left(x-i\sqrt{3}\right)} + \frac{7i\sqrt{3}}{108\left(x+i\sqrt{3}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x-i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

For the pole at $x = -i\sqrt{3}$ let b be the coefficient of $\frac{1}{\left(x+i\sqrt{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-5x^4 + 18x^2 - 81}{36\left(x^3 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$
$-i\sqrt{3}$	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{6}$	$\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{5}{6} - \left(\frac{5}{6}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{1}{6x - 6i\sqrt{3}} + \frac{1}{6x + 6i\sqrt{3}} + (0) \\ &= \frac{1}{2x} + \frac{1}{6x - 6i\sqrt{3}} + \frac{1}{6x + 6i\sqrt{3}} \\ &= \frac{1}{2x} + \frac{x}{3x^2 + 9}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{1}{6x - 6i\sqrt{3}} + \frac{1}{6x + 6i\sqrt{3}}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6\,(x - i\sqrt{3})^2} - \frac{1}{6\,(x + i\sqrt{3})^2}\right) + \left(\frac{1}{2x} + \frac{}{6x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{6x - 6i\sqrt{3}} + \frac{1}{6x + 6i\sqrt{3}}\right) dx} \\ &= (x^2 + 3)^{\frac{1}{6}} \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+3x}{3x^4+9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{6}-\frac{5\ln\left(x^2+3\right)}{6}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{6}}\left(x^2+3\right)^{\frac{5}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+3x}{3x^4+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}-\frac{5\ln\left(x^2+3\right)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{1}{x\left(x^2+3\right)^{\frac{1}{3}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{\left(x^2+3\right)^{\frac{2}{3}}}\left(\int \frac{1}{x\left(x^2+3\right)^{\frac{1}{3}}}dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{1}{x(x^2+3)^{\frac{1}{3}}} dx\right)}{(x^2+3)^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{1}{x(x^2+3)^{\frac{1}{3}}} dx\right)}{(x^2+3)^{\frac{2}{3}}}$$

Verified OK.

2.568.1 Maple step by step solution

Let's solve

$(3x^4+9x^2)\, y'' + (11x^3+3x)\, y' + (5x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x^2+1)y}{3x^2(x^2+3)} - \frac{(11x^2+3)y'}{3x(x^2+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(11x^2+3)y'}{3x(x^2+3)} + \frac{(5x^2+1)y}{3x^2(x^2+3)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{11x^2+3}{3x(x^2+3)}, P_3(x) = \frac{5x^2+1}{3x^2(x^2+3)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{9}$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$3x^2(x^2+3)\, y'' + x(11x^2+3)\, y' + (5x^2+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+3r)^2\, x^r + a_1(2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-2}(3k+3r-1)\,(k+r-1)\right) x\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+3r)^2 = 0$

- Values of r that satisfy the indicial equation

$r = \frac{1}{3}$

- Each term must be 0

  $a_1(2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(\frac{a_{k-2}(k+r-1)}{3}+a_k\left(k+r-\frac{1}{3}\right)\right) = 0$

- Shift index using $k- >k+2$

  $9\left(k+\frac{5}{3}+r\right)\left(\frac{a_k(k+r+1)}{3}+a_{k+2}\left(k+\frac{5}{3}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{3k+3r+5}$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{a_k\left(k+\frac{4}{3}\right)}{3k+6}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
         -> solution has integrals; searching for one without integrals...
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
         <- hypergeometric solution without integrals succesful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(3*x^2*(3+x^2)*diff(y(x),x$2)+x*(3+11*x^2)*diff(y(x),x)+(1+5*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{(x^2+3)^{\frac{2}{3}}} + \frac{c_2 x^{\frac{1}{3}} \left(\int \frac{1}{(x^2+3)^{\frac{1}{3}} x} dx\right)}{(x^2+3)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 94

```
DSolve[3*x^2*(3+x^2)*y'[x]+x*(3+11*x^2)*y'[x]+(1+5*x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \rightarrow \frac{c_1 \exp\left(\frac{1}{3}\text{RootSum}\left[3\#1^3+11\#1^2+9\#1+3\&, \frac{3\#1^2 \log(x-\#1)-4\#1 \log(x-\#1)+9\log(x-\#1)}{9\#1^2+22\#1+9}\&\right]\right)}{\sqrt[3]{x}}$$

$$y(x) \rightarrow 0$$

2.569 problem 583

2.569.1 Maple step by step solution 5343

Internal problem ID [8059]
Internal file name [OUTPUT/6992_Sunday_June_05_2022_05_23_46_PM_40613799/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 583.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y'' - 3x(-2x^2+7)\,y' + (2x^2+25)\,y = 0}$$

Writing the ode as

$$9x^2y'' + (6x^3 - 21x)\,y' + (2x^2+25)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 6x^3 - 21x \\ C &= 2x^2 + 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 24x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1083: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{3} - \frac{1}{x} - \frac{15}{8x^3} - \frac{45}{8x^5} - \frac{2835}{128x^7} - \frac{12555}{128x^9} - \frac{477495}{1024x^{11}} - \frac{2380185}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{3}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{3}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{9}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4 - 24x^2 - 9}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{x^2}{9} - \frac{2}{3}\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{9} - \frac{2}{3} - \frac{1}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is $-\frac{2}{3}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{2}{3}\right) - (0) \\
&= -\frac{2}{3}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{3} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{2}{3}}{\frac{1}{3}} - 1\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 24x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{3}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{x}{3}\right) \\
&= \frac{1}{2x} - \frac{x}{3} \\
&= \frac{1}{2x} - \frac{x}{3}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{3}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{3}\right) + \left(\frac{1}{2x} - \frac{x}{3}\right)^2 - \left(\frac{4x^4 - 24x^2 - 9}{36x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{x}{3}\right) dx} \\ &= \sqrt{x}\, e^{-\frac{x^2}{6}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6x^3 - 21x}{9x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{6} + \frac{7 \ln(x)}{6}} \\ &= z_1 \left(x^{\frac{7}{6}} e^{-\frac{x^2}{6}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{5}{3}} e^{-\frac{x^2}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{6x^3-21x}{9x^2}dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\frac{x^2}{3}+\frac{7\ln(x)}{3}}}{(y_1)^2} dx$$
$$= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}}\right) + c_2 \left(x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} - \frac{c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} - \frac{c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \text{expIntegral}_1\left(-\frac{x^2}{3}\right)}{2}$$

Verified OK.

2.569.1 Maple step by step solution

Let's solve

$9x^2y'' + (6x^3 - 21x)\, y' + (2x^2 + 25)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(2x^2+25)y}{9x^2} - \frac{(2x^2-7)y'}{3x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(2x^2-7)y'}{3x} + \frac{(2x^2+25)y}{9x^2} = 0$$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

$$\left[P_2(x) = \frac{2x^2-7}{3x}, P_3(x) = \frac{2x^2+25}{9x^2}\right]$$

  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{7}{3}$$

  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{25}{9}$$

  - ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$9x^2y'' + 3x(2x^2-7)\, y' + (2x^2+25)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - ◦ Shift index using $k- >k + 1 - m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-5+3r)^2\, x^r + a_1(-2+3r)^2\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-5+3r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = \frac{5}{3}$

- Each term must be 0

  $a_1(-2+3r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(3k+3r-5)^2 + 2a_{k-2}(3k+3r-5) = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(3k+3r+1)^2 + 2a_k(3k+3r+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{3k+3r+1}$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{2a_k}{3k+6}$

- Solution for $r = \frac{5}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{2a_k}{3k+6}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(9*x^2*diff(y(x),x$2)-3*x*(7-2*x^2)*diff(y(x),x)+(25+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} + c_2 x^{\frac{5}{3}} \mathrm{e}^{-\frac{x^2}{3}} \operatorname{expIntegral}_1\left(-\frac{x^2}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 39

```
DSolve[9*x^2*y''[x]-3*x*(7-2*x^2)*y'[x]+(25+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{3}} x^{5/3} \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{x^2}{3}\right) + 2c_1\right)$$

2.570 problem 584

2.570.1 Maple step by step solution 5353

Internal problem ID [8060]
Internal file name [`OUTPUT/6993_Sunday_June_05_2022_05_23_49_PM_22208101/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 584.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(1-x^2)\,y' + (x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^3-x)\,y' + (x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 4x^2 - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 4x^2 - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1085: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{1}{x} - \frac{5}{4x^3} - \frac{5}{2x^5} - \frac{105}{16x^7} - \frac{155}{8x^9} - \frac{1965}{32x^{11}} - \frac{3265}{16x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 4x^2 - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 1\right) + \left(-\frac{1}{4x^2}\right) \\ &= \frac{x^2}{4} - 1 - \frac{1}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 1\right) &&= -\frac{3}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 1\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 4x^2 - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{1}{2x} - \frac{x}{2} \\ &= \frac{1}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{1}{2}\right) + \left(\frac{1}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 4x^2 - 1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{2}\right) dx} \\ &= \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3 - x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{x^2}{4}} \\ &= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-x}{x^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \, \mathrm{e}^{-\frac{x^2}{2}} \right) + c_2 \left(x \, \mathrm{e}^{-\frac{x^2}{2}} \left(-\frac{\text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \, \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x \, \mathrm{e}^{-\frac{x^2}{2}} \, \text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x \, \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x \, \mathrm{e}^{-\frac{x^2}{2}} \, \text{expIntegral}_1\left(-\frac{x^2}{2}\right)}{2}
$$

Verified OK.

2.570.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 - x) y' + (x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2+1)y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x^2-1)\,y' + (x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 + a_{k-2}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-1) + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(k+r+1)\,(a_{k+2}(k+r+1) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{k+r+1}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{k+2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2 x\,\mathrm{e}^{-\frac{x^2}{2}} \operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right)$$

✓ Solution by Mathematica
Time used: 0.013 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-x*(1-x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{x^2}{2}} x \left(c_1 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) + 2c_2\right)$$

2.571 problem 585

2.571.1 Maple step by step solution 5363

Internal problem ID [8061]
Internal file name [OUTPUT/6994_Sunday_June_05_2022_05_23_52_PM_53591490/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 585.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\,y''+3xy'+(1+4x)\,y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right)y''+3xy'+(1+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2\\ B &= 3x\\ C &= 1+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 32x^2 + 16x - 1 \\
t &= 4\left(2x^2 - x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1087: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{x} - \frac{1}{4x^2} + \frac{15}{4\left(x - \frac{1}{2}\right)^2} - \frac{3}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{32x^2 + 16x - 1}{4\left(2x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} + (-)(0) \\ &= \frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)} \\ &= \frac{-1-4x}{4x^2-2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)(0) + \left(\left(-\frac{1}{2x^2} + \frac{3}{2\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right)^2 - \left(\frac{32x^2+16x-1}{4\left(2x^2-x\right)^2}\right)\right)$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} - \frac{3}{2\left(x-\frac{1}{2}\right)}\right) dx} \\ &= \frac{\sqrt{x}}{(2x-1)^{\frac{3}{2}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{-2x^3+x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2} + \frac{3\ln(2x-1)}{2}} \\ &= z_1 \left(\frac{(2x-1)^{\frac{3}{2}}}{x^{\frac{3}{2}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{-2x^3+x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3\ln(x)+3\ln(2x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \left(\frac{8x^3}{3} + 6x + \frac{1}{2} - 6x^2 - \ln(x)\right)}{x}$$

Verified OK.

2.571.1 Maple step by step solution

Let's solve

$\left(-2x^3+x^2\right)y''+3xy'+\left(1+4x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{(1+4x)y}{x^2(2x-1)}+\frac{3y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{3y'}{x(2x-1)}-\frac{(1+4x)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=-\frac{3}{x(2x-1)},P_3(x)=-\frac{1+4x}{x^2(2x-1)}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=3$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=1$

- ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)-3xy'+(-1-4x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum\limits_{k=m}^{\infty} a_{k-m}x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum\limits_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum\limits_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum\limits_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(1+r)^{2} x^{r}+\left(\sum\limits_{k=1}^{\infty} \left(-a_k(k+r+1)^{2}+2a_{k-1}(k+r)\,(k-3+r)\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=-1$

- Each term in the series must be 0, giving the recursion relation

  $-a_k(k+r+1)^{2}+2a_{k-1}(k+r)\,(k-3+r)=0$

- Shift index using $k->k+1$

  $-a_{k+1}(k+2+r)^{2}+2a_k(k+r+1)\,(k+r-2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r+1)(k+r-2)}{(k+2+r)^{2}}$

- Recursion relation for $r=-1$; series terminates at $k=3$

  $a_{k+1}=\frac{2a_k k(k-3)}{(k+1)^{2}}$

- Apply recursion relation for $k=0$

  $a_1=0$

- Apply recursion relation for $k=1$

  $a_2=-a_1$

- Express in terms of a_0

  $a_2 = 0$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{9}$
- Express in terms of a_0

  $a_3 = 0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot 0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+3*x*diff(y(x),x)+(1+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} - \frac{c_2(-8x^3 + 18x^2 + 3\ln(x) - 18x)}{3x}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 36

```
DSolve[x^2*(1-2*x)*y''[x]+3*x*y'[x]+(1+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{3}c_2\left(4x^2 - 9x + 9\right) + \frac{c_1}{x} + \frac{c_2\log(x)}{x}$$

2.572 problem 586

2.572.1 Maple step by step solution 5373

Internal problem ID [8062]
Internal file name [OUTPUT/6995_Sunday_June_05_2022_05_23_55_PM_57287442/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 586.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)\,y''+(1-x)\,y'+y=0}$$

Writing the ode as

$$(x^2+x)\,y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1089: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 - 10x - 1}{4(x^2+x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\,\mathrm{e}^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\,\mathrm{e}^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\ln(1+x)} \\
&= z_1 \left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x-1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4)
$$

Verified OK.

2.572.1 Maple step by step solution

Let's solve

$(x^2+x)\,y''+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(1+x)}+\frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-1)y'}{x(1+x)}+\frac{y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=-\frac{x-1}{x(1+x)}, P_3(x)=\frac{1}{x(1+x)}\right]$

  ○ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-2$

  ○ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ○ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)\,y''+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $$a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = -\frac{a_0}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $$y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$$

- Revert the change of variables $u = 1 + x$

  $$\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x*(1+x)*diff(y(x),x$2)+(1-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-1) + c_2(x\ln(x) - \ln(x) - 4)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 23

```
DSolve[x*(1+x)*y''[x]+(1-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

2.573 problem 587

2.573.1 Maple step by step solution 5382

Internal problem ID [8063]
Internal file name [OUTPUT/6996_Sunday_June_05_2022_05_23_58_PM_8689071/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 587.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)\,y''-x(3-5x)\,y'+(4-5x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-3x\right)y'+(4-5x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+x^2 \\ B &= 5x^2-3x \\ C &= 4-5x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 6x - 1}{4(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 - 6x - 1$$
$$t = 4(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 6x - 1}{4(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1091: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(x-1)^2} - \frac{1}{4x^2} + \frac{2}{x-1}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 6x - 1}{4(x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 - 6x - 1}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{5}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{x - 1} + (0) \\
&= \frac{1}{2x} + \frac{2}{x - 1} \\
&= \frac{-1 + 5x}{2x\left(x - 1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{x-1}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{2}{(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{x-1}\right)^2 - \left(\frac{15x^2 - 6x - 1}{4(x^2-x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{2}{x-1}\right) dx} \\ &= \sqrt{x}\,(x-1)^2 \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2-3x}{-x^3+x^2}\, dx} \\ &= z_1 e^{\frac{3\ln(x)}{2} + \ln(x-1)} \\ &= z_1 \left(x^{\frac{3}{2}}(x-1)\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2(x-1)^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-3x}{-x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x)+2\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) - \frac{1}{3(x-1)^3} - \frac{1}{x-1} + \frac{1}{2(x-1)^2} - \ln(x-1) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2(x-1)^3\right) + c_2\left(x^2(x-1)^3\left(\ln(x) - \frac{1}{3(x-1)^3} - \frac{1}{x-1} + \frac{1}{2(x-1)^2} - \ln(x-1)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 (x-1)^3 + c_2 \left(-\ln(x-1)(x-1)^3 + \ln(x)(x-1)^3 - x^2 + \frac{5x}{2} - \frac{11}{6} \right) x^2(1)
$$

Verification of solutions

$$
y = c_1 x^2 (x-1)^3 + c_2 \left(-\ln(x-1)(x-1)^3 + \ln(x)(x-1)^3 - x^2 + \frac{5x}{2} - \frac{11}{6} \right) x^2
$$

Verified OK.

2.573.1 Maple step by step solution

Let's solve

$(-x^3 + x^2) y'' + (5x^2 - 3x) y' + (4 - 5x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x-4)y}{x^2(x-1)} + \frac{(5x-3)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{(5x-3)y'}{x(x-1)} + \frac{(5x-4)y}{x^2(x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{5x-3}{x(x-1)}, P_3(x) = \frac{5x-4}{x^2(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1) - x(5x-3)\,y' + y(5x-4) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-2)^2 + a_{k-1}(k+r-2)(k-6+r) = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+r-1)^2 + a_k(k+r-1)(k+r-5) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-5)}{k+r-1}$$

- Recursion relation for $r=2$; series terminates at $k=3$

$$a_{k+1} = \frac{a_k(k-3)}{k+1}$$

- Apply recursion relation for $k=0$

$$a_1 = -3a_0$$

- Apply recursion relation for $k=1$

$$a_2 = -a_1$$

- Express in terms of a_0

$$a_2 = 3a_0$$

- Apply recursion relation for $k=2$

$$a_3 = -\frac{a_2}{3}$$

- Express in terms of a_0

$$a_3 = -a_0$$

- Terminating series solution of the ODE for $r=2$. Use reduction of order to find the second li

$$y = a_0 \cdot \left(-x^3 + 3x^2 - 3x + 1\right)$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 89

```
dsolve(x^2*(1-x)*diff(y(x),x$2)-x*(3-5*x)*diff(y(x),x)+(4-5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 \left(x^3 - 3x^2 + 3x - 1\right) + c_2 x^2 \left(x^3 \ln(x) - \ln(x-1)\, x^3 - 3x^2 \ln(x) + 3\ln(x-1)\, x^2 + 3x \ln(x) - 3\ln(x-1)\, x - x^2 - \ln(x) + \ln(x-1) + \frac{5x}{2} - \frac{11}{6}\right)$$

✓ Solution by Mathematica
Time used: 0.075 (sec). Leaf size: 76

```
DSolve[x^2*(1-x)*y''[x]-x*(3-5*x)*y'[x]+(4-5*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{1}{6}x^2\left(6c_1x^3 - 18c_1x^2 - 6c_2x^2 + 18c_1x + 15c_2x - 6c_2(x-1)^3\log(x-1) + 6c_2(x-1)^3\log(x) - 6c_1 - 11c_2\right)$$

2.574 problem 588

2.574.1 Maple step by step solution 5392

Internal problem ID [8064]
Internal file name [`OUTPUT/6997_Sunday_June_05_2022_05_24_01_PM_82579885/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 588.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\, y''-x(9x^2+1)\, y'+(25x^2+1)\, y=0}$$

Writing the ode as

$$\left(x^4+x^2\right) y''+\left(-9x^3-x\right) y'+\left(25x^2+1\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= -9x^3-x \\ C &= 25x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 - 98x^2 - 1$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1093: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{(x-i)^2} + \frac{6}{(x+i)^2} + \frac{6i}{x-i} - \frac{6i}{x+i} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 - 98x^2 - 1}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	3	-2
$-i$	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{2} - \left(-\frac{7}{2}\right) \\
&= 4
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} + (-)(0) \\
&= \frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \\
&= \frac{1}{2x} - \frac{4x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \right) (4x^3 + 3x^2 a_3 + 2a_2 x + a_1) + \left(\left(-\frac{1}{2x^2} + \frac{2}{(x-i)^2} + \frac{}{(x - \cdots} \right. \right.$$
$$\overline{(x^2+1)(x^4 a \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0, a_2 = -4, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 4x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^4 - 4x^2 + 1)\, e^{\int \left(\frac{1}{2x} - \frac{2}{x-i} - \frac{2}{x+i} \right) dx} \\
&= (x^4 - 4x^2 + 1)\, e^{\frac{\ln(x)}{2} - 2\ln(x^2+1)} \\
&= \frac{(x^4 - 4x^2 + 1)\sqrt{x}}{(x^2+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^3-x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}+2\ln(x^2+1)} \\
&= z_1\left(\sqrt{x}\,(x^2+1)^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^5 - 4x^3 + x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^3-x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)+4\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) + \frac{-6x^2+3}{x^4-4x^2+1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5-4x^3+x\right) + c_2\left(x^5-4x^3+x\left(\ln(x)+\frac{-6x^2+3}{x^4-4x^2+1}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\left(x^5-4x^3+x\right) + c_2\left(\ln(x)\left(x^4-4x^2+1\right)-6x^2+3\right)x \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1\left(x^5-4x^3+x\right) + c_2\left(\ln(x)\left(x^4-4x^2+1\right)-6x^2+3\right)x
$$

Verified OK.

2.574.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y''+(-9x^3-x)\,y'+(25x^2+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(25x^2+1)y}{x^2(x^2+1)}+\frac{(9x^2+1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(9x^2+1)y'}{x(x^2+1)}+\frac{(25x^2+1)y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{9x^2+1}{x(x^2+1)}, P_3(x)=\frac{25x^2+1}{x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=1$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\,y''-x(9x^2+1)\,y'+(25x^2+1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..4$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)^{2} x^{r}+a_{1} r^{2} x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r-1)^{2}+a_{k-2}(k-7+r)^{2}\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $a_{1} r^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r-1)^{2}+a_{k-2}(k-7+r)^{2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)^{2}+a_{k}(k+r-5)^{2}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r-5)^{2}}{(k+1+r)^{2}}$

- Recursion relation for $r = 1$; series terminates at $k = 4$

  $a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k(k-4)^2}{(k+2)^2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(1+9*x^2)*diff(y(x),x)+(1+25*x^2)*y(x)=0,y(x), singsol=a
```

$$y(x) = c_1 x(x^4 - 4x^2 + 1) + c_2(x^4 \ln(x) - 4x^2 \ln(x) - 6x^2 + \ln(x) + 3)\, x$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 43

```
DSolve[x^2*(1+x^2)*y''[x]-x*(1+9*x^2)*y'[x]+(1+25*x^2)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to c_1(x^5 - 4x^3 + x) + c_2 x(-6x^2 + (x^4 - 4x^2 + 1)\log(x) + 3)$$

2.575 problem 589

2.575.1 Maple step by step solution 5402

Internal problem ID [8065]
Internal file name [`OUTPUT/6998_Sunday_June_05_2022_05_24_05_PM_39246584/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 589.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2y''+3x(1-x^2)\,y'+(7x^2+1)\,y=0}$$

Writing the ode as

$$9x^2y''+(-3x^3+3x)\,y'+(7x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= -3x^3+3x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 36x^2 - 9}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 36x^2 - 9 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 36x^2 - 9}{36x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1095: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{36} - 1 - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{6} - \frac{3}{x} - \frac{111}{4x^3} - \frac{999}{2x^5} - \frac{180819}{16x^7} - \frac{2292705}{8x^9} - \frac{249239511}{32x^{11}} - \frac{3548540907}{16x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{6}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{36}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 36x^2 - 9}{36x^2} \\
&= Q + \frac{R}{36x^2} \\
&= \left(\frac{x^2}{36} - 1\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{36} - 1 - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{6}} - 1\right) &= -\frac{7}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{6}} - 1\right) &= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 36x^2 - 9}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{6}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{6}\right) \\ &= \frac{1}{2x} - \frac{x}{6} \\ &= \frac{1}{2x} - \frac{x}{6} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2x} - \frac{x}{6}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{6}\right) + \left(\frac{1}{2x} - \frac{x}{6}\right)^2 - \left(\frac{x^4 - 36x^2 - 9}{36x^2}\right)\right) = 0$$

$$\frac{x^2 a_1 + 2(6 + a_0)\, x + 3a_1}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -6, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 6$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (x^2 - 6)\, \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{x}{6}\right) dx} \\ &= (x^2 - 6)\, \mathrm{e}^{-\frac{x^2}{12} + \frac{\ln(x)}{2}} \\ &= (x^2 - 6)\, \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{12}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x^3 + 3x}{9x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{6} + \frac{x^2}{12}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{x^2}{12}}}{x^{\frac{1}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{3}}\left(x^2 - 6\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+3x}{9x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{6}-\frac{\ln(x)}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{3}}\left(x^2-6\right)\right) + c_2\left(x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{3}}\left(x^2-6\right) + c_2 x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^{\frac{1}{3}}\left(x^2-6\right) + c_2 x^{\frac{1}{3}}\left(x^2-6\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{6}}}{x\left(x^2-6\right)^2} dx\right)$$

Verified OK.

2.575.1 Maple step by step solution

Let's solve

$9x^2y'' + (-3x^3 + 3x)\,y' + (7x^2 + 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{9x^2} + \frac{(x^2-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{3x} + \frac{(7x^2+1)y}{9x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^2-1}{3x}, P_3(x) = \frac{7x^2+1}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{9}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9x^2y'' - 3x(x^2 - 1)\,y' + (7x^2 + 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+3r)^{2} x^{r}+a_{1}(2+3r)^{2} x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(3k+3r-1)^{2}-a_{k-2}(3k-13+3r)\right) x^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)^{2}=0$

- Values of r that satisfy the indicial equation

  $r=\frac{1}{3}$

- Each term must be 0

  $a_{1}(2+3r)^{2}=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(3k+3r-1)^{2}+(-3k+13-3r)\, a_{k-2}=0$

- Shift index using $k- >k+2$

  $a_{k+2}(3k+5+3r)^{2}+a_{k}(-3k-3r+7)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(3k+3r-7)}{(3k+5+3r)^{2}}$

- Recursion relation for $r=\frac{1}{3}$; series terminates at $k=2$

  $a_{k+2}=\frac{a_{k}(3k-6)}{(3k+6)^{2}}$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{a_k(3k-6)}{(3k+6)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*(1-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{3}}(x^2-6) + c_2 x^{\frac{1}{3}}(x^2-6)\left(\int \frac{e^{\frac{x^2}{6}}}{(x^2-6)^2 x} dx\right)$$

✓ Solution by Mathematica

Time used: 0.309 (sec). Leaf size: 53

```
DSolve[9*x^2*y''[x]+3*x*(1-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{72}\sqrt[3]{x}\left(c_2(x^2-6)\operatorname{ExpIntegralEi}\left(\frac{x^2}{6}\right)+72c_1(x^2-6)-6c_2e^{\frac{x^2}{6}}\right)$$

2.576 problem 590

2.576.1 Maple step by step solution 5412

Internal problem ID [8066]
Internal file name [`OUTPUT/6999_Sunday_June_05_2022_05_24_08_PM_78906789/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 590.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x(x^2+1)\,y''+(1-x^2)\,y'-8yx=0}$$

Writing the ode as

$$(x^3+x)\,y''+(1-x^2)\,y'-8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^3+x \\ B &= 1-x^2 \\ C &= -8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^4 + 22x^2 - 1}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^4 + 22x^2 - 1$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^4 + 22x^2 - 1}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1097: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{35x^4 + 22x^2 - 1}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (0) \\ &= \frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\ &= \frac{1}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{3}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(\frac{1}{2x} + \frac{3}{2\,(x-i)} + \frac{3}{2(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)dx} \\ &= \sqrt{x}\left(x^2+1\right)^{\frac{3}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x^2}{x^3+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{x^2+1}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x^2}{x^3+x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\ln(x)+\frac{1}{2x^2+2}-\frac{\ln\left(x^2+1\right)}{2}+\frac{1}{4\left(x^2+1\right)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+1\right)^2\right) + c_2\left(\left(x^2+1\right)^2\left(\ln(x)+\frac{1}{2x^2+2}-\frac{\ln\left(x^2+1\right)}{2}+\frac{1}{4\left(x^2+1\right)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+1\right)^2 + c_2\left(\ln(x)\left(x^2+1\right)^2+\frac{x^2}{2}+\frac{3}{4}-\frac{\ln\left(x^2+1\right)\left(x^2+1\right)^2}{2}\right) \tag{1}
$$

Verification of solutions

$$y = c_1 (x^2+1)^2 + c_2 \left(\ln(x)\,(x^2+1)^2 + \frac{x^2}{2} + \frac{3}{4} - \frac{\ln(x^2+1)\,(x^2+1)^2}{2}\right)$$

Verified OK.

2.576.1 Maple step by step solution

Let's solve

$(x^3+x)\,y'' + (1-x^2)\,y' - 8yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{8y}{x^2+1} + \frac{(x^2-1)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-1)y'}{x(x^2+1)} - \frac{8y}{x^2+1} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x^2-1}{x(x^2+1)}, P_3(x) = -\frac{8}{x^2+1}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x^2+1)\,y'' + (1-x^2)\,y' - 8yx = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}(k+r+1)\,(k-5+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1(1+r)^2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left((a_{k-1} + a_{k+1})\, k - 5a_{k-1} + a_{k+1}\right)(k+1) = 0$$

- Shift index using $k->k+1$

$$((a_k + a_{k+2})(k+1) - 5a_k + a_{k+2})(k+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k(k-4)}{k+2}$$

- Recursion relation for $r = 0$; series terminates at $k = 4$

$$a_{k+2} = -\frac{a_k(k-4)}{k+2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k-4)}{k+2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(x*(1+x^2)*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)-8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 2x^2 + 1) + c_2\left(-\frac{\ln(x^2+1)\,x^4}{2} + x^4\ln(x) - \ln(x^2+1)\,x^2 + 2x^2\ln(x) + \frac{x^2}{2} - \frac{\ln(x^2+1)}{2} + \ln(x) + \frac{3}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 55

```
DSolve[x*(1+x^2)*y''[x]+(1-x^2)*y'[x]-8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x^2+1)^2 + \frac{1}{4}c_2\left(2x^2 + 4(x^2+1)^2\log(x) - 2(x^2+1)^2\log\left(x^2+1\right) + 3\right)$$

2.577 problem 591

2.577.1 Maple step by step solution 5423

Internal problem ID [8067]
Internal file name [OUTPUT/7000_Sunday_June_05_2022_05_24_11_PM_30772382/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 591.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 2x(-x^2+4)\,y' + (7x^2+1)\,y = 0}$$

Writing the ode as

$$4x^2y'' + (-2x^3+8x)\,y' + (7x^2+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -2x^3+8x \\ C &= 7x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 40x^2 - 4}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 40x^2 - 4 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 40x^2 - 4}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1099: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{4} - \frac{5}{x} - \frac{101}{2x^3} - \frac{1010}{x^5} - \frac{50601}{2x^7} - \frac{710030}{x^9} - \frac{21351501}{x^{11}} - \frac{672670100}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{4}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{16}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 40x^2 - 4}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(\frac{x^2}{16} - \frac{5}{2}\right) + \left(-\frac{1}{4x^2}\right) \\
&= \frac{x^2}{16} - \frac{5}{2} - \frac{1}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is $-\frac{5}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{4}} - 1\right) &= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 40x^2 - 4}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{4}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(\frac{x}{4}\right) \\ &= \frac{1}{2x} - \frac{x}{4} \\ &= \frac{1}{2x} - \frac{x}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12x^2 + 6xa_3 + 2a_2) + 2\left(\frac{1}{2x} - \frac{x}{4}\right)(4x^3 + 3a_3x^2 + 2a_2x + a_1) + \left(\left(-\frac{1}{2x^2} - \frac{1}{4}\right) + \left(\frac{1}{2x} - \frac{x}{4}\right)^2 - \left(\frac{x^4 \cdots}{}\right.\right.$$

$$\frac{x^4a_3 + 2(16 + a_2)\, x^3 + 3(a_1 + 6a_3)\, x^2 + 4(a_0 + 2a \cdots}{2x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 32, a_1 = 0, a_2 = -16, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 16x^2 + 32$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (x^4 - 16x^2 + 32)\, e^{\int \left(\frac{1}{2x} - \frac{x}{4}\right) dx} \\ &= (x^4 - 16x^2 + 32)\, e^{-\frac{x^2}{8} + \frac{\ln(x)}{2}} \\ &= (x^4 - 16x^2 + 32)\, \sqrt{x}\, e^{-\frac{x^2}{8}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3 + 8x}{4x^2}\, dx} \\ &= z_1 e^{\frac{x^2}{8} - \ln(x)} \\ &= z_1 \left(\frac{e^{\frac{x^2}{8}}}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^4 - 16x^2 + 32}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+8x}{4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{4}-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x\left(x^4 - 16x^2 + 32\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}}\right) + c_2 \left(\frac{x^4 - 16x^2 + 32}{\sqrt{x}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x\left(x^4 - 16x^2 + 32\right)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32)\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x(x^4-16x^2+32)^2} dx\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32)\left(\int \frac{\mathrm{e}^{\frac{x^2}{4}}}{x(x^4-16x^2+32)^2} dx\right)}{\sqrt{x}}$$

Verified OK.

2.577.1 Maple step by step solution

Let's solve

$4x^2y'' + (-2x^3+8x)\,y' + (7x^2+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(7x^2+1)y}{4x^2} + \frac{(x^2-4)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2-4)y'}{2x} + \frac{(7x^2+1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x^2-4}{2x}, P_3(x) = \frac{7x^2+1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 2x(x^2-4)\,y' + (7x^2+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)^2 x^r + a_1(3+2r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)^2 - a_{k-2}(2k-11+2r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = -\frac{1}{2}$

- Each term must be 0

  $a_1(3+2r)^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+1)^2 + (-2k+11-2r)\, a_{k-2} = 0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+5+2r)^2 + a_k(-2k-2r+7) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(2k+2r-7)}{(2k+5+2r)^2}$

- Recursion relation for $r = -\frac{1}{2}$; series terminates at $k = 4$

  $a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{a_k(2k-8)}{(2k+4)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(4*x^2*diff(y(x),x$2)+2*x*(4-x^2)*diff(y(x),x)+(1+7*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4 - 16x^2 + 32)}{\sqrt{x}} + \frac{c_2(x^4 - 16x^2 + 32)\left(\int \frac{e^{\frac{x^2}{4}}}{x(x^4-16x^2+32)^2} dx\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 68

```
DSolve[4*x^2*y''[x]+2*x*(4-x^2)*y'[x]+(1+7*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_2(x^4-16x^2+32)\,\text{ExpIntegralEi}\left(\frac{x^2}{4}\right)-4c_2e^{\frac{x^2}{4}}(x^2-12)+2048c_1(x^4-16x^2+32)}{2048\sqrt{x}}$$

2.578 problem 592

2.578.1 Maple step by step solution 5432

Internal problem ID [8068]
Internal file name [`OUTPUT/7001_Sunday_June_05_2022_05_24_14_PM_87583963/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 592.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(1+x)\,y''+8x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+8x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 4x^3+4x^2\\ B &= 8x^2\\ C &= 1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{8x^2}{4x^3+4x^2} \, dx} \\ &= z_1 e^{-\ln(1+x)} \\ &= z_1 \left(\frac{1}{1+x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{1+x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2}{4x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(1+x)}}{(y_1)^2} \, dx \\ &= y_1 (\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{1+x}\right) + c_2 \left(\frac{\sqrt{x}}{1+x} (\ln(x))\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 \sqrt{x}}{1+x} + \frac{c_2 \sqrt{x}\, \ln(x)}{1+x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 \sqrt{x}}{1+x} + \frac{c_2 \sqrt{x}\, \ln(x)}{1+x}
$$

Verified OK.

2.578.1 Maple step by step solution

Let's solve

$(4x^3 + 4x^2)\, y'' + 8x^2 y' + (1+x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{1+x} - \frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{1+x} + \frac{y}{4x^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{1+x}, P_3(x) = \frac{1}{4x^2}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 2$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- ○ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\,y'' + 8x^2y' + (1+x)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (8u^2 - 16u + 8)\left(\frac{d}{du}y(u)\right) + uy(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u \cdot y(u)$ to series expansion

  $u \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+1}$

- ○ Shift index using $k- > k-1$

  $u \cdot y(u) = \sum_{k=1}^{\infty} a_{k-1} u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ○ Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$4a_0r(1+r)\,u^{-1+r}+(4a_1(1+r)\,(2+r)-8a_0r(1+r))\,u^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+r+1)\,(k+2+r)-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,0\}$

- Each term must be 0

  $4a_1(1+r)\,(2+r)-8a_0r(1+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k-1}(2k-1+2r)^2-8\left(\left(-\frac{k}{2}-\frac{r}{2}-1\right)a_{k+1}+a_k(k+r)\right)(k+r+1)=0$

- Shift index using $k->k+1$

  $a_k(2k+2r+1)^2-8\left(\left(-\frac{k}{2}-\frac{3}{2}-\frac{r}{2}\right)a_{k+2}+a_{k+1}(k+r+1)\right)(k+2+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+8kra_k-16kra_{k+1}+4r^2a_k-8r^2a_{k+1}+4ka_k-24ka_{k+1}+4ra_k-24ra_{k+1}+a_k-16a_{k+1}}{4(k+3+r)(k+2+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-8ka_{k+1}+a_k}{4(k+2)(k+1)}$

- Solution for $r=-1$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-8ka_{k+1}+a_k}{4(k+2)(k+1)},0=0\right]$

- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k-1},a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}-4ka_k-8ka_{k+1}+a_k}{4(k+2)(k+1)},0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-24ka_{k+1}+a_k-16a_{k+1}}{4(k+3)(k+2)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-24ka_{k+1}+a_k-16a_{k+1}}{4(k+3)(k+2)},8a_1=0\right]$

- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k,a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+4ka_k-24ka_{k+1}+a_k-16a_{k+1}}{4(k+3)(k+2)},8a_1=0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^k\right), a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} - 4k a_k - 8k a_{k+1} + a_k}{4(k+2)(k+1)}, 0 = 0, b_{k+2}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+8*x^2*diff(y(x),x)+(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\sqrt{x}}{x+1} + \frac{c_2\sqrt{x}\,\ln(x)}{x+1}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 24

```
DSolve[4*x^2*(1+x)*y''[x]+8*x^2*y'[x]+(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}(c_2\log(x) + c_1)}{x+1}$$

2.579 problem 593

2.579.1 Maple step by step solution 5441

Internal problem ID [8069]
Internal file name [OUTPUT/7002_Sunday_June_05_2022_05_24_17_PM_25774129/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 593.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(x+3)\,y''+3x(3+7x)\,y'+(3+4x)\,y=0}$$

Writing the ode as

$$\left(9x^3+27x^2\right)y''+\left(21x^2+9x\right)y'+(3+4x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^3+27x^2 \\ B &= 21x^2+9x \\ C &= 3+4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{21x^2+9x}{9x^3+27x^2} \, dx} \\ &= z_1 e^{-\ln(x+3) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(x+3)\, x^{\frac{1}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{x+3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+9x}{9x^3+27x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x+3) - \frac{\ln(x)}{3}}}{(y_1)^2} \, dx \\ &= y_1(\ln(x) - 1) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{3}}}{x+3}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{x+3}(\ln (x) - 1)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}} (\ln (x) - 1)}{x+3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}} (\ln (x) - 1)}{x+3}
$$

Verified OK.

2.579.1 Maple step by step solution

Let's solve

$(9x^3 + 27x^2) y'' + (21x^2 + 9x) y' + (3 + 4x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3+4x)y}{9x^2(x+3)} - \frac{(3+7x)y'}{3x(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+7x)y'}{3x(x+3)} + \frac{(3+4x)y}{9x^2(x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{3+7x}{3x(x+3)}, P_3(x) = \frac{3+4x}{9x^2(x+3)}\right]$

  ◦ $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

  $\left((x+3) \cdot P_2(x)\right)\bigg|_{x=-3} = 2$

  ◦ $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

$$\left((x+3)^2 \cdot P_3(x)\right)\Bigg|_{x=-3} = 0$$

- $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $9x^2(x+3)\,y'' + 3x(3+7x)\,y' + (3+4x)\,y = 0$

- Change variables using $x = u-3$ so that the regular singular point is at $u = 0$

  $(9u^3 - 54u^2 + 81u)\left(\frac{d^2}{du^2}y(u)\right) + (21u^2 - 117u + 162)\left(\frac{d}{du}y(u)\right) + (-9+4u)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- > k-m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$81a_0r(1+r)\,u^{-1+r}+\left(81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(81a_{k+1}(k+r+1)\,(k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$81r(1+r)=0$$
- Values of r that satisfy the indicial equation
$$r\in\{-1,0\}$$
- Each term must be 0
$$81a_1(1+r)\,(2+r)-9a_0(1+r)\,(1+6r)=0$$
- Each term in the series must be 0, giving the recursion relation
$$81a_{k+1}(k+r+1)\,(k+2+r)-54a_k(k+r+1)\left(k+r+\frac{1}{6}\right)+a_{k-1}(3k-1+3r)^2=0$$
- Shift index using $k->k+1$
$$81a_{k+2}(k+2+r)\,(k+3+r)-54a_{k+1}(k+2+r)\left(k+\frac{7}{6}+r\right)+a_k(3k+3r+2)^2=0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+18kra_k-108kra_{k+1}+9r^2a_k-54r^2a_{k+1}+12ka_k-171ka_{k+1}+12ra_k-171ra_{k+1}+4a_k-126a_{k+1}}{81(k+2+r)(k+3+r)}$$
- Recursion relation for $r=-1$
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)}$$
- Solution for $r=-1$
$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$$
- Revert the change of variables $u=x+3$
$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^{k-1},a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}-6ka_k-63ka_{k+1}+a_k-9a_{k+1}}{81(k+1)(k+2)},0=0\right]$$
- Recursion relation for $r=0$
$$a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)}$$
- Solution for $r=0$
$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$$
- Revert the change of variables $u=x+3$
$$\left[y=\sum_{k=0}^{\infty}a_k(x+3)^k,a_{k+2}=-\frac{9k^2a_k-54k^2a_{k+1}+12ka_k-171ka_{k+1}+4a_k-126a_{k+1}}{81(k+2)(k+3)},162a_1-9a_0=0\right]$$
- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+3)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k}\right), a_{k+2} = -\frac{9k^2 a_k - 54k^2 a_{k+1} - 6k a_k - 63k a_{k+1} + a_k - 9a_{k+1}}{81(k+1)(k+2)}, 0 =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(9*x^2*(3+x)*diff(y(x),x$2)+3*x*(3+7*x)*diff(y(x),x)+(3+4*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{x+3} + \frac{c_2 x^{\frac{1}{3}} \ln(x)}{x+3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 24

```
DSolve[9*x^2*(3+x)*y''[x]+3*x*(3+7*x)*y'[x]+(3+4*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x+3}$$

2.580 problem 594

2.580.1 Maple step by step solution 5450

Internal problem ID [8070]
Internal file name [`OUTPUT/7003_Sunday_June_05_2022_05_24_20_PM_17272265/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 594.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-x^2+2)\,y''-x(3x^2+2)\,y'+(-x^2+2)\,y=0}$$

Writing the ode as

$$(-x^4+2x^2)\,y''+(-3x^3-2x)\,y'+(-x^2+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2\\ B &= -3x^3-2x \\ C &= -x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3x^3-2x}{-x^4+2x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(x^2-2)} \\ &= z_1 \left(\frac{\sqrt{x}}{x^2-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x^2-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3-2x}{-x^4+2x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x) - 2\ln(x^2-2)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x^2-2}\right) + c_2\left(\frac{x}{x^2-2}(\ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 x}{x^2-2} + \frac{c_2 x \ln(x)}{x^2-2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{x^2-2} + \frac{c_2 x \ln(x)}{x^2-2}
$$

Verified OK.

2.580.1 Maple step by step solution

Let's solve

$(-x^4+2x^2)\,y''+(-3x^3-2x)\,y'+(-x^2+2)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x^2}-\frac{(3x^2+2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x^2+2)y'}{x(x^2-2)}+\frac{y}{x^2}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{3x^2+2}{x(x^2-2)}, P_3(x)=\frac{1}{x^2}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = 1$$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x^{2}(x^{2}-2)+x(3x^{2}+2)\, y'+y(x^{2}-2)=0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- ◦ Shift index using $k- >k + 2 - m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0(-1+r)^2\, x^r - 2a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(-2a_k(k+r-1)^2 + a_{k-2}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-2(-1+r)^2=0$

- Values of r that satisfy the indicial equation

  $r=1$

- Each term must be 0

  $-2a_1r^2=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $-2\left(a_k-\frac{a_{k-2}}{2}\right)(k+r-1)^2=0$

- Shift index using $k->k+2$

  $-2\left(a_{k+2}-\frac{a_k}{2}\right)(k+r+1)^2=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{2}$

- Recursion relation for $r=1$

  $a_{k+2}=\frac{a_k}{2}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+1},a_{k+2}=\frac{a_k}{2},a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(x^2*(2-x^2)*diff(y(x),x$2)-x*(2+3*x^2)*diff(y(x),x)+(2-x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x^2 - 2} + \frac{c_2 x \ln(x)}{x^2 - 2}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 23

```
DSolve[x^2*(2-x^2)*y''[x]-x*(2+3*x^2)*y'[x]+(2-x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{x(c_2 \log(x) + c_1)}{x^2 - 2}$$

2.581 problem 595

2.581.1 Maple step by step solution 5459

Internal problem ID [8071]
Internal file name [`OUTPUT/7004_Sunday_June_05_2022_05_24_23_PM_33791003/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 595.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{16x^2(x^2+1)\,y''+8x(9x^2+1)\,y'+(49x^2+1)\,y=0}$$

Writing the ode as

$$(16x^4+16x^2)\,y''+(72x^3+8x)\,y'+(49x^2+1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^4+16x^2 \\ B &= 72x^3+8x \\ C &= 49x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{72x^3+8x}{16x^4+16x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4} - \ln(x^2+1)} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}}(x^2+1)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{4}}}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{72x^3+8x}{16x^4+16x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-2\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) - \frac{1}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{4}}}{x^2+1}\right) + c_2 \left(\frac{x^{\frac{1}{4}}}{x^2+1}\left(\ln(x) - \frac{1}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}} (2\ln(x) - 1)}{2x^2+2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}} (2\ln(x) - 1)}{2x^2+2}
$$

Verified OK.

2.581.1 Maple step by step solution

Let's solve

$(16x^4 + 16x^2) y'' + (72x^3 + 8x) y' + (49x^2 + 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x^2+1)y}{16x^2(x^2+1)} - \frac{(9x^2+1)y'}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(9x^2+1)y'}{2x(x^2+1)} + \frac{(49x^2+1)y}{16x^2(x^2+1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{9x^2+1}{2x(x^2+1)}, P_3(x) = \frac{49x^2+1}{16x^2(x^2+1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2(x^2+1)\, y'' + 8x(9x^2+1)\, y' + (49x^2+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+4r)^2\,x^r + a_1(3+4r)^2\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(4k+4r-1)^2 + a_{k-2}(4k+4r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+4r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{4}$$

- Each term must be 0

$$a_1(3+4r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(4k+4r-1)^2\,(a_k + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(4k+4r+7)^2\,(a_{k+2} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -a_k$$

- Recursion relation for $r = \frac{1}{4}$

$$a_{k+2} = -a_k$$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -a_k, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(16*x^2*(1+x^2)*diff(y(x),x$2)+8*x*(1+9*x^2)*diff(y(x),x)+(1+49*x^2)*y(x)=0,y(x), sing
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{x^2+1} + \frac{c_2 x^{\frac{1}{4}} \ln(x)}{x^2+1}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 26

```
DSolve[16*x^2*(1+x^2)*y''[x]+8*x*(1+9*x^2)*y'[x]+(1+49*x^2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{\sqrt[4]{x}(c_2 \log(x) + c_1)}{x^2+1}$$

2.582 problem 596

2.582.1 Maple step by step solution 5468

Internal problem ID [8072]
Internal file name [`OUTPUT/7005_Sunday_June_05_2022_05_24_26_PM_48644773/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 596.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(4+3x)\,y''-x(4-3x)\,y'+4y=0}$$

Writing the ode as

$$\left(3x^3+4x^2\right)y''+\left(3x^2-4x\right)y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^3+4x^2 \\ B &= 3x^2-4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3x^2-4x}{3x^3+4x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \ln(4+3x)} \\ &= z_1 \left(\frac{\sqrt{x}}{4+3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{4+3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2-4x}{3x^3+4x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(x) - 2\ln(4+3x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(\frac{x}{4+3x}\right) + c_2\left(\frac{x}{4+3x}(\ln(x))\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1x}{4+3x} + \frac{c_2x\ln(x)}{4+3x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1x}{4+3x} + \frac{c_2x\ln(x)}{4+3x}$$

Verified OK.

2.582.1 Maple step by step solution

Let's solve

$(3x^3+4x^2)\,y''+(3x^2-4x)\,y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{4y}{x^2(4+3x)}-\frac{(3x-4)y'}{x(4+3x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x-4)y'}{x(4+3x)}+\frac{4y}{x^2(4+3x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{3x-4}{x(4+3x)}, P_3(x)=\frac{4}{x^2(4+3x)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(4+3x)\, y'' + x(3x-4)\, y' + 4y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(4a_k(k+r-1)^2 + 3a_{k-1}(k+r-1)^2\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $4(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)^2\,(4a_k + 3a_{k-1}) = 0$

- Shift index using $k->k+1$

  $(k+r)^{2}(4a_{k+1}+3a_k)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_k}{4}$
- Recursion relation for $r=1$

  $a_{k+1}=-\frac{3a_k}{4}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=-\frac{3a_k}{4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*(4+3*x)*diff(y(x),x$2)-x*(4-3*x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 x}{3x+4}+\frac{c_2 x\ln(x)}{3x+4}$$

✓ <u>Solution by Mathematica</u>

Time used: 0.033 (sec). Leaf size: 22

```
DSolve[x^2*(4+3*x)*y''[x]-x*(4-3*x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{3x + 4}$$

2.583 problem 597

2.583.1 Maple step by step solution 5477

Internal problem ID [8073]
Internal file name [OUTPUT/7006_Sunday_June_05_2022_05_24_29_PM_29778049/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 597.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+3x+1)\,y''+8x^2(3+2x)\,y'+(9x^2+3x+1)\,y=0}$$

Writing the ode as

$$\left(4x^4+12x^3+4x^2\right)y''+\left(16x^3+24x^2\right)y'+\left(9x^2+3x+1\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+12x^3+4x^2 \\ B &= 16x^3+24x^2 \\ C &= 9x^2+3x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\, dx} \\ &= z_1 e^{-\ln(x^2+3x+1)} \\ &= z_1\left(\frac{1}{x^2+3x+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{x^2+3x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{16x^3+24x^2}{4x^4+12x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(x^2+3x+1)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\sqrt{x}}{x^2+3x+1}\right) + c_2\left(\frac{\sqrt{x}}{x^2+3x+1}(\ln(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2\sqrt{x}\,\ln(x)}{x^2+3x+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2\sqrt{x}\,\ln(x)}{x^2+3x+1}
$$

Verified OK.

2.583.1 Maple step by step solution

Let's solve

$(4x^4+12x^3+4x^2)\,y''+(16x^3+24x^2)\,y'+(9x^2+3x+1)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)}-\frac{2(3+2x)y'}{x^2+3x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(3+2x)y'}{x^2+3x+1}+\frac{(9x^2+3x+1)y}{4x^2(x^2+3x+1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(3+2x)}{x^2+3x+1}, P_3(x)=\frac{9x^2+3x+1}{4x^2(x^2+3x+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=0$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+3x+1)\, y'' + 8x^2(3+2x)\, y' + (9x^2+3x+1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 2..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- > k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)^2 x^r + \left(a_1(1+2r)^2 + 3a_0(1+2r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)^2 + 3a_{k-1}(2k+2r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term must be 0

$$a_1(1+2r)^2 + 3a_0(1+2r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -3a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(2k+2r-1)^2 (a_k + 3a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k- >k+2$

$$(2k+2r+3)^2 (a_{k+2} + 3a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -3a_{k+1} - a_k$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -3a_{k+1} - a_k$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -3a_{k+1} - a_k, a_1 = -3a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(4*x^2*(1+3*x+x^2)*diff(y(x),x$2)+8*x^2*(3+2*x)*diff(y(x),x)+(1+3*x+9*x^2)*y(x)=0,y(x)
```

$$y(x) = \frac{c_1\sqrt{x}}{x^2+3x+1} + \frac{c_2\sqrt{x}\,\ln(x)}{x^2+3x+1}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 29

```
DSolve[4*x^2*(1+3*x+x^2)*y''[x]+8*x^2*(3+2*x)*y'[x]+(1+3*x+9*x^2)*y[x]==0,y[x],x,IncludeSing
```

$$y(x) \to \frac{\sqrt{x}(c_2\log(x)+c_1)}{x^2+3x+1}$$

2.584 problem 598

2.584.1 Maple step by step solution 5486

Internal problem ID [8074]
Internal file name [OUTPUT/7007_Sunday_June_05_2022_05_24_34_PM_83134208/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 598.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x)^2 y''-x\left(-3x^2+2x+1\right) y'+\left(x^2+1\right) y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+\left(3x^3-2x^2-x\right) y'+\left(x^2+1\right) y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= 3x^3-2x^2-x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^3 - 2x^2 - x}{x^2 (x-1)^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2} - 2\ln(x-1)} \\
&= z_1 \left(\frac{\sqrt{x}}{(x-1)^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{(x-1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^3-2x^2-x}{x^2(x-1)^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x)-4\ln(x-1)}}{(y_1)^2} dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x}{(x-1)^2}\right) + c_2 \left(\frac{x}{(x-1)^2}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln(x)}{(x-1)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln(x)}{(x-1)^2}
$$

Verified OK.

2.584.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (3x^3 - 2x^2 - x)\, y' + (x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{x^2(x-1)^2} - \frac{y'(3x+1)}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'(3x+1)}{x(x-1)} + \frac{(x^2+1)y}{x^2(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

$$\left[P_2(x) = \frac{3x+1}{x(x-1)}, P_3(x) = \frac{x^2+1}{x^2(x-1)^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$$y''x^2(x-1)^2 + x(3x+1)(x-1)y' + (x^2+1)y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + (-2a_0r^2 + a_1r^2)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)^2 - 2a_{k-1}(k+r-1)^2 + a_{k-2}(k+r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$-2a_0r^2 + a_1r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 2a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)^2 (a_k - 2a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(k+r+1)^2 (a_{k+2} - 2a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = 2a_{k+1} - a_k$$

- Recursion relation for $r = 1$

$$a_{k+2} = 2a_{k+1} - a_k$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = 2a_{k+1} - a_k, a_1 = 2a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*(1-x)^2*diff(y(x),x$2)-x*(1+2*x-3*x^2)*diff(y(x),x)+(1+x^2)*y(x)=0,y(x), singsol=
```

$$y(x) = \frac{c_1 x}{(x-1)^2} + \frac{c_2 x \ln (x)}{(x-1)^2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 20

```
DSolve[x^2*(1-x)^2*y''[x]-x*(1+2*x-3*x^2)*y'[x]+(1+x^2)*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \frac{x(c_2 \log(x) + c_1)}{(x-1)^2}$$

2.585 problem 599

2.585.1 Maple step by step solution 5495

Internal problem ID [8075]
Internal file name [`OUTPUT/7008_Sunday_June_05_2022_05_24_37_PM_15441132/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 599.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{9x^2(x^2+x+1)\,y''+3x(13x^2+7x+1)\,y'+(25x^2+4x+1)\,y=0}$$

Writing the ode as

$$\left(9x^4+9x^3+9x^2\right)y''+\left(39x^3+21x^2+3x\right)y'+\left(25x^2+4x+1\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^4+9x^3+9x^2 \\ B &= 39x^3+21x^2+3x \\ C &= 25x^2+4x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2} \, dx} \\ &= z_1 e^{-\ln(x^2+x+1) - \frac{\ln(x)}{6}} \\ &= z_1 \left(\frac{1}{(x^2+x+1)\, x^{\frac{1}{6}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{3}}}{x^2 + x + 1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{39x^3+21x^2+3x}{9x^4+9x^3+9x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-2\ln(x^2+x+1)-\frac{\ln(x)}{3}}}{(y_1)^2}\, dx$$
$$= y_1 \left(\ln(x) + \frac{13}{24}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{\frac{1}{3}}}{x^2+x+1}\right) + c_2 \left(\frac{x^{\frac{1}{3}}}{x^2+x+1}\left(\ln(x) + \frac{13}{24}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{3}}}{x^2+x+1} + \frac{c_2 x^{\frac{1}{3}}(24\ln(x)+13)}{24x^2+24x+24} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{3}}}{x^2+x+1} + \frac{c_2 x^{\frac{1}{3}}(24\ln(x)+13)}{24x^2+24x+24}$$

Verified OK.

2.585.1 Maple step by step solution

Let's solve

$(9x^4+9x^3+9x^2)\, y'' + (39x^3+21x^2+3x)\, y' + (25x^2+4x+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} - \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(13x^2+7x+1)y'}{3x(x^2+x+1)} + \frac{(25x^2+4x+1)y}{9x^2(x^2+x+1)} = 0$$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $$\left[P_2(x) = \frac{13x^2+7x+1}{3x(x^2+x+1)}, P_3(x) = \frac{25x^2+4x+1}{9x^2(x^2+x+1)}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{3}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{1}{9}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$9x^2(x^2+x+1)\, y'' + 3x(13x^2+7x+1)\, y' + (25x^2+4x+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)^2 x^r + \left(a_1(2+3r)^2 + a_0(2+3r)^2\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-1)^2 + a_{k-1}(3k+3r-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{3}$$

- Each term must be 0

$$a_1(2+3r)^2 + a_0(2+3r)^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -a_0$$

- Each term in the series must be 0, giving the recursion relation

$$(3k+3r-1)^2 (a_k + a_{k-1} + a_{k-2}) = 0$$

- Shift index using $k->k+2$

$$(3k+3r+5)^2 (a_{k+2} + a_{k+1} + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -a_{k+1} - a_k$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -a_{k+1} - a_k$$

- Solution for $r = \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -a_{k+1} - a_k, a_1 = -a_0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(9*x^2*(1+x+x^2)*diff(y(x),x$2)+3*x*(1+7*x+13*x^2)*diff(y(x),x)+(1+4*x+25*x^2)*y(x)=0,
```

$$y(x) = \frac{c_1 x^{\frac{1}{3}}}{x^2 + x + 1} + \frac{c_2 x^{\frac{1}{3}} \ln(x)}{x^2 + x + 1}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 27

```
DSolve[9*x^2*(1+x+x^2)*y''[x]+3*x*(1+7*x+13*x^2)*y'[x]+(1+4*x+25*x^2)*y[x]==0,y[x],x,Include
```

$$y(x) \to \frac{\sqrt[3]{x}(c_2 \log(x) + c_1)}{x^2 + x + 1}$$

2.586 problem 600

2.586.1 Maple step by step solution 5505

Internal problem ID [8076]
Internal file name [OUTPUT/7009_Sunday_June_05_2022_05_24_40_PM_59052262/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 600.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x+2)\,y''-x(4-7x)\,y'-(5-3x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+(-5+3x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=7x^2-4x\\C&=-5+3x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 32x + 128$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{2x} + \frac{5}{2(x+2)} + \frac{45}{16(x+2)^2} + \frac{2}{x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 32x + 128}{16(x^2+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 32x + 128}{16\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{3}{4} - \left(\frac{3}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{5}{4\left(x+2\right)} + \frac{2}{x} + (0) \\
&= -\frac{5}{4\left(x+2\right)} + \frac{2}{x} \\
&= \frac{3x+16}{4x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{4(x+2)} + \frac{2}{x}\right)(0) + \left(\left(\frac{5}{4(x+2)^2} - \frac{2}{x^2}\right) + \left(-\frac{5}{4(x+2)} + \frac{2}{x}\right)^2 - \left(\frac{-3x^2 - 32x + 128}{16(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{5}{4(x+2)} + \frac{2}{x}\right) dx} \\ &= \frac{x^2}{(x+2)^{\frac{5}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{7x^2 - 4x}{2x^3 + 4x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{9\ln(x+2)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(x+2)^{\frac{9}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x^2-4x}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{9\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 2\sqrt{x+2}\,(33x^2+52x+32)}{48x^3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}}\right) \\
&\quad + c_2\left(\frac{x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}}\left(\frac{-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 2\sqrt{x+2}\,(33x^2+52x+32)}{48x^3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} \\
&\quad + \frac{c_2\left(-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 66x^2\sqrt{x+2} - 104\sqrt{x+2}\,x - 64\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} \\
&\quad + \frac{c_2\left(-15\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 - 66x^2\sqrt{x+2} - 104\sqrt{x+2}\,x - 64\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}
\end{aligned}
$$

Verified OK.

2.586.1 Maple step by step solution

Let's solve

$\left(2x^3+4x^2\right)y''+\left(7x^2-4x\right)y'+\left(-5+3x\right)y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-5+3x)y}{2x^2(x+2)}-\frac{(7x-4)y'}{2x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(7x-4)y'}{2x(x+2)}+\frac{(-5+3x)y}{2x^2(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{7x-4}{2x(x+2)}, P_3(x)=\frac{-5+3x}{2x^2(x+2)}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{9}{2}$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)y''+x(7x-4)y'+(-5+3x)y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $\left(2u^3-8u^2+8u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(7u^2-32u+36\right)\left(\frac{d}{du}y(u)\right)+\left(-11+3u\right)y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0r(7+2r)\, u^{-1+r} + \left(4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r(7+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, -\frac{7}{2}\right\}$$

- Each term must be 0

  $$4a_1(1+r)\,(9+2r) - a_0(8r^2+24r+11) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + \left(4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 24a_k + a_{k-1} + 44a_{k+1}\right) k + 2(-4a_k +$$

- Shift index using $k->k+1$

  $$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + \left(4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 24a_{k+1} + a_k + 44a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2 a_k - 8r^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 5ra_k - 40ra_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 4kr + 2r^2 + 15k + 15r + 22)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a_1 - 11a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a_1 - 11a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}$$

- Solution for $r = -\frac{7}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}, -20a_1 - 25a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{7}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 9ka_k + 16ka_{k+1} + 10a_k - a_{k+1}}{4(2k^2 + k - 6)}, -20a_1 - 25a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{7}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 5ka_k - 40ka_{k+1} + 3a_k - 43a_{k+1}}{4(2k^2 + 15k + 22)}, 36a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 80

```
dsolve(2*x^2*(2+x)*diff(y(x),x$2)-x*(4-7*x)*diff(y(x),x)-(5-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{5}{2}}}{(x+2)^{\frac{7}{2}}} - \frac{c_2\sqrt{2}\left(33\sqrt{2}\,\sqrt{x+2}\,x^2 + 15\,\mathrm{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)x^3 + 52\sqrt{2}\,\sqrt{x+2}\,x + 32\sqrt{2}\,\sqrt{x+2}\right)}{48\sqrt{x}\,(x+2)^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 92

```
DSolve[2*x^2*(2+x)*y''[x]-x*(4-7*x)*y'[x]-(5-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{15\sqrt{2}c_2x^3\mathrm{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right) - 48c_1x^3 + 66c_2\sqrt{x+2}x^2 + 104c_2\sqrt{x+2}x + 64c_2\sqrt{x+2}}{48\sqrt{x}(x+2)^{7/2}}$$

2.587 problem 601

2.587.1 Maple step by step solution 5515

Internal problem ID [8077]
Internal file name [OUTPUT/7010_Sunday_June_05_2022_05_24_43_PM_95181498/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 601.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1-2x)\, y''+x(8-9x)\, y'+(6-3x)\, y=0}$$

Writing the ode as

$$\left(-2x^3+x^2\right) y''+\left(-9x^2+8x\right) y'+(6-3x)\, y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+x^2 \\ B &= -9x^2+8x \\ C &= 6-3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^2 - 20x + 24$$
$$t = 4\left(2x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2} + \frac{19}{x} + \frac{77}{16\left(x - \frac{1}{2}\right)^2} - \frac{19}{x - \frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{77}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{11}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{7}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{21}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{21x^2-20x+24}{4\left(2x^2-x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2
$\frac{1}{2}$	2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{7}{4}-\left(\frac{3}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)} + (0) \\
&= -\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)} \\
&= \frac{4 + 3x}{4x^2 - 2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right)(1) + \left(\left(\frac{2}{x^2} - \frac{11}{4\left(x - \frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right)^2 - \left(\frac{21x^2 - 20x + 24}{4\left(2x^2 - x\right)^2}\right)\right)$$

$$\frac{4 - 3a_0}{x\left(2x - 1\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{4}{3}\right) e^{\int \left(-\frac{2}{x} + \frac{11}{4\left(x - \frac{1}{2}\right)}\right) dx} \\
&= \left(x + \frac{4}{3}\right) e^{-2\ln(x) + \frac{11\ln(2x-1)}{4}} \\
&= \frac{\left(x + \frac{4}{3}\right)\left(2x - 1\right)^{\frac{11}{4}}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-9x^2+8x}{-2x^3+x^2}\,dx} \\
&= z_1 e^{-4\ln(x)+\frac{7\ln(2x-1)}{4}} \\
&= z_1 \left(\frac{(2x-1)^{\frac{7}{4}}}{x^4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-9x^2+8x}{-2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-8\ln(x)+\frac{7\ln(2x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-231x^3+198x^2-66x+8}{(2x-1)^{\frac{9}{2}}\,(1155x+1540)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}\right) + c_2\left(\frac{(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6}\left(\frac{-231x^3+198x^2-66x+8}{(2x-1)^{\frac{9}{2}}\,(1155x+1540)}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1(4+3x)\,(2x-1)^{\frac{9}{2}}}{3x^6} + \frac{c_2(-231x^3+198x^2-66x+8)}{1155x^6} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(4+3x)(2x-1)^{\frac{9}{2}}}{3x^6} + \frac{c_2(-231x^3+198x^2-66x+8)}{1155x^6}$$

Verified OK.

2.587.1 Maple step by step solution

Let's solve

$(-2x^3+x^2)\,y''+(-9x^2+8x)\,y'+(6-3x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3(x-2)y}{x^2(2x-1)}-\frac{(9x-8)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(9x-8)y'}{x(2x-1)}+\frac{3(x-2)y}{x^2(2x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{9x-8}{x(2x-1)}, P_3(x)=\frac{3(x-2)}{x^2(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=8$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=6$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x^2(2x-1)+x(9x-8)\,y'+(3x-6)\,y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k + 2 - m + r)\, (k + 1 - m + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 (6 + r)\, (1 + r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_k (k + r + 6)\, (k + r + 1) + a_{k-1} (k + 2 + r)\, (2k - 1 + 2r)\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(6 + r)\, (1 + r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, -1\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(k + 2 + r)\left(k + r - \frac{1}{2}\right) a_{k-1} - a_k (k + r + 6)\, (k + r + 1) = 0$$

- Shift index using $k- >k + 1$

$$2(k + r + 3)\left(k + \frac{1}{2} + r\right) a_k - a_{k+1} (k + 7 + r)\, (k + 2 + r) = 0$$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r+3)(2k+2r+1)a_k}{(k+7+r)(k+2+r)}$

- Recursion relation for $r = -6$; series terminates at $k = 3$

  $a_{k+1} = \frac{(k-3)(2k-11)a_k}{(k+1)(k-4)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{33a_0}{4}$

- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$

- Express in terms of a_0

  $a_2 = \frac{99a_0}{4}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{7a_2}{6}$

- Express in terms of a_0

  $a_3 = -\frac{231a_0}{8}$

- Terminating series solution of the ODE for $r = -6$. Use reduction of order to find the second

  $y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right)$

- Recursion relation for $r = -1$

  $a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{(k+2)(2k-1)a_k}{(k+6)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(-\frac{231}{8}x^3 + \frac{99}{4}x^2 - \frac{33}{4}x + 1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), b_{k+1} = \frac{(k+2)(2k-1)b_k}{(k+6)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(x^2*(1-2*x)*diff(y(x),x$2)+x*(8-9*x)*diff(y(x),x)+(6-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(231x^3 - 198x^2 + 66x - 8)}{x^6} + \frac{c_2(3x+4)(-1+2x)^{\frac{9}{2}}}{x^6}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 49

```
DSolve[x^2*(1-2*x)*y''[x]+x*(8-9*x)*y'[x]+(6-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{c_2(231x^3 - 198x^2 + 66x - 8) + 385c_1(3x+4)(1-2x)^{9/2}}{1155x^6}$$

2.588 problem 602

2.588.1 Maple step by step solution 5525

Internal problem ID [8078]
Internal file name [OUTPUT/7011_Sunday_June_05_2022_05_24_47_PM_7324665/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 602.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+x(10x^2+3)\,y'-(-14x^2+15)\,y=0}$$

Writing the ode as

$$\left(x^4+x^2\right)y''+\left(10x^3+3x\right)y'+\left(14x^2-15\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 10x^3+3x \\ C &= 14x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 24x^4 + 66x^2 + 63 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{99i}{16(x-i)} - \frac{99i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{24x^4 + 66x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= 3 - (3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} + (0) \\
&= \frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)} \\
&= \frac{9}{2x} - \frac{3x}{2x^2+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right)(0) + \left(\left(-\frac{9}{2x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2}\right) + \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{9}{2x} - \frac{3}{4(x-i)} - \frac{3}{4(x+i)}\right) dx} \\
&= \frac{x^{\frac{9}{2}}}{(x^2+1)^{\frac{3}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{10x^3+3x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}-\frac{7\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}\left(x^2+1\right)^{\frac{7}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{10x^3+3x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)-\frac{7\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{-3\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8+\left(3x^6-2x^4-24x^2-16\right)\sqrt{x^2+1}}{128x^8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}\right) \\
&\quad + c_2\left(\frac{x^3}{\left(x^2+1\right)^{\frac{5}{2}}}\left(\frac{-3\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right)x^8+\left(3x^6-2x^4-24x^2-16\right)\sqrt{x^2+1}}{128x^8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{3c_2\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^8 - (x^2+2)\left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)\sqrt{x^2+1}\right)}{128\,(x^2+1)^{\frac{5}{2}}\, x^5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{3c_2\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^8 - (x^2+2)\left(x^4 - \frac{8}{3}x^2 - \frac{8}{3}\right)\sqrt{x^2+1}\right)}{128\,(x^2+1)^{\frac{5}{2}}\, x^5}$$

Verified OK.

2.588.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y''+(10x^3+3x)\,y'+(14x^2-15)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(14x^2-15)y}{x^2(x^2+1)}-\frac{(10x^2+3)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(10x^2+3)y'}{x(x^2+1)}+\frac{(14x^2-15)y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{10x^2+3}{x(x^2+1)}, P_3(x)=\frac{14x^2-15}{x^2(x^2+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-15$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2(x^2+1)\,y'' + x(10x^2+3)\,y' + (14x^2-15)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(5+r)\,(-3+r)\,x^r + a_1(6+r)\,(-2+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+5)\,(k+r-3) + a_{k-2}(k+r-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(5+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-5, 3\}$

- Each term must be 0

$$a_1(6+r)(-2+r)=0$$

- Solve for the dependent coefficient(s)

  $$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+5)\left(a_k(k+r-3)+a_{k-2}(k+r)\right)=0$$

- Shift index using $k->k+2$

  $$(k+r+7)\left(a_{k+2}(k+r-1)+a_k(k+r+2)\right)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{a_k(k+r+2)}{k+r-1}$$

- Recursion relation for $r=-5$

  $$a_{k+2}=-\frac{a_k(k-3)}{k-6}$$

- Series not valid for $r=-5$, division by 0 in the recursion relation at $k=6$

  $$a_{k+2}=-\frac{a_k(k-3)}{k-6}$$

- Recursion relation for $r=3$

  $$a_{k+2}=-\frac{a_k(k+5)}{k+2}$$

- Solution for $r=3$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+3},a_{k+2}=-\frac{a_k(k+5)}{k+2},a_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(3+10*x^2)*diff(y(x),x)-(15-14*x^2)*y(x)=0,y(x), singsol
```

$$y(x) = \frac{c_1 x^3}{(x^2+1)^{\frac{5}{2}}} - \frac{c_2 \left(-3 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^8 + 3\sqrt{x^2+1}\, x^6 - 2x^4\sqrt{x^2+1} - 24\sqrt{x^2+1}\, x^2 - 16\sqrt{x^2+1}\right)}{128 x^5 (x^2+1)^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 75

```
DSolve[x^2*(1+x^2)*y''[x]+x*(3+10*x^2)*y'[x]-(15-14*x^2)*y[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{c_2\left(\sqrt{x^2+1}(3x^6-2x^4-24x^2-16) - 3x^8 \operatorname{arctanh}\left(\sqrt{x^2+1}\right)\right) + 128 c_1 x^8}{128 x^5 (x^2+1)^{5/2}}$$

2.589 problem 603

2.589.1 Maple step by step solution 5535

Internal problem ID [8079]
Internal file name [OUTPUT/7012_Sunday_June_05_2022_05_24_50_PM_73280530/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 603.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(-2x^2+1)\,y''+x(-13x^2+7)\,y'-14x^2y=0}$$

Writing the ode as

$$(-2x^4+x^2)\,y''+(-13x^3+7x)\,y'-14x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^4+x^2 \\ B &= -13x^3+7x \\ C &= -14x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^4 - 68x^2 + 35$$
$$t = 4\left(2x^3 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^3 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{9}{64\left(x - \frac{\sqrt{2}}{2}\right)^2} + \frac{9}{64\left(x + \frac{\sqrt{2}}{2}\right)^2} - \frac{279\sqrt{2}}{64\left(x - \frac{\sqrt{2}}{2}\right)} + \frac{279\sqrt{2}}{64\left(x + \frac{\sqrt{2}}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = \frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^4 - 68x^2 + 35}{4\left(2x^3 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$
$-\frac{\sqrt{2}}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-x^2+5}{4x^3-2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right)(0) + \left(\left(\frac{5}{2x^2} - \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)^2} - \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)^2} \right) + \left(-\frac{5}{2x} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{5}{2x} + \frac{9}{8\left(x-\frac{\sqrt{2}}{2}\right)} + \frac{9}{8\left(x+\frac{\sqrt{2}}{2}\right)} \right) dx} \\
&= \frac{\left(2x-\sqrt{2}\right)^{\frac{9}{8}} \left(2x+\sqrt{2}\right)^{\frac{9}{8}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-13x^3+7x}{-2x^4+x^2}\,dx} \\
&= z_1 e^{\frac{\ln\left(2x^2-1\right)}{8}-\frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{\left(2x^2-1\right)^{\frac{1}{8}}}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-13x^3+7x}{-2x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(2x^2-1\right)}{4}-7\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{\left(5x^4-20x^2+8\right)2^{\frac{3}{4}}}{120\left(2x^2-1\right)^{\frac{5}{4}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}\right) + c_2\left(\frac{2\,2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6}\left(\frac{\left(5x^4-20x^2+8\right)2^{\frac{3}{4}}}{120\left(2x^2-1\right)^{\frac{5}{4}}}\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{2c_1 2^{\frac{1}{8}}\left(2x^2-1\right)^{\frac{5}{4}}}{x^6} + \frac{c_2 2^{\frac{7}{8}}\left(5x^4-20x^2+8\right)}{60x^6} \tag{1}$$

Verification of solutions

$$y = \frac{2c_1 2^{\frac{1}{8}} (2x^2-1)^{\frac{5}{4}}}{x^6} + \frac{c_2 2^{\frac{7}{8}} (5x^4 - 20x^2 + 8)}{60x^6}$$

Verified OK.

2.589.1 Maple step by step solution

Let's solve

$(-2x^4 + x^2)\, y'' + (-13x^3 + 7x)\, y' - 14x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{14y}{2x^2-1} - \frac{(13x^2-7)y'}{x(2x^2-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(13x^2-7)y'}{x(2x^2-1)} + \frac{14y}{2x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{13x^2-7}{x(2x^2-1)}, P_3(x) = \frac{14}{2x^2-1}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $14yx + (13x^2 - 7)\, y' + xy''(2x^2 - 1) = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(6+r)\, x^{-1+r} - a_1(1+r)\,(7+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-a_{k+1}(k+r+1)\,(k+7+r) + a_{k-1}(2k+5+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(6+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-6, 0\}$$

- Each term must be 0

$$-a_1(1+r)\,(7+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r+\frac{5}{2}\right) a_{k-1} - \frac{a_{k+1}(k+7+r)}{2}\right)(k+r+1) = 0$$

- Shift index using $k->k+1$

  $$2\left(\left(k+\frac{7}{2}+r\right)a_k-\frac{a_{k+2}(k+8+r)}{2}\right)(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{(2k+2r+7)a_k}{k+8+r}$$

- Recursion relation for $r=-6$

  $$a_{k+2}=\frac{(2k-5)a_k}{k+2}$$

- Solution for $r=-6$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k-6},a_{k+2}=\frac{(2k-5)a_k}{k+2},5a_1=0\right]$$

- Recursion relation for $r=0$

  $$a_{k+2}=\frac{(2k+7)a_k}{k+8}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^k,a_{k+2}=\frac{(2k+7)a_k}{k+8},-7a_1=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-6}\right)+\left(\sum_{k=0}^{\infty}b_kx^k\right),a_{k+2}=\frac{(2k-5)a_k}{k+2},5a_1=0,b_{k+2}=\frac{(2k+7)b_k}{k+8},-7b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(x^2*(1-2*x^2)*diff(y(x),x$2)+x*(7-13*x^2)*diff(y(x),x)-14*x^2*y(x)=0,y(x), singsol=al
```

$$y(x) = \frac{c_1(5x^4 - 20x^2 + 8)}{x^6} + \frac{c_2(2x^2 - 1)^{\frac{5}{4}}}{x^6}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 43

```
DSolve[x^2*(1-2*x^2)*y''[x]+x*(7-13*x^2)*y'[x]-14*x^2*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{15c_1(1 - 2x^2)^{5/4} + c_2(-5x^4 + 20x^2 - 8)}{15x^6}$$

2.590 problem 604

2.590.1 Maple step by step solution 5545

Internal problem ID [8080]
Internal file name [`OUTPUT/7013_Sunday_June_05_2022_05_24_53_PM_5571297/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 604.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(1+x)\,y''+4x(2x+1)\,y'-(3x+1)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(8x^2+4x\right)y'+(-3x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 8x^2+4x \\ C &= -3x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4 + 3x}{4x\,(1 + x)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 + 3x \\
t &= 4x(1 + x)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{4 + 3x}{4x\,(1 + x)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(1+x)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{1}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4+3x}{4x(1+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{4+3x}{4x\left(1+x\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= \frac{3}{2}-\left(\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{x} + \frac{1}{2x+2} \\
&= \frac{1}{x} + \frac{1}{2x+2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{x} + \frac{1}{2x+2}\right)(0) + \left(\left(-\frac{1}{x^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{x} + \frac{1}{2x+2}\right)^2 - \left(\frac{4+3x}{4x(1+x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} + \frac{1}{2x+2}\right) dx} \\
&= \sqrt{1+x}\, x
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8x^2+4x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x(1+x))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x(1+x)}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(1+x))}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}\right) + c_2 \left(\frac{\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}}\left(-\frac{1}{x} - \ln(x) + \ln(1+x)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}} + \frac{c_2(-x\ln(x) + \ln(1+x)\,x - 1)\sqrt{1+x}}{\sqrt{x\,(1+x)}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{1+x}\,x}{\sqrt{x\,(1+x)}} + \frac{c_2(-x\ln(x) + \ln(1+x)\,x - 1)\sqrt{1+x}}{\sqrt{x\,(1+x)}}$$

Verified OK.

2.590.1 Maple step by step solution

Let's solve

$(4x^3 + 4x^2)\, y'' + (8x^2 + 4x)\, y' + (-3x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+1)y}{4x^2(1+x)} - \frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x(1+x)} - \frac{(3x+1)y}{4x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x+1}{x(1+x)}, P_3(x) = -\frac{3x+1}{4x^2(1+x)}\right]$

  ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\, y'' + 4x(2x+1)\, y' + (-3x - 1)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2} y(u)\right) + (8u^2 - 12u + 4)\left(\frac{d}{du} y(u)\right) + (-3u + 2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$4a_0 r^2 u^{-1+r} + \left(4a_1(1+r)^2 - 2a_0(4r^2 + 2r - 1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+1+r)^2 - 2a_k(4k^2 + 8kr -$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$4a_1(1+r)^2 - 2a_0(4r^2 + 2r - 1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(4k^2 - 4k - 3)\, a_{k-1} + (-8k^2 - 4k + 2)\, a_k + 4a_{k+1}(k+1)^2 = 0$$

- Shift index using $k- >k + 1$

  $$\left(4(k+1)^2 - 4k - 7\right) a_k + \left(-8(k+1)^2 - 4k - 2\right) a_{k+1} + 4a_{k+2}(k+2)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}, 4a_1 + 2a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{4k^2 a_k - 8k^2 a_{k+1} + 4k a_k - 20k a_{k+1} - 3a_k - 10a_{k+1}}{4(k+2)^2}, 4a_1 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)-(1+3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + \frac{c_2(x\ln(x) - \ln(x+1)x + 1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 32

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(1+2*x)*y'[x]-(1+3*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{c_1 x + c_2(-x\log(x) + x\log(x+1) - 1)}{\sqrt{x}}$$

2.591 problem 605

2.591.1 Maple step by step solution 5555

Internal problem ID [8081]
Internal file name [`OUTPUT/7014_Sunday_June_05_2022_05_24_56_PM_72178685/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 605.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(3x+2)\,y''+x(4+21x)\,y'-(1-9x)\,y=0}$$

Writing the ode as

$$\left(6x^3+4x^2\right)y''+\left(21x^2+4x\right)y'+(9x-1)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^3+4x^2 \\ B &= 21x^2+4x \\ C &= 9x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-27x - 48}{16x\,(3x + 2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -27x - 48 \\ t &= 16x(3x + 2)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-27x - 48}{16x\,(3x + 2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(3x+2)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = -\frac{2}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{5}{16\left(x+\frac{2}{3}\right)^2} + \frac{3}{4\left(x+\frac{2}{3}\right)}$$

For the pole at $x = -\frac{2}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{2}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-27x - 48}{16x\left(3x+2\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-27x - 48}{16x\left(3x+2\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
$-\frac{2}{3}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= \frac{3}{4} - \left(\frac{3}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} + (0) \\ &= \frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)} \\ &= \frac{9x+8}{12x^{2}+8x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4\left(x+\frac{2}{3}\right)^2}\right) + \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right)^2 - \left(\frac{-27x-48}{16x\left(3x+2\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4\left(x+\frac{2}{3}\right)}\right) dx} \\ &= \frac{x}{\left(3x+2\right)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{21x^2+4x}{6x^3+4x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{5\ln(3x+2)}{4}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\left(3x+2\right)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(3x+2\right)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{21x^2+4x}{6x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)-\frac{5\ln(3x+2)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-3\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right) x - 2\sqrt{3x+2}}{2x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(3x+2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x}}{(3x+2)^{\frac{3}{2}}} \left(\frac{-3\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right) x - 2\sqrt{3x+2}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}} - \frac{3c_2\left(\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right) x + \frac{2\sqrt{3x+2}}{3}\right)}{2\,(3x+2)^{\frac{3}{2}}\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}} - \frac{3c_2\left(\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right) x + \frac{2\sqrt{3x+2}}{3}\right)}{2\,(3x+2)^{\frac{3}{2}}\sqrt{x}}
$$

Verified OK.

2.591.1 Maple step by step solution

Let's solve

$(6x^3 + 4x^2) y'' + (21x^2 + 4x) y' + (9x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x-1)y}{2x^2(3x+2)} - \frac{(4+21x)y'}{2x(3x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4+21x)y'}{2x(3x+2)} + \frac{(9x-1)y}{2x^2(3x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{4+21x}{2x(3x+2)}, P_3(x) = \frac{9x-1}{2x^2(3x+2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2(3x + 2) y'' + x(4 + 21x) y' + (9x - 1) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+2r)\,(-1+2r)\, x^{r}+\left(\sum_{k=1}^{\infty} \left(a_{k}(2k+2r+1)\,(2k+2r-1)+3a_{k-1}(2k+2r+1)\,(k+r)\right) x^{k+}\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(\left(k+r-\frac{1}{2}\right) a_{k}+\frac{3a_{k-1}(k+r)}{2}\right)\left(k+r+\frac{1}{2}\right)=0$

- Shift index using $k->k+1$

  $4\left(\left(k+r+\frac{1}{2}\right) a_{k+1}+\frac{3a_{k}(k+r+1)}{2}\right)\left(k+\frac{3}{2}+r\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{3a_{k}(k+r+1)}{2k+2r+1}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{3a_{k}\left(k+\frac{1}{2}\right)}{2k}$

- Solution for $r=-\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{3a_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{3a_k\left(k+\frac{1}{2}\right)}{2k}, b_{k+1} = -\frac{3b_k\left(k+\frac{3}{2}\right)}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(2*x^2*(2+3*x)*diff(y(x),x$2)+x*(4+21*x)*diff(y(x),x)-(1-9*x)*y(x)=0,y(x), singsol=all
```

$$y(x) = \frac{c_1\sqrt{x}}{(3x+2)^{\frac{3}{2}}} + \frac{c_2\sqrt{2}\left(\sqrt{2}\,\sqrt{3x+2} + 3\,\operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{3x+2}}{2}\right)x\right)}{2\sqrt{x}\,(3x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 64

```
DSolve[2*x^2*(2+3*x)*y''[x]+x*(4+21*x)*y'[x]-(1-9*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{3\sqrt{2}c_2 x \operatorname{arctanh}\left(\sqrt{\frac{3x}{2}+1}\right) - 2c_1 x + 2c_2\sqrt{3x+2}}{2\sqrt{x}(3x+2)^{3/2}}$$

2.592 problem 606

2.592.1 Maple step by step solution 5566

Internal problem ID [8082]
Internal file name [`OUTPUT/7015_Sunday_June_05_2022_05_25_00_PM_30007777/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 606.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(x+2)\,y' - (2-3x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+2x\right)y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+2x \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{x} - \frac{2}{x^2} - \frac{8}{x^3} - \frac{36}{x^4} - \frac{176}{x^5} - \frac{912}{x^6} - \frac{4928}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-8x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-8x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned}b &= (-2) - (0) \\ &= -2\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 8x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{2}{x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{2}{x} - \frac{1}{2} \\ &= -\frac{x-4}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{2}{x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-8x+8}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2}\right) dx} \\ &= x^2 \mathrm{e}^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2+2x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2} - \ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x^3 - \mathrm{e}^{x}(x^2+x+2)}{6x^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x}\right) + c_2\left(x\,\mathrm{e}^{-x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x^3 - \mathrm{e}^{x}(x^2+x+2)}{6x^3}\right)\right)
\end{aligned}$$

<ins>Summary</ins>
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x^3\mathrm{e}^{-x} - x^2 - x - 2)}{6x^2} \tag{1}$$

<ins>Verification of solutions</ins>

$$y = c_1 x\,\mathrm{e}^{-x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x^3\mathrm{e}^{-x} - x^2 - x - 2)}{6x^2}$$

Verified OK.

2.592.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + 2x)\, y' + (3x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{x^2} - \frac{(x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+2)y'}{x} + \frac{(3x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x+2}{x}, P_3(x) = \frac{3x-2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(x+2)\, y' + (3x - 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + a_{k-1}(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r+2)\,(a_k(k+r-1) + a_{k-1}) = 0$$

- Shift index using $k->k+1$

  $$(k+r+3)\,(a_{k+1}(k+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+r}$$

- Recursion relation for $r = -2$

  $$a_{k+1} = -\frac{a_k}{k-2}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $$a_{k+1} = -\frac{a_k}{k-2}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve(x^2*diff(y(x),x$2)+x*(2+x)*diff(y(x),x)-(2-3*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} x + \frac{c_2 e^{-x} (\text{expIntegral}_1 (-x) x^3 + e^x x^2 + x e^x + 2 e^x)}{6x^2}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 46

```
DSolve[x^2*y''[x]+x*(2+x)*y'[x]-(2-3*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x} (c_2 (x^3 \text{ExpIntegralEi}(x) - e^x (x^2 + x + 2)) + 6c_1 x^3)}{6x^2}$$

2.593 problem 607

2.593.1 Maple step by step solution 5574

Internal problem ID [8083]
Internal file name [OUTPUT/7016_Sunday_June_05_2022_05_25_03_PM_5040625/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 607.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(1+x)\,y''+4x(3+8x)\,y'-(5-49x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(32x^2+12x\right)y'+(49x-5)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 32x^2+12x \\ C &= 49x-5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 8x + 8$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{6}{x} + \frac{15}{4(1+x)^2} + \frac{2}{x^2} + \frac{6}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 8x + 8}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 - 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2\left(1 + x\right)} + \frac{2}{x} + (-)\left(0\right) \\
&= -\frac{3}{2\left(1 + x\right)} + \frac{2}{x} \\
&= \frac{4 + x}{2x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\left(1+x\right)} + \frac{2}{x}\right)(0) + \left(\left(\frac{3}{2\left(1+x\right)^2} - \frac{2}{x^2}\right) + \left(-\frac{3}{2\left(1+x\right)} + \frac{2}{x}\right)^2 - \left(\frac{-x^2 - 8x + 8}{4\left(x^2+x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(1+x)} + \frac{2}{x}\right) dx} \\
&= \frac{x^2}{\left(1+x\right)^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{32x^2+12x}{4x^3+4x^2} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2} - \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}\left(1+x\right)^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{\left(1+x\right)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{32x^2+12x}{4x^3+4x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-3\ln(x)-5\ln(1+x)}}{(y_1)^2}\,dx$$
$$= y_1\left(-\frac{3}{x}-\frac{3}{2x^2}+\ln(x)-\frac{1}{3x^3}\right)$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$
$$= c_1\left(\frac{\sqrt{x}}{(1+x)^4}\right) + c_2\left(\frac{\sqrt{x}}{(1+x)^4}\left(-\frac{3}{x}-\frac{3}{2x^2}+\ln(x)-\frac{1}{3x^3}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}}{(1+x)^4} + \frac{c_2(6x^3\ln(x) - 18x^2 - 9x - 2)}{6x^{\frac{5}{2}}(1+x)^4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}}{(1+x)^4} + \frac{c_2(6x^3\ln(x) - 18x^2 - 9x - 2)}{6x^{\frac{5}{2}}(1+x)^4}$$

Verified OK.

2.593.1 Maple step by step solution

Let's solve

$(4x^3+4x^2)\,y'' + (32x^2+12x)\,y' + (49x-5)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(49x-5)y}{4x^2(1+x)} - \frac{(3+8x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(3+8x)y'}{x(1+x)} + \frac{(49x-5)y}{4x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{3+8x}{x(1+x)}, P_3(x) = \frac{49x-5}{4x^2(1+x)}\right]$

- ◦ $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 5$

- ◦ $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\, y'' + 4x(3+8x)\, y' + (49x-5)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (32u^2 - 52u + 20)\left(\frac{d}{du}y(u)\right) + (49u - 54)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(4+r)\,u^{-1+r} + \left(4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27)\right)u^r + \left(\sum_{k=1}^{\infty} \left(4a_{k+1}(k+1+r)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(4+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-4, 0\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+r) - 2a_0(4r^2+22r+27) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4a_{k+1}(k+1+r)\,(k+5+r) - 2a_k(4k^2+8kr+4r^2+22k+22r+27) + a_{k-1}(2k+5+2r)^2 = 0$$

- Shift index using $k- >k+1$

$$4a_{k+2}(k+2+r)\,(k+6+r) - 2a_{k+1}\left(4(k+1)^2+8(k+1)\,r+4r^2+22k+49+22r\right) + a_k(2k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+8kra_k-16kra_{k+1}+4r^2a_k-8r^2a_{k+1}+28ka_k-60ka_{k+1}+28ra_k-60ra_{k+1}+49a_k-106a_{k+1}}{4(k+2+r)(k+6+r)}$$

- Recursion relation for $r = -4$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}-4ka_k+4ka_{k+1}+a_k+6a_{k+1}}{4(k-2)(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+28ka_k-60ka_{k+1}+49a_k-106a_{k+1}}{4(k+2)(k+6)}, 20a_1-54a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(3+8*x)*diff(y(x),x)-(5-49*x)*y(x)=0,y(x), singsol=all
```

$$y(x)=\frac{c_1\sqrt{x}}{(x+1)^4}+\frac{c_2(6x^3\ln(x)-18x^2-9x-2)}{6(x+1)^4 x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 52

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(3+8*x)*y'[x]-(5-49*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x)\to \frac{6c_1x^3+6c_2x^3\log(x)-18c_2x^2-9c_2x-2c_2}{6x^{5/2}(x+1)^4}$$

2.594 problem 608

2.594.1 Maple step by step solution 5584

Internal problem ID [8084]
Internal file name [OUTPUT/7017_Sunday_June_05_2022_05_25_06_PM_64920767/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 608.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''-x(3+10x)\,y'+30yx=0}$$

The ODE is

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30yx=0$$

Or

$$x\left(x^2y''-10xy'+xy''+30y-3y'\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(x^2+x\right)y''+(-3-10x)\,y'+30y=0$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2-3x\right)y'+30yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-10x^2-3x\\C&=30x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-48x + 15}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -48x + 15 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-48x + 15}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 1 \\
&= 3
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{39}{2x} + \frac{63}{4(1 + x)^2} + \frac{15}{4x^2} + \frac{39}{2(1 + x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-48x + 15}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right)\\
&= 0 - (-1)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_\infty\\
&= -\frac{7}{2(1+x)} + \frac{5}{2x} + (0)\\
&= -\frac{7}{2(1+x)} + \frac{5}{2x}\\
&= -\frac{2x-5}{2x(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)(1) + \left(\left(\frac{7}{2(1+x)^2} - \frac{5}{2x^2}\right) + \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right)^2 - \left(\frac{-48x+15}{4(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{5+2a_0}{x(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{5}{2}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{2}\right) \mathrm{e}^{\int \left(-\frac{7}{2(1+x)} + \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{2}\right) \mathrm{e}^{\frac{5\ln(x)}{2} - \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{2}\right) x^{\frac{5}{2}}}{(1+x)^{\frac{7}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-10x^2 - 3x}{x^2(1+x)} \, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} + \frac{7\ln(1+x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} (1+x)^{\frac{7}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^5 - \frac{5}{2} x^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2 - 3x}{x^2(1+x)} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3\ln(x) + 7\ln(1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(x - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x) - \frac{823543}{6250(2x - 5)}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(x^5 - \frac{5}{2}x^4\left(x - \frac{1}{25x^4} - \frac{52}{125x^3} - \frac{1354}{625x^2} - \frac{27708}{3125x} + 12\ln(x) - \frac{823543}{6250(2x-5)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(12\ln(x)\,x^5 + x^6 - 30\ln(x)\,x^4 - \frac{5x^5}{2} - \frac{299x^4}{4} + 20x^3 + 5x^2 + x + \frac{1}{10}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\left(x^5 - \frac{5}{2}x^4\right) \\
&\quad + c_2\left(12\ln(x)\,x^5 + x^6 - 30\ln(x)\,x^4 - \frac{5x^5}{2} - \frac{299x^4}{4} + 20x^3 + 5x^2 + x + \frac{1}{10}\right)
\end{aligned}
$$

Verified OK. `{x <> 0}`

2.594.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y'' + (-10x^2 - 3x)\,y' + 30yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{30y}{x(1+x)} + \frac{(3+10x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3+10x)y'}{x(1+x)} + \frac{30y}{x(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=-\frac{3+10x}{x(1+x)}, P_3(x)=\frac{30}{x(1+x)}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=-7$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)\,y''+(-3-10x)\,y'+30y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(7-10u)\left(\frac{d}{du}y(u)\right)+30y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(-8+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6)\right) u^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-8+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 8\}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k-7+r) + a_k(k+r-5)\,(k+r-6) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-5)(k+r-6)}{(k+1+r)(k-7+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 5$

  $a_{k+1} = \frac{a_k(k-5)(k-6)}{(k+1)(k-7)}$
- Apply recursion relation for $k = 0$

  $a_1 = -\frac{30a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{3}$
- Express in terms of a_0

  $a_2 = \frac{50a_0}{7}$
- Apply recursion relation for $k = 2$

  $a_3 = -\frac{4a_2}{5}$
- Express in terms of a_0

  $a_3 = -\frac{40a_0}{7}$
- Apply recursion relation for $k = 3$

  $a_4 = -\frac{3a_3}{8}$
- Express in terms of a_0

  $a_4 = \frac{15a_0}{7}$
- Apply recursion relation for $k = 4$

  $a_5 = -\frac{2a_4}{15}$
- Express in terms of a_0

$a_5 = -\frac{2a_0}{7}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{30}{7}u + \frac{50}{7}u^2 - \frac{40}{7}u^3 + \frac{15}{7}u^4 - \frac{2}{7}u^5\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right)\right]$

- Recursion relation for $r = 8$

  $a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}$

- Solution for $r = 8$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+8}, a_{k+1} = \frac{a_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(\frac{5}{7}x^4 - \frac{2}{7}x^5\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+8}\right), b_{k+1} = \frac{b_k(k+3)(k+2)}{(k+9)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(3+10*x)*diff(y(x),x)+30*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^5 - \frac{5}{2}x^4\right) + c_2\left(3x^5\ln(x) + \frac{x^6}{4} - \frac{15x^4\ln(x)}{2} - \frac{5x^5}{8} - \frac{299x^4}{16} + 5x^3 + \frac{5x^2}{4} + \frac{x}{4} + \frac{1}{40}\right)$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 68

```
DSolve[x^2*(1+x)*y''[x]-x*(3+10*x)*y'[x]+30*x*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1\left(x^5 - \frac{5x^4}{2}\right) + \frac{1}{20}c_2\left(20x^6 - 50x^5 - 1495x^4 + 120(2x-5)x^4\log(x) + 400x^3 + 100x^2 + 20x + 2\right)$$

2.595 problem 609

2.595.1 Maple step by step solution 5596

Internal problem ID [8085]
Internal file name [`OUTPUT/7018_Sunday_June_05_2022_05_25_09_PM_84391809/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 609.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2} y'' + x(1 + x) y' - 3(x + 3) y = 0}$$

Writing the ode as

$$x^{2} y'' + \left(x^{2} + x\right) y' + \left(-3x - 9\right) y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2} \\ B &= x^{2} + x \\ C &= -3x - 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 14x + 35$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 14x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{7}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{7}{2x} - \frac{7}{2x^2} + \frac{49}{2x^3} - \frac{735}{4x^4} + \frac{5831}{4x^5} - \frac{48363}{4x^6} + \frac{415373}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 14x + 35}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{14x + 35}{4x^2}\right) \\ &= \frac{1}{4} + \frac{14x + 35}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 14. Dividing this by leading coefficient in t which is 4 gives $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 14x + 35}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{7}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^+) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{7}{2x} \\
&= \frac{7 + x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{7}{2x}\right)(0) + \left(\left(-\frac{7}{2x^2}\right) + \left(\frac{1}{2} + \frac{7}{2x}\right)^2 - \left(\frac{x^2 + 14x + 35}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{7}{2x}\right) dx} \\
&= x^{\frac{7}{2}} \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6}{720x^6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^3) + c_2 \left(x^3 \left(\frac{(x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6}{720x^6}\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x^3 + \frac{c_2((x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6)}{720x^3} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^3 + \frac{c_2((x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\, x^6)}{720x^3}$$

Verified OK.

2.595.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2+x)\,y' + (-3x-9)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x+3)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} - \frac{3(x+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = -\frac{3(x+3)}{x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -9$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1+x)\,y' + (-3x-9)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r))\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k-4+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r-3) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r-3)}{(k+4+r)(k-2+r)}$

- Recursion relation for $r = -3$; series terminates at $k = 6$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$

- Series not valid for $r = -3$, division by 0 in the recursion relation at $k = 5$

  $a_{k+1} = -\frac{a_k(k-6)}{(k+1)(k-5)}$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k k}{(k+7)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(x^2*diff(y(x),x$2)+x*(1+x)*diff(y(x),x)-3*(3+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^3 - \frac{c_2(-\operatorname{expIntegral}_1(x)\, x^6 + \mathrm{e}^{-x} x^5 - \mathrm{e}^{-x} x^4 + 2\,\mathrm{e}^{-x} x^3 - 6x^2 \mathrm{e}^{-x} + 24\,\mathrm{e}^{-x} x - 120\,\mathrm{e}^{-x})}{720x^3}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 60

```
DSolve[x^2*y''[x]+x*(1+x)*y'[x]-3*(3+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 e^{-x}(e^x x^6 \operatorname{ExpIntegralEi}(-x) + x^5 - x^4 + 2x^3 - 6x^2 + 24x - 120)}{720x^3} + c_1 x^3$$

2.596 problem 610

2.596.1 Maple step by step solution 5605

Internal problem ID [8086]
Internal file name [OUTPUT/7019_Sunday_June_05_2022_05_25_13_PM_16749410/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 610.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''+x(9+13x)\,y'+(5x+7)\,y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(13x^2+9x\right)y'+(5x+7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=13x^2+9x\\C&=5x+7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 77x^2 + 86x + 35$$
$$t = 4\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{2x} + \frac{35}{4x^2} + \frac{45}{16\left(x + \frac{1}{2}\right)^2} + \frac{27}{2\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{5}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{77x^2 + 86x + 35}{4\left(2x^2 + x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{77}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{11}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{7}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{77x^2+86x+35}{4\left(2x^2+x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{4}$	$-\frac{7}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=-\frac{7}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= -\frac{7}{4}-\left(-\frac{15}{4}\right)\\
&= 2
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} + (-)\left(0\right) \\ &= -\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)} \\ &= \frac{-15x-5}{4x^2+2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)(2x+a_1) + \left(\left(\frac{5}{2x^2} + \frac{5}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{77x^2+86x}{4(2x^2+}\right.\right.$$

$$\frac{(11a_1-8)x+26a_0}{2x^2+x}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{20}{143}, a_1 = \frac{8}{11}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + \frac{8}{11}x + \frac{20}{143}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{\int \left(-\frac{5}{2x} - \frac{5}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= \left(x^2 + \frac{8}{11}x + \frac{20}{143}\right) e^{-\frac{5\ln(x)}{2} - \frac{5\ln(2x+1)}{4}} \\ &= \frac{x^2 + \frac{8}{11}x + \frac{20}{143}}{x^{\frac{5}{2}}(2x+1)^{\frac{5}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{13x^2+9x}{2x^3+x^2}\,dx} \\
&= z_1 e^{-\frac{9\ln(x)}{2}+\frac{5\ln(2x+1)}{4}} \\
&= z_1\left(\frac{(2x+1)^{\frac{5}{4}}}{x^{\frac{9}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{13x^2+9x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-9\ln(x)+\frac{5\ln(2x+1)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(5005x^3-6435x^2+5148x-2860)(2x+1)^{\frac{7}{2}}}{45045x^2+32760x+6300}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\right) \\
&\quad + c_2\left(\frac{x^2+\frac{8}{11}x+\frac{20}{143}}{x^7}\left(\frac{(5005x^3-6435x^2+5148x-2860)(2x+1)^{\frac{7}{2}}}{45045x^2+32760x+6300}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(x^2 + \frac{8}{11}x + \frac{20}{143}\right)}{x^7} + \frac{c_2(2x+1)^{\frac{7}{2}}\left(35x^3 - 45x^2 + 36x - 20\right)}{315x^7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(x^2 + \frac{8}{11}x + \frac{20}{143}\right)}{x^7} + \frac{c_2(2x+1)^{\frac{7}{2}}\left(35x^3 - 45x^2 + 36x - 20\right)}{315x^7}$$

Verified OK.

2.596.1 Maple step by step solution

Let's solve

$(2x^3 + x^2)\, y'' + (13x^2 + 9x)\, y' + (5x + 7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(5x+7)y}{x^2(2x+1)} - \frac{(9+13x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(9+13x)y'}{x(2x+1)} + \frac{(5x+7)y}{x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{9+13x}{x(2x+1)}, P_3(x) = \frac{5x+7}{x^2(2x+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 9$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 7$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(2x+1)\,y''+x(9+13x)\,y'+(5x+7)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m\cdot y''$ to series expansion for $m=2..3$

    $x^m\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(7+r)\,(1+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k+r+7)\,(k+r+1)+a_{k-1}(k+4+r)\,(2k-1+2r)\right)x^{k+r}\right)=$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(7+r)\,(1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-7,-1\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+4+r)\,a_{k-1}+a_k(k+r+7)\,(k+r+1)=0$

- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)\left(k+r+5\right)a_{k}+a_{k+1}\left(k+8+r\right)\left(k+2+r\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{(2k+2r+1)(k+r+5)a_{k}}{(k+8+r)(k+2+r)}$
- Recursion relation for $r=-7$; series terminates at $k=2$

  $a_{k+1}=-\frac{(2k-13)(k-2)a_{k}}{(k+1)(k-5)}$
- Apply recursion relation for $k=0$

  $a_{1}=\frac{26a_{0}}{5}$
- Apply recursion relation for $k=1$

  $a_{2}=\frac{11a_{1}}{8}$
- Express in terms of a_{0}

  $a_{2}=\frac{143a_{0}}{20}$
- Terminating series solution of the ODE for $r=-7$. Use reduction of order to find the second

  $y=a_{0}\cdot\left(\frac{143}{20}x^{2}+\frac{26}{5}x+1\right)$
- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{(2k-1)(k+4)a_{k}}{(k+7)(k+1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k-1},a_{k+1}=-\frac{(2k-1)(k+4)a_{k}}{(k+7)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_{0}\cdot\left(\frac{143}{20}x^{2}+\frac{26}{5}x+1\right)+\left(\sum_{k=0}^{\infty}b_{k}x^{k-1}\right),b_{k+1}=-\frac{(2k-1)(k+4)b_{k}}{(k+7)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)+x*(9+13*x)*diff(y(x),x)+(7+5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(143x^2 + 104x + 20)}{x^7} + \frac{c_2(35x^3 - 45x^2 + 36x - 20)\,(2x + 1)^{\frac{7}{2}}}{x^7}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 58

```
DSolve[x^2*(1+2*x)*y''[x]+x*(9+13*x)*y'[x]+(7+5*x)*y[x]==0,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{c_1(13x(11x + 8) + 20)}{143x^7} + \frac{c_2(35x^3 - 45x^2 + 36x - 20)\,(2x + 1)^{7/2}}{315x^7}$$

2.597 problem 611

2.597.1 Maple step by step solution 5615

Internal problem ID [8087]
Internal file name [`OUTPUT/7020_Sunday_June_05_2022_05_25_16_PM_87982949/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 611.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2(2x+1)\,y''-2x(-x+4)\,y'-(5x+7)\,y=0}$$

Writing the ode as

$$\left(8x^3+4x^2\right)y''+\left(2x^2-8x\right)y'+(-5x-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=8x^3+4x^2\\B&=2x^2-8x\\C&=-5x-7\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 33x^2 + 132x + 60$$
$$t = 16\left(2x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{27}{4x} + \frac{15}{4x^2} + \frac{9}{64\left(x + \frac{1}{2}\right)^2} + \frac{27}{4\left(x + \frac{1}{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{9}{8}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{8}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading

coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{33x^2 + 132x + 60}{16\left(2x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
$-\frac{1}{2}$	2	0	$\frac{9}{8}$	$-\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{3}{8}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= -\frac{3}{8} - \left(-\frac{3}{8}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)} \\
&= -\frac{3(x+2)}{4x(2x+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{9}{8\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{33x^2+132x+60}{16\left(2x^2+x\right)^2}\right.\right.$$

$$0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{9}{8\left(x+\frac{1}{2}\right)}\right) dx} \\
&= \frac{(2x+1)^{\frac{9}{8}}}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2-8x}{8x^3+4x^2} \, dx} \\
&= z_1 e^{\ln(x) - \frac{9\ln(2x+1)}{8}} \\
&= z_1 \left(\frac{x}{(2x+1)^{\frac{9}{8}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-8x}{8x^3+4x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)-\frac{9\ln(2x+1)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\frac{2}{7}x^3 - \frac{4}{7}x^2 - \frac{16}{7}x - \frac{32}{35}}{(2x+1)^{\frac{5}{4}}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}} \left(\frac{\frac{2}{7}x^3 - \frac{4}{7}x^2 - \frac{16}{7}x - \frac{32}{35}}{(2x+1)^{\frac{5}{4}}}\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2\left(x^3 - 2x^2 - 8x - \frac{16}{5}\right)}{7\sqrt{x}\,(2x+1)^{\frac{5}{4}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2\left(x^3 - 2x^2 - 8x - \frac{16}{5}\right)}{7\sqrt{x}\,(2x+1)^{\frac{5}{4}}}$$

Verified OK.

2.597.1 Maple step by step solution

Let's solve

$(8x^3 + 4x^2)\, y'' + (2x^2 - 8x)\, y' + (-5x - 7)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(5x+7)y}{4x^2(2x+1)} - \frac{(x-4)y'}{2x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-4)y'}{2x(2x+1)} - \frac{(5x+7)y}{4x^2(2x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{x-4}{2x(2x+1)}, P_3(x) = -\frac{5x+7}{4x^2(2x+1)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{7}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(2x + 1)\, y'' + 2x(x - 4)\, y' + (-5x - 7)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-7+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (2k+2r+1)\,(2k+2r-7) + a_{k-1}(2k-1+2r)\,(4k-9+4r\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-7+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{7}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$8\left(k-\frac{1}{2}+r\right)\left(k-\frac{9}{4}+r\right) a_{k-1} + 4\left(k+r-\frac{7}{2}\right)\left(k+r+\frac{1}{2}\right) a_k = 0$$

- Shift index using $k->k+1$

  $$8\left(k+r+\frac{1}{2}\right)\left(k-\frac{5}{4}+r\right) a_k + 4\left(k-\frac{5}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+1} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{(2k+2r+1)(4k+4r-5)a_k}{(2k-5+2r)(2k+3+2r)}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+1} = -\frac{2k(4k-7)a_k}{(2k-6)(2k+2)}$$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 3$

$$a_{k+1} = -\frac{2k(4k-7)a_k}{(2k-6)(2k+2)}$$

- Recursion relation for $r = \frac{7}{2}$

$$a_{k+1} = -\frac{(2k+8)(4k+9)a_k}{(2k+2)(2k+10)}$$

- Solution for $r = \frac{7}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{2}}, a_{k+1} = -\frac{(2k+8)(4k+9)a_k}{(2k+2)(2k+10)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(4*x^2*(1+2*x)*diff(y(x),x$2)-2*x*(4-x)*diff(y(x),x)-(7+5*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2(5x^3 - 10x^2 - 40x - 16)}{(2x+1)^{\frac{5}{4}}\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 47

```
DSolve[4*x^2*(1+2*x)*y''[x]-2*x*(4-x)*y'[x]-(7+5*x)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\frac{2c_2(5x^3-10x^2-40x-16)}{(2x+1)^{5/4}} + 35c_1}{35\sqrt{x}}$$

2.598 problem 612

2.598.1 Maple step by step solution 5625

Internal problem ID [8088]
Internal file name [OUTPUT/7021_Sunday_June_05_2022_05_25_19_PM_20548882/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 612.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3x^2(x+3)\,y''-x(15+x)\,y'-20y=0}$$

Writing the ode as

$$\left(3x^3+9x^2\right)y''+\left(-x^2-15x\right)y'-20y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=3x^3+9x^2\\ B&=-x^2-15x\\ C&=-20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 7x^2 + 450x + 1215$$
$$t = 36\left(x^2 + 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 450x + 1215}{36\left(x^2 + 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -3$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{10}{9x} + \frac{10}{9(x+3)} + \frac{15}{4x^2} - \frac{2}{9(x+3)^2}$$

For the pole at $x = -3$ let b be the coefficient of $\frac{1}{(x+3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 450x + 1215}{36(x^2 + 3x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{6} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 450x + 1215}{36 \left(x^2 + 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-3	2	0	$\frac{2}{3}$	$\frac{1}{3}$
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{6}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -\frac{1}{6} - \left(-\frac{7}{6}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3x + 9} - \frac{3}{2x} + (-)(0) \\ &= \frac{1}{3x + 9} - \frac{3}{2x} \\ &= -\frac{7x + 27}{6x (x + 3)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x+9} - \frac{3}{2x}\right)(1) + \left(\left(-\frac{1}{3\left(x+3\right)^2} + \frac{3}{2x^2}\right) + \left(\frac{1}{3x+9} - \frac{3}{2x}\right)^2 - \left(\frac{7x^2 + 450x + 1215}{36\left(x^2+3x\right)^2}\right)\right) = \frac{-27 + 7a_0}{3x\left(x+3\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{27}{7}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{27}{7}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{27}{7}\right) e^{\int \left(\frac{1}{3x+9} - \frac{3}{2x}\right) dx} \\
&= \left(x + \frac{27}{7}\right) e^{-\frac{3\ln(x)}{2} + \frac{\ln(x+3)}{3}} \\
&= \frac{\left(x + \frac{27}{7}\right)\left(x+3\right)^{\frac{1}{3}}}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-15x}{3x^3+9x^2} \, dx} \\
&= z_1 e^{-\frac{2\ln(x+3)}{3} + \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{x^{\frac{5}{6}}}{\left(x+3\right)^{\frac{2}{3}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-15x}{3x^3+9x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4\ln(x+3)}{3}+\frac{5\ln(x)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{21(x+3)^{\frac{1}{3}}\left(x^2-36x-243\right)}{28x+108}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}\right) + c_2 \left(\frac{7x+27}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}}\left(\frac{21(x+3)^{\frac{1}{3}}\left(x^2-36x-243\right)}{28x+108}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(7x+27)}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}} + \frac{3c_2(x^2-36x-243)}{4x^{\frac{2}{3}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(7x+27)}{7x^{\frac{2}{3}}\left(x+3\right)^{\frac{1}{3}}} + \frac{3c_2(x^2-36x-243)}{4x^{\frac{2}{3}}}$$

Verified OK.

2.598.1 Maple step by step solution

Let's solve

$(3x^3 + 9x^2)\, y'' + (-x^2 - 15x)\, y' - 20y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(15+x)y'}{3x(x+3)} + \frac{20y}{3x^2(x+3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(15+x)y'}{3x(x+3)} - \frac{20y}{3x^2(x+3)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{15+x}{3x(x+3)}, P_3(x) = -\frac{20}{3x^2(x+3)}\right]$

  ◦ $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

    $\left((x+3) \cdot P_2(x)\right)\bigg|_{x=-3} = \frac{4}{3}$

  ◦ $(x+3)^2 \cdot P_3(x)$ is analytic at $x = -3$

    $\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = 0$

  ◦ $x = -3$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -3$

- Multiply by denominators

  $3x^2(x+3)\, y'' - x(15+x)\, y' - 20y = 0$

- Change variables using $x = u - 3$ so that the regular singular point is at $u = 0$

  $(3u^3 - 18u^2 + 27u)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2 - 9u + 36)\left(\frac{d}{du}y(u)\right) - 20y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
- Shift index using $k- > k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- > k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $9a_0 r(1+3r)\, u^{-1+r} + (9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20))\, u^r + \left(\sum_{k=1}^{\infty} (9a_{k+1}(k+1+r)\,(3k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9r(1+3r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{3}\right\}$
- Each term must be 0

  $9a_1(1+r)\,(4+3r) - a_0(18r^2-9r+20) = 0$
- Each term in the series must be 0, giving the recursion relation

  $3(-6a_k + a_{k-1} + 9a_{k+1})\, k^2 + (6(-6a_k + a_{k-1} + 9a_{k+1})\, r + 9a_k - 10a_{k-1} + 63a_{k+1})\, k + 3(-6a_k +$
- Shift index using $k- > k+1$

  $3(-6a_{k+1} + a_k + 9a_{k+2})\,(k+1)^2 + (6(-6a_{k+1} + a_k + 9a_{k+2})\, r + 9a_{k+1} - 10a_k + 63a_{k+2})\,(k+1)$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} + 6kra_k - 36kra_{k+1} + 3r^2a_k - 18r^2a_{k+1} - 4ka_k - 27ka_{k+1} - 4ra_k - 27ra_{k+1} - 29a_{k+1}}{9(3k^2+6kr+3r^2+13k+13r+14)}$
- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{3k^2a_k - 18k^2a_{k+1} - 4ka_k - 27ka_{k+1} - 29a_{k+1}}{9(3k^2+13k+14)}$
- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-4k a_k-27k a_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1-20a_0=0\right]$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-4k a_k-27k a_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1-20a_0=0\right]$$

- Recursion relation for $r=-\frac{1}{3}$

$$a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-6k a_k-15k a_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}$$

- Solution for $r=-\frac{1}{3}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-6k a_k-15k a_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{1}{3}}, a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-6k a_k-15k a_{k+1}+\frac{5}{3}a_k-22a_{k+1}}{9(3k^2+11k+10)}, 18a_1-25a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+3)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{1}{3}}\right), a_{k+2}=-\frac{3k^2 a_k-18k^2 a_{k+1}-4k a_k-27k a_{k+1}-29a_{k+1}}{9(3k^2+13k+14)}, 36a_1 -$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(3*x^2*(3+x)*diff(y(x),x$2)-x*(15+x)*diff(y(x),x)-20*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 36x - 243)}{x^{\frac{2}{3}}} + \frac{c_2(7x + 27)}{x^{\frac{2}{3}} (x + 3)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 43

```
DSolve[3*x^2*(3+x)*y''[x]-x*(15+x)*y'[x]-20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{21c_2(x^2 - 36x - 243) + \frac{4c_1(7x+27)}{\sqrt[3]{x+3}}}{28x^{2/3}}$$

2.599 problem 613

2.599.1 Maple step by step solution 5635

Internal problem ID [8089]
Internal file name [OUTPUT/7022_Sunday_June_05_2022_05_25_22_PM_4430714/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 613.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''+x(1-10x)\,y'-(9-10x)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-10x^2+x\right)y'+(10x-9)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -10x^2+x \\ C &= 10x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 80x^2 - 28x + 35$$
$$t = 4\left(x^2+x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{80x^2 - 28x + 35}{4\left(x^2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{49}{2x} + \frac{143}{4(1+x)^2} + \frac{35}{4x^2} + \frac{49}{2(1+x)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{143}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{13}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{11}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{80x^2 - 28x + 35}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{80x^2 - 28x + 35}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{13}{2}$	$-\frac{11}{2}$
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 5$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= 5 - (4) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{13}{2\left(1 + x\right)} - \frac{5}{2x} + (0) \\ &= \frac{13}{2\left(1 + x\right)} - \frac{5}{2x} \\ &= \frac{8x - 5}{2x\left(1 + x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)(1) + \left(\left(-\frac{13}{2(1+x)^2} + \frac{5}{2x^2}\right) + \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right)^2 - \left(\frac{80x^2 - 28x + 35}{4(x^2+x)^2}\right)\right) = \frac{-5 - 8a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{5}{8}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{5}{8}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{5}{8}\right) e^{\int \left(\frac{13}{2(1+x)} - \frac{5}{2x}\right) dx} \\
&= \left(x - \frac{5}{8}\right) e^{-\frac{5\ln(x)}{2} + \frac{13\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{5}{8}\right)(1+x)^{\frac{13}{2}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x^2+x}{x^2(1+x)} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \frac{11\ln(1+x)}{2}} \\
&= z_1 \left(\frac{(1+x)^{\frac{11}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x^2+x}{x^2(1+x)}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+11\ln(1+x)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{-\frac{8}{9}x^4-\frac{32}{45}x^3-\frac{16}{55}x^2-\frac{32}{495}x-\frac{8}{1287}}{\left(8x-5\right)\left(1+x\right)^{12}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3}\right) + c_2\left(\frac{\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3}\left(\frac{-\frac{8}{9}x^4-\frac{32}{45}x^3-\frac{16}{55}x^2-\frac{32}{495}x-\frac{8}{1287}}{\left(8x-5\right)\left(1+x\right)^{12}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3} + \frac{c_2(-715x^4-572x^3-234x^2-52x-5)}{6435x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(x-\frac{5}{8}\right)\left(1+x\right)^{12}}{x^3} + \frac{c_2(-715x^4-572x^3-234x^2-52x-5)}{6435x^3}$$

Verified OK.

2.599.1 Maple step by step solution

Let's solve

$x^2(1+x)\, y'' + (-10x^2+x)\, y' + (10x-9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x-9)y}{x^2(1+x)} + \frac{(10x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(10x-1)y'}{x(1+x)} + \frac{(10x-9)y}{x^2(1+x)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{10x-1}{x(1+x)}, P_3(x) = \frac{10x-9}{x^2(1+x)}\right]$

- ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -11$

- ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\, y'' - x(10x-1)\, y' + (10x-9)\, y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right) + (-10u^2+21u-11)\left(\frac{d}{du}y(u)\right) + (10u-19)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- ◦ Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-12+r)\, u^{-1+r} + (a_1(1+r)\,(-11+r) - a_0(2r^2 - 23r + 19))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-12+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 12\}$

- Each term must be 0

  $a_1(1+r)\,(-11+r) - a_0(2r^2 - 23r + 19) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 23a_k - 13a_{k-1} - 10a_{k+1})\, k + (-2a_k + a$

- Shift index using $k- >k + 1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 23a_{k+1} - 13a_k - 10a_{k+2})\,(k+1)$ -

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 11k a_k + 19k a_{k+1} - 11r a_k + 19r a_{k+1} + 10a_k + 2a_{k+1}}{k^2 + 2kr + r^2 - 8k - 8r - 20}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11k a_k + 19k a_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 10$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 11k a_k + 19k a_{k+1} + 10a_k + 2a_{k+1}}{k^2 - 8k - 20}$$

- Recursion relation for $r = 12$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}$$

- Solution for $r = 12$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1 + x)^{k+12}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 13k a_k - 29k a_{k+1} + 22a_k - 58a_{k+1}}{k^2 + 16k + 28}, 13a_1 - 31a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 74

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x*(1-10*x)*diff(y(x),x)-(9-10*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{x^3} + \frac{c_2\left(x^{13} + \frac{91}{8}x^{12} + \frac{117}{2}x^{11} + \frac{715}{4}x^{10} + \frac{715}{2}x^9 + \frac{3861}{8}x^8 + 429x^7 + \frac{429}{2}x^6\right)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 51

```
DSolve[x^2*(1+x)*y''[x]+x*(1-10*x)*y'[x]-(9-10*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{6435c_1(x+1)^{12}(8x-5) - 8c_2(715x^4 + 572x^3 + 234x^2 + 52x + 5)}{51480x^3}$$

2.600 problem 614

2.600.1 Maple step by step solution 5645

Internal problem ID [8090]
Internal file name [OUTPUT/7023_Sunday_June_05_2022_05_25_25_PM_4274740/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 614.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\,y''+3x^2y'-(-x+6)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= 3x^2 \\ C &= x-6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 20x + 24$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{x} + \frac{3}{4(1+x)^2} + \frac{6}{x^2} + \frac{7}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 + 20x + 24}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 20x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2(1+x)} - \frac{2}{x} + (-)(0) \\
&= \frac{3}{2(1+x)} - \frac{2}{x} \\
&= -\frac{4 + x}{2x(1+x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2(1+x)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{3}{2(1+x)^2} + \frac{2}{x^2}\right) + \left(\frac{3}{2(1+x)} - \frac{2}{x}\right)^2 - \left(\frac{-x^2+20x+24}{4(x^2+x)^2}\right)\right) = 0$$

$$\frac{-4+a_0}{x(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 4 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (4+x)\, e^{\int \left(\frac{3}{2(1+x)} - \frac{2}{x}\right) dx} \\ &= (4+x)\, e^{-2\ln(x) + \frac{3\ln(1+x)}{2}} \\ &= \frac{(4+x)(1+x)^{\frac{3}{2}}}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x^2}{x^2(1+x)}\, dx} \\ &= z_1 e^{-\frac{3\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{(1+x)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{4+x}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27+27x} + \frac{256}{108+27x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{4+x}{x^2}\right) + c_2 \left(\frac{4+x}{x^2}\left(\ln(1+x) - \frac{1}{18(1+x)^2} + \frac{14}{27+27x} + \frac{256}{108+27x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(4+x)}{x^2} + \frac{c_2\left(6(1+x)^2(4+x)\ln(1+x) + 60x^2 + 129x + 68\right)}{6x^2(1+x)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(4+x)}{x^2} + \frac{c_2\left(6(1+x)^2(4+x)\ln(1+x) + 60x^2 + 129x + 68\right)}{6x^2(1+x)^2}$$

Verified OK.

2.600.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{1+x}-\frac{(x-6)y}{x^2(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{1+x}+\frac{(x-6)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{1+x},P_3(x)=\frac{x-6}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=3$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''+3x^2y'+(x-6)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(3u^2-6u+3)\left(\frac{d}{du}y(u)\right)+(u-7)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(2+r)\, u^{-1+r} + \left(a_1(1+r)\,(3+r) - a_0(2r^2+4r+7)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+3+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 0\}$$

- Each term must be 0

  $$a_1(1+r)\,(3+r) - a_0(2r^2+4r+7) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(k+r)^2 + a_{k+1}(k+r+1)\,(k+3+r) - 2\left(k^2 + (2r+2)\,k + r^2 + 2r + \frac{7}{2}\right) a_k = 0$$

- Shift index using $k- > k + 1$

  $$a_k(k+r+1)^2 + a_{k+2}(k+r+2)\,(k+4+r) - 2\left((k+1)^2 + (2r+2)\,(k+1) + r^2 + 2r + \frac{7}{2}\right) a_{k+1}$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} + 2k a_k - 8k a_{k+1} + 2r a_k - 8r a_{k+1} + a_k - 13a_{k+1}}{(k+r+2)(k+4+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2k a_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 2k a_k + a_k - 5a_{k+1}}{k(k+2)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 8k a_{k+1} + a_k - 13a_{k+1}}{(k+2)(k+4)}, 3a_1 - 7a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 78

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+3*x^2*diff(y(x),x)-(6-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^3+6x^2+9x+4)}{x^2(x+1)^2} + \frac{c_2\left(\ln(x+1)\,x^3+6\ln(x+1)\,x^2+9\ln(x+1)\,x+10x^2+4\ln(x+1)+\frac{43x}{2}+\frac{34}{3}\right)}{x^2(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 49

```
DSolve[x^2*(1+x)*y''[x]+3*x^2*y'[x]-(6-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\frac{c_2(60x^2+129x+68)}{(x+1)^2}+6c_1(x+4)+6c_2(x+4)\log(x+1)}{6x^2}$$

2.601 problem 615

2.601.1 Maple step by step solution 5655

Internal problem ID [8091]
Internal file name [OUTPUT/7024_Sunday_June_05_2022_05_25_29_PM_64302104/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 615.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(2x+1)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0}$$

Writing the ode as

$$\left(2x^3+x^2\right)y''+\left(-28x^2-6x\right)y'+(6+100x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+x^2\\B&=-28x^2-6x\\C&=6+100x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{24x^2 - 16x + 6}{(2x^2 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 24x^2 - 16x + 6$$
$$t = (2x^2 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{24x^2 - 16x + 6}{(2x^2 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{40}{x} + \frac{6}{x^2} + \frac{20}{\left(x+\frac{1}{2}\right)^2} + \frac{40}{x+\frac{1}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{24x^2 - 16x + 6}{(2x^2+x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=3\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{24x^2-16x+6}{(2x^2+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2
$-\frac{1}{2}$	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=3$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\
&= 3-(3)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + \frac{5}{x + \frac{1}{2}} + (0) \\
&= -\frac{2}{x} + \frac{5}{x + \frac{1}{2}} \\
&= \frac{6x - 2}{2x^2 + x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)(0) + \left(\left(\frac{2}{x^2} - \frac{5}{\left(x + \frac{1}{2}\right)^2}\right) + \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right)^2 - \left(\frac{24x^2 - 16x + 6}{\left(2x^2 + x\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{2}{x} + \frac{5}{x + \frac{1}{2}}\right) dx} \\
&= \frac{(2x + 1)^5}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-28x^2 - 6x}{2x^3 + x^2} \, dx} \\
&= z_1 e^{3 \ln(x) + 4 \ln(2x + 1)} \\
&= z_1 \left(x^3 (2x + 1)^4\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x(2x+1)^9$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-28x^2-6x}{2x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6\ln(x)+8\ln(2x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-2016x^4-672x^3-144x^2-18x-1}{20160\,(2x+1)^9}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x(2x+1)^9\right) + c_2 \left(x(2x+1)^9 \left(\frac{-2016x^4-672x^3-144x^2-18x-1}{20160\,(2x+1)^9}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x(2x+1)^9 + c_2 \left(-\frac{1}{10}x^5 - \frac{1}{30}x^4 - \frac{1}{140}x^3 - \frac{1}{1120}x^2 - \frac{1}{20160}x\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(2x+1)^9 + c_2 \left(-\frac{1}{10}x^5 - \frac{1}{30}x^4 - \frac{1}{140}x^3 - \frac{1}{1120}x^2 - \frac{1}{20160}x\right)$$

Verified OK.

2.601.1 Maple step by step solution

Let's solve

$(2x^3+x^2)\,y''+(-28x^2-6x)\,y'+(6+100x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3+50x)y}{x^2(2x+1)}+\frac{2(3+14x)y'}{x(2x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(3+14x)y'}{x(2x+1)}+\frac{2(3+50x)y}{x^2(2x+1)}=0$

☐ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=-\frac{2(3+14x)}{x(2x+1)}, P_3(x)=\frac{2(3+50x)}{x^2(2x+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=-6$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=6$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(2x+1)\,y''-2x(3+14x)\,y'+(6+100x)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x^{m}\cdot y'$ to series expansion for $m=1..2$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty}a_{k}(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m=2..3$

  $x^{m}\cdot y''=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+r)\,(-6+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k}(k+r-1)\,(k+r-6)+2a_{k-1}(k+r-6)\,(k-11+r)\right)x^{k+r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-6+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,6\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-6)\left((2k+2r-22)\,a_{k-1}+a_{k}(k+r-1)\right)=0$

- Shift index using $k->k+1$

  $(k+r-5)\left((2k+2r-20)\,a_{k}+a_{k+1}(k+r)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2(k+r-10)a_{k}}{k+r}$

- Recursion relation for $r=1$; series terminates at $k=9$

  $a_{k+1}=-\frac{2(k-9)a_{k}}{k+1}$

- Recursion relation that defines the terminating series solution of the ODE for $r=1$

$$\left[y = \sum_{k=0}^{8} a_k x^{k+1}, a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$$

- Recursion relation for $r = 6$; series terminates at $k = 4$

  $a_{k+1} = -\frac{2(k-4)a_k}{k+6}$

- Apply recursion relation for $k = 0$

  $a_1 = \frac{4a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = \frac{6a_1}{7}$

- Express in terms of a_0

  $a_2 = \frac{8a_0}{7}$

- Apply recursion relation for $k = 2$

  $a_3 = \frac{a_2}{2}$

- Express in terms of a_0

  $a_3 = \frac{4a_0}{7}$

- Apply recursion relation for $k = 3$

  $a_4 = \frac{2a_3}{9}$

- Express in terms of a_0

  $a_4 = \frac{8a_0}{63}$

- Terminating series solution of the ODE for $r = 6$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right)$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{8} a_k x^{k+1}\right) + b_0 \cdot \left(1 + \frac{4}{3}x + \frac{8}{7}x^2 + \frac{4}{7}x^3 + \frac{8}{63}x^4\right), a_{k+1} = -\frac{2(k-9)a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(x^2*(1+2*x)*diff(y(x),x$2)-2*x*(3+14*x)*diff(y(x),x)+(6+100*x)*y(x)=0,y(x), singsol=a
```

$$y(x) = c_1 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1) + c_2 x\left(x^9 + \frac{9}{2}x^8 + 9x^7 + \frac{21}{2}x^6 + \frac{63}{8}x^5\right)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 44

```
DSolve[x^2*(1+2*x)*y''[x]-2*x*(3+14*x)*y'[x]+(6+100*x)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to c_1 x(2x+1)^9 - \frac{c_2 x(2016x^4 + 672x^3 + 144x^2 + 18x + 1)}{20160}$$

2.602 problem 616

2.602.1 Maple step by step solution 5665

Internal problem ID [8092]
Internal file name [OUTPUT/7025_Sunday_June_05_2022_05_25_32_PM_8909608/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 616.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0}$$

Writing the ode as

$$x^2(1+x)\,y''+\left(-11x^2-6x\right)y'+(6+32x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^2(1+x)\\B&=-11x^2-6x\\C&=6+32x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15x^2 + 4x + 24$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{11}{x} + \frac{35}{4(1+x)^2} + \frac{6}{x^2} + \frac{11}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 4x + 24}{4(x^2+x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15x^2 + 4x + 24}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} + (0) \\
&= \frac{7}{2\left(1 + x\right)} - \frac{2}{x} \\
&= \frac{3x - 4}{2x\left(1 + x\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{2\left(1+x\right)} - \frac{2}{x}\right)(1) + \left(\left(-\frac{7}{2\left(1+x\right)^2} + \frac{2}{x^2}\right) + \left(\frac{7}{2\left(1+x\right)} - \frac{2}{x}\right)^2 - \left(\frac{15x^2 + 4x + 24}{4\left(x^2+x\right)^2}\right)\right) = 0$$

$$\frac{-4 - 3a_0}{x\left(1+x\right)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{4}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{4}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{7}{2(1+x)} - \frac{2}{x}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{-2\ln(x) + \frac{7\ln(1+x)}{2}} \\
&= \frac{\left(x - \frac{4}{3}\right)\left(1+x\right)^{\frac{7}{2}}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-11x^2 - 6x}{x^2(1+x)} \, dx} \\
&= z_1 e^{3\ln(x) + \frac{5\ln(1+x)}{2}} \\
&= z_1 \left(x^3 (1+x)^{\frac{5}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x - \frac{4}{3}\right) x(1+x)^6$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-11x^2-6x}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{6\ln(x)+5\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{-\frac{3}{4}x^3 - \frac{9}{10}x^2 - \frac{9}{20}x - \frac{3}{35}}{(1+x)^6\,(3x-4)}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\left(x - \frac{4}{3}\right) x(1+x)^6\right) + c_2 \left(\left(x - \frac{4}{3}\right) x(1+x)^6 \left(\frac{-\frac{3}{4}x^3 - \frac{9}{10}x^2 - \frac{9}{20}x - \frac{3}{35}}{(1+x)^6\,(3x-4)}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x - \frac{4}{3}\right) x(1+x)^6 + c_2 \left(-\frac{1}{4}x^4 - \frac{3}{10}x^3 - \frac{3}{20}x^2 - \frac{1}{35}x\right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(x - \frac{4}{3}\right) x(1+x)^6 + c_2 \left(-\frac{1}{4}x^4 - \frac{3}{10}x^3 - \frac{3}{20}x^2 - \frac{1}{35}x\right)$$

Verified OK.

2.602.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+(-11x^2-6x)\,y'+(6+32x)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2(3+16x)y}{x^2(1+x)}+\frac{(6+11x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(6+11x)y'}{x(1+x)}+\frac{2(3+16x)y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{6+11x}{x(1+x)},P_3(x)=\frac{2(3+16x)}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1}=-5$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''-x(6+11x)\,y'+(6+32x)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(-11u^2+16u-5)\left(\frac{d}{du}y(u)\right)+(-26+32u)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-6+r)\, u^{-1+r} + (a_1(1+r)\,(-5+r) - 2a_0(r^2 - 9r + 13))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k-5$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-6+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 6\}$$

- Each term must be 0

  $$a_1(1+r)\,(-5+r) - 2a_0(r^2 - 9r + 13) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + 2((-2a_k + a_{k-1} + a_{k+1})\, r + 9a_k - 7a_{k-1} - 2a_{k+1})\, k + (-2a_k + a_{k-1} -$$

- Shift index using $k- >k + 1$

  $$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + 2((-2a_{k+1} + a_k + a_{k+2})\, r + 9a_{k+1} - 7a_k - 2a_{k+2})\,(k+1) + (-$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 12ka_k + 14ka_{k+1} - 12ra_k + 14ra_{k+1} + 32a_k - 10a_{k+1}}{k^2 + 2kr + r^2 - 2k - 2r - 8}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 12ka_k + 14ka_{k+1} + 32a_k - 10a_{k+1}}{k^2 - 2k - 8}$$

- Recursion relation for $r = 6$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}$$

- Solution for $r = 6$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+6}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 10ka_{k+1} - 4a_k + 2a_{k+1}}{k^2 + 10k + 16}, 7a_1 + 10a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-x*(6+11*x)*diff(y(x),x)+(6+32*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x(35x^3 + 42x^2 + 21x + 4) + c_2 x\left(x^7 + \frac{14}{3}x^6 + 7x^5\right)$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 45

```
DSolve[x^2*(1+x)*y''[x]-x*(6+11*x)*y'[x]+(6+32*x)*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}c_1 x(x+1)^6(3x-4) - \frac{1}{140}c_2 x\left(35x^3 + 42x^2 + 21x + 4\right)$$

2.603 problem 617

2.603.1 Maple step by step solution 5674

Internal problem ID [8093]
Internal file name [`OUTPUT/7026_Sunday_June_05_2022_05_25_35_PM_49873359/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 617.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(1+x)\,y''+4x(1+4x)\,y'-(49+27x)\,y=0}$$

Writing the ode as

$$\left(4x^3+4x^2\right)y''+\left(16x^2+4x\right)y'+(-27x-49)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3+4x^2 \\ B &= 16x^2+4x \\ C &= -27x-49 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 35x^2 + 80x + 48$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{4}{x} + \frac{3}{4(1+x)^2} + \frac{12}{x^2} + \frac{4}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{35x^2 + 80x + 48}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{35}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= \frac{7}{2} - \left(\frac{7}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(1 + x\right)} + \frac{4}{x} + (0) \\ &= -\frac{1}{2\left(1 + x\right)} + \frac{4}{x} \\ &= \frac{7x + 8}{2x\left(1 + x\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(1+x\right)} + \frac{4}{x}\right)(0) + \left(\left(\frac{1}{2\left(1+x\right)^2} - \frac{4}{x^2}\right) + \left(-\frac{1}{2\left(1+x\right)} + \frac{4}{x}\right)^2 - \left(\frac{35x^2 + 80x + 48}{4\left(x^2 + x\right)^2}\right)\right) =$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{4}{x}\right) dx} \\ &= \frac{x^4}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{16x^2 + 4x}{4x^3 + 4x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} - \frac{3\ln(1+x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}\,(1+x)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{7}{2}}}{(1+x)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{16x^2+4x}{4x^3+4x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-\ln(x)-3\ln(1+x)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{-7x-6}{42x^7}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{x^{\frac{7}{2}}}{(1+x)^2}\right) + c_2 \left(\frac{x^{\frac{7}{2}}}{(1+x)^2} \left(\frac{-7x-6}{42x^7}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{7}{2}}}{(1+x)^2} + \frac{c_2(-7x-6)}{42x^{\frac{7}{2}}(1+x)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{7}{2}}}{(1+x)^2} + \frac{c_2(-7x-6)}{42x^{\frac{7}{2}}(1+x)^2}$$

Verified OK.

2.603.1 Maple step by step solution

Let's solve

$(4x^3 + 4x^2) y'' + (16x^2 + 4x) y' + (-27x - 49) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(49+27x)y}{4x^2(1+x)} - \frac{(1+4x)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(1+4x)y'}{x(1+x)} - \frac{(49+27x)y}{4x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{1+4x}{x(1+x)}, P_3(x) = -\frac{49+27x}{4x^2(1+x)}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 3$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $4x^2(1+x)\, y'' + 4x(1+4x)\, y' + (-27x - 49)\, y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^3 - 8u^2 + 4u)\left(\frac{d^2}{du^2}y(u)\right) + (16u^2 - 28u + 12)\left(\frac{d}{du}y(u)\right) + (-27u - 22)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty} a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..3$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k - >k +2-m$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty} a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$4 a_{0} r \left(2+r \right) u^{-1+r}+\left(4 a_{1} \left(1+r \right) \left(3+r \right)-2 a_{0} \left(4 r^{2}+10 r +11\right)\right) u^{r}+\left(\sum_{k =1}^{\infty} \left(4 a_{k +1} \left(k +1+r \right) \left(k +\right.\right.\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$4 r \left(2+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-2, 0\right\}$$

- Each term must be 0

$$4 a_{1} \left(1+r \right) \left(3+r \right)-2 a_{0} \left(4 r^{2}+10 r +11\right)=0$$

- Each term in the series must be 0, giving the recursion relation

$$4 \left(-2 a_{k}+a_{k -1}+a_{k +1}\right) k^{2}+4 \left(2 \left(-2 a_{k}+a_{k -1}+a_{k +1}\right) r -5 a_{k}+a_{k -1}+4 a_{k +1}\right) k +4 \left(-2 a_{k}+a_{k -}\right.$$

- Shift index using $k - >k +1$

$$4 \left(-2 a_{k +1}+a_{k}+a_{k +2}\right) \left(k +1\right)^{2}+4 \left(2 \left(-2 a_{k +1}+a_{k}+a_{k +2}\right) r -5 a_{k +1}+a_{k}+4 a_{k +2}\right) \left(k +1\right)+4 \left(\right.$$

- Recursion relation that defines series solution to ODE

$$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+8 k r a_{k}-16 k r a_{k +1}+4 r^{2} a_{k}-8 r^{2} a_{k +1}+12 k a_{k}-36 k a_{k +1}+12 r a_{k}-36 r a_{k +1}-27 a_{k}-50 a_{k +1}}{4 \left(k^{2}+2 k r +r^{2}+6 k +6 r +8\right)}$$

- Recursion relation for $r =-2$

$$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}-4 k a_{k}-4 k a_{k +1}-35 a_{k}-10 a_{k +1}}{4 \left(k^{2}+2 k \right)}$$

- Series not valid for $r =-2$, division by 0 in the recursion relation at $k =0$

$$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}-4 k a_{k}-4 k a_{k +1}-35 a_{k}-10 a_{k +1}}{4 \left(k^{2}+2 k \right)}$$

- Recursion relation for $r =0$

$$a_{k +2}=-\frac{4 k^{2} a_{k}-8 k^{2} a_{k +1}+12 k a_{k}-36 k a_{k +1}-27 a_{k}-50 a_{k +1}}{4 \left(k^{2}+6 k +8\right)}$$

- Solution for $r =0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1-22a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k, a_{k+2}=-\frac{4k^2a_k-8k^2a_{k+1}+12ka_k-36ka_{k+1}-27a_k-50a_{k+1}}{4(k^2+6k+8)}, 12a_1-22a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(4*x^2*(1+x)*diff(y(x),x$2)+4*x*(1+4*x)*diff(y(x),x)-(49+27*x)*y(x)=0,y(x), singsol=al
```

$$y(x)=\frac{c_1(7x+6)}{(x+1)^2\, x^{\frac{7}{2}}}+\frac{c_2 x^{\frac{7}{2}}}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 36

```
DSolve[4*x^2*(1+x)*y''[x]+4*x*(1+4*x)*y'[x]-(49+27*x)*y[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x)\to\frac{42c_1x^7-7c_2x-6c_2}{42x^{7/2}(x+1)^2}$$

2.604 problem 618

2.604.1 Maple step by step solution 5684

Internal problem ID [8094]
Internal file name [`OUTPUT/7027_Sunday_June_05_2022_05_25_38_PM_66086525/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 618.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-2x^2+7)\,y'+12y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(2x^3-7x)\,y'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= 2x^3-7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-30x^2 + 15}{4 (x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -30x^2 + 15 \\ t &= 4(x^3 + x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-30x^2 + 15}{4 (x^3 + x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{45}{16(x-i)^2} + \frac{45}{16(x+i)^2} + \frac{75i}{16(x-i)} - \frac{75i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-30x^2 + 15}{4\left(x^3 + x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{9}{4}$	$-\frac{5}{4}$
$-i$	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} + (0) \\
&= \frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)} \\
&= \frac{5}{2x(x^2+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right)(0) + \left(\left(-\frac{5}{2x^2} + \frac{5}{4(x-i)^2} + \frac{5}{4(x+i)^2}\right) + \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{2x} - \frac{5}{4(x-i)} - \frac{5}{4(x+i)}\right) dx} \\
&= \frac{x^{\frac{5}{2}}}{(x^2+1)^{\frac{5}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3-7x}{x^4+x^2}\, dx} \\
&= z_1 e^{\frac{7\ln(x)}{2} - \frac{9\ln(x^2+1)}{4}} \\
&= z_1 \left(\frac{x^{\frac{7}{2}}}{(x^2+1)^{\frac{9}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^6}{(x^2+1)^{\frac{7}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3-7x}{x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{7\ln(x)-\frac{9\ln\left(x^2+1\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-15\ \mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 + (8x^4-9x^2-2)\sqrt{x^2+1}}{8x^4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^6}{(x^2+1)^{\frac{7}{2}}}\right) \\
&\quad + c_2\left(\frac{x^6}{(x^2+1)^{\frac{7}{2}}}\left(\frac{-15\ \mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 + (8x^4-9x^2-2)\sqrt{x^2+1}}{8x^4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} \\
&\quad + \frac{c_2 x^2\left(8\sqrt{x^2+1}\, x^4 - 15\ \mathrm{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 - 9x^2\sqrt{x^2+1} - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} + \frac{c_2 x^2 \left(8\sqrt{x^2+1}\, x^4 - 15 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) x^4 - 9x^2\sqrt{x^2+1} - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}$$

Verified OK.

2.604.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y'' + (2x^3-7x)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y}{x^2(x^2+1)} - \frac{(2x^2-7)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2-7)y'}{x(x^2+1)} + \frac{12y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2-7}{x(x^2+1)}, P_3(x) = \frac{12}{x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $(x \cdot P_2(x))\Big|_{x=0} = -7$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 12$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1)\,y'' + x(2x^2-7)\,y' + 12y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-6+r)\, x^r + a_1(-1+r)\,(-5+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-6) + a_{k-2}(k +$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-6+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$
- Each term must be 0

  $a_1(-1+r)\,(-5+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_k(k+r-6) + a_{k-2}(k+r-1)) = 0$
- Shift index using $k->k+2$

  $(k+r)\,(a_{k+2}(k-4+r) + a_k(k+r+1)) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k(k+r+1)}{k-4+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k(k+3)}{k-2}$
- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k(k+7)}{k+2}$
- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k(k+7)}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(7-2*x^2)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^6}{(x^2+1)^{\frac{7}{2}}} + \frac{c_2 x^2 \left(8x^4\sqrt{x^2+1} - 15x^4 \operatorname{arctanh}\left(\frac{1}{\sqrt{x^2+1}}\right) - 9\sqrt{x^2+1}\, x^2 - 2\sqrt{x^2+1}\right)}{8\,(x^2+1)^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 88

```
DSolve[x^2*(1+x^2)*y''[x]-x*(7-2*x^2)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{-15c_2x^6\operatorname{arctanh}\left(\sqrt{x^2+1}\right)-2c_2\sqrt{x^2+1}x^2+8x^6\left(c_2\sqrt{x^2+1}+c_1\right)-9c_2\sqrt{x^2+1}x^4}{8\left(x^2+1\right)^{7/2}}$$

2.605 problem 619

2.605.1 Maple step by step solution 5694

Internal problem ID [8095]
Internal file name [`OUTPUT/7028_Sunday_June_05_2022_05_25_41_PM_39963627/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 619.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(-x^2 + 7)\, y' + 12y = 0}$$

Writing the ode as

$$x^2y'' + (x^3 - 7x)\, y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3 - 7x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 12x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^4 - 12x^2 + 15$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{x^2}{4} - 3 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{x} - \frac{21}{4x^3} - \frac{63}{2x^5} - \frac{3465}{16x^7} - \frac{13041}{8x^9} - \frac{417501}{32x^{11}} - \frac{1744659}{16x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^4 - 12x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{x^2}{4} - 3\right) + \left(\frac{15}{4x^2}\right) \\ &= \frac{x^2}{4} - 3 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -3. Now b can be found.

$$\begin{aligned} b &= (-3) - (0) \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{\frac{1}{2}} - 1\right) &&= -\frac{7}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{\frac{1}{2}} - 1\right) &&= \frac{5}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 12x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	$-\frac{7}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{x}{2}\right) \\ &= \frac{5}{2x} - \frac{x}{2} \\ &= \frac{5}{2x} - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} - \frac{x}{2}\right)(0) + \left(\left(-\frac{5}{2x^2} - \frac{1}{2}\right) + \left(\frac{5}{2x} - \frac{x}{2}\right)^2 - \left(\frac{x^4 - 12x^2 + 15}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} - \frac{x}{2}\right) dx} \\ &= x^{\frac{5}{2}} e^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3 - 7x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} + \frac{7 \ln(x)}{2}} \\ &= z_1 \left(x^{\frac{7}{2}} e^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^6 e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-7x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}+7\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 - 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 - 4\,\mathrm{e}^{\frac{x^2}{2}}}{16x^4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^6 \mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x^6 \mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 - 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 - 4\,\mathrm{e}^{\frac{x^2}{2}}}{16x^4} \right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x^2 \left(\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 \mathrm{e}^{-\frac{x^2}{2}} + 2x^2 + 4\right)}{16} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 x^2 \left(\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 \mathrm{e}^{-\frac{x^2}{2}} + 2x^2 + 4\right)}{16}
$$

Verified OK.

2.605.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^3 - 7x)\, y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{12y}{x^2} - \frac{(x^2-7)y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(x^2-7)y'}{x} + \frac{12y}{x^2} = 0$$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{x^2-7}{x}, P_3(x) = \frac{12}{x^2}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = -7$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 12$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$x^2y'' + x(x^2-7)\, y' + 12y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)(-6+r)x^r + a_1(-1+r)(-5+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-6) + a_{k-2}(k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)(-6+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{2, 6\}$

- Each term must be 0

  $a_1(-1+r)(-5+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)(a_k(k+r-6) + a_{k-2}) = 0$

- Shift index using $k- > k+2$

  $(k+r)(a_{k+2}(k-4+r) + a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{k-4+r}$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Series not valid for $r = 2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{k-2}$

- Recursion relation for $r = 6$

  $a_{k+2} = -\frac{a_k}{k+2}$

- Solution for $r = 6$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+6}, a_{k+2} = -\frac{a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
dsolve(x^2*diff(y(x),x$2)-x*(7-x^2)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^6 \mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2 x^2 \mathrm{e}^{-\frac{x^2}{2}} \left(\operatorname{expIntegral}_1\left(-\frac{x^2}{2}\right) x^4 + 2\,\mathrm{e}^{\frac{x^2}{2}} x^2 + 4\,\mathrm{e}^{\frac{x^2}{2}}\right)}{16}$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 61

```
DSolve[x^2*y''[x]-x*(7-x^2)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16} c_2 e^{-\frac{x^2}{2}} x^6 \operatorname{ExpIntegralEi}\left(\frac{x^2}{2}\right) - \frac{1}{8} c_2 \left(x^2+2\right) x^2 + c_1 e^{-\frac{x^2}{2}} x^6$$

2.606 problem 620

2.606.1 Maple step by step solution 5705

Internal problem ID [8096]
Internal file name [`OUTPUT/7029_Sunday_June_05_2022_05_25_45_PM_41662593/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 620.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(2x^2+1)\,y' - (-10x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (2x^3+x)\,y' + (10x^2-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^3+x \\ C &= 10x^2-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^4 - 32x^2 + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 - 8 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{4}{x} - \frac{61}{8x^3} - \frac{61}{2x^5} - \frac{19337}{128x^7} - \frac{26779}{32x^9} - \frac{5083557}{1024x^{11}} - \frac{7896633}{256x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 - 32x^2 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(x^2 - 8\right) + \left(\frac{3}{4x^2}\right) \\ &= x^2 - 8 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is -8. Now b can be found.

$$\begin{aligned} b &= (-8) - (0) \\ &= -8 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-8}{1} - 1\right) &&= -\frac{9}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-8}{1} - 1\right) &&= \frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 - 32x^2 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$-\frac{9}{2}$	$\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{7}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{7}{2} - \left(\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} + (-)(x) \\
&= \frac{3}{2x} - x \\
&= \frac{3}{2x} - x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{3}{2x} - x\right)(2x + a_1) + \left(\left(-\frac{3}{2x^2} - 1\right) + \left(\frac{3}{2x} - x\right)^2 - \left(\frac{4x^4 - 32x^2 + 3}{4x^2}\right)\right) = 0$$

$$\frac{2x^2 a_1 + (4a_0 + 8)\, x + 3a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{3}{2x} - x\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{x^2}{2} + \frac{3\ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{\frac{3}{2}} e^{-\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^3 + x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x^2}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x^2-2)\, x\, \mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x^2-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{x^2}}{x^3\,(x^2-2)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((x^2-2)\, x\, \mathrm{e}^{-x^2}\right) + c_2 \left((x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3\,(x^2-2)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 (x^2-2)\, x\, \mathrm{e}^{-x^2} + c_2 (x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3\,(x^2-2)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 (x^2-2)\, x\, \mathrm{e}^{-x^2} + c_2 (x^2-2)\, x\, \mathrm{e}^{-x^2} \left(\int \frac{\mathrm{e}^{x^2}}{x^3\,(x^2-2)^2} dx\right)$$

Verified OK.

2.606.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^3 + x)\,y' + (10x^2 - 1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(10x^2-1)y}{x^2} - \frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+1)y'}{x} + \frac{(10x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x^2+1}{x}, P_3(x) = \frac{10x^2-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x^2 + 1)\,y' + (10x^2 - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$
- Convert $x^{m}\cdot y'$ to series expansion for $m=1..3$

  $x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-1+r)\, x^{r}+a_{1}(2+r)\, r\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_{k}(k+r+1)\,(k+r-1)+2a_{k-2}(k+3+r)\right) x^{k}$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{-1,1\}$
- Each term must be 0

  $a_{1}(2+r)\, r=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)\,(k+r-1)+2a_{k-2}(k+3+r)=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r+1)+2a_{k}(k+r+5)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{2a_{k}(k+r+5)}{(k+3+r)(k+r+1)}$
- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{2a_{k}(k+4)}{(k+2)k}$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{2a_k(k+6)}{(k+4)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{2a_k(k+4)}{(k+2)k}, a_1 = 0, b_{k+2} = -\frac{2b_k(k+6)}{(k+4)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(x^2*diff(y(x),x$2)+x*(1+2*x^2)*diff(y(x),x)-(1-10*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,\mathrm{e}^{-x^2}\left(x^2-2\right) + c_2 x\,\mathrm{e}^{-x^2}\left(x^2-2\right)\left(\int \frac{\mathrm{e}^{x^2}}{\left(x^2-2\right)^2 x^3}dx\right)$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 68

```
DSolve[x^2*y''[x]+x*(1+2*x^2)*y'[x]-(1-10*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{e^{-x^2}\left(c_2(x^2-2)\,x^2\,\text{ExpIntegralEi}\,(x^2)+4c_1x^4-x^2\left(c_2e^{x^2}+8c_1\right)+c_2e^{x^2}\right)}{4x}$$

2.607 problem 621

2.607.1 Maple step by step solution 5716

Internal problem ID [8097]
Internal file name [`OUTPUT/7030_Sunday_June_05_2022_05_25_48_PM_73756053/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 621.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(-2x^2+1)\,y' - 4(2x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (-2x^3+x)\,y' + (-8x^2-4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^3 + x \\ C &= -8x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 24x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1159: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + 6 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{x} - \frac{21}{8x^3} + \frac{63}{8x^5} - \frac{3465}{128x^7} + \frac{13041}{128x^9} - \frac{417501}{1024x^{11}} + \frac{1744659}{1024x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 24x^2 + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(x^2 + 6\right) + \left(\frac{15}{4x^2}\right) \\ &= x^2 + 6 + \frac{15}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 6. Now b can be found.

$$\begin{aligned} b &= (6) - (0) \\ &= 6 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{6}{1} - 1\right) &&= \frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{6}{1} - 1\right) &&= -\frac{7}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4 + 24x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$\frac{5}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\ &= \frac{5}{2} - \left(\frac{5}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (x) \\ &= \frac{5}{2x} + x \\ &= \frac{5}{2x} + x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{2x} + x\right)(0) + \left(\left(-\frac{5}{2x^2} + 1\right) + \left(\frac{5}{2x} + x\right)^2 - \left(\frac{4x^4 + 24x^2 + 15}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{5}{2x} + x\right) dx} \\ &= x^{\frac{5}{2}} e^{\frac{x^2}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^3 + x}{x^2} \, dx} \\ &= z_1 e^{\frac{x^2}{2} - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{\frac{x^2}{2}}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 e^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(x^2)\,x^4 + x^2\mathrm{e}^{-x^2} - \mathrm{e}^{-x^2}}{4x^4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\mathrm{e}^{x^2}\right) + c_2\left(x^2\mathrm{e}^{x^2}\left(\frac{-\operatorname{expIntegral}_1(x^2)\,x^4 + x^2\mathrm{e}^{-x^2} - \mathrm{e}^{-x^2}}{4x^4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2\mathrm{e}^{x^2} + \frac{c_2\left(-\operatorname{expIntegral}_1(x^2)\,\mathrm{e}^{x^2}x^4 + x^2 - 1\right)}{4x^2} \tag{1}$$

Verification of solutions

$$y = c_1 x^2\mathrm{e}^{x^2} + \frac{c_2\left(-\operatorname{expIntegral}_1(x^2)\,\mathrm{e}^{x^2}x^4 + x^2 - 1\right)}{4x^2}$$

Verified OK.

2.607.1 Maple step by step solution

Let's solve

$x^2y'' + (-2x^3 + x)\,y' + (-8x^2 - 4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4(2x^2+1)y}{x^2} + \frac{(2x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2-1)y'}{x} - \frac{4(2x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x^2-1}{x}, P_3(x) = -\frac{4(2x^2+1)}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -4$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(2x^2-1)\, y' + (-8x^2-4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-2+r)x^r + a_1(3+r)(-1+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-2) - 2a_{k-2}(k+r\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term must be 0

  $a_1(3+r)(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+2)(a_k(k+r-2) - 2a_{k-2}) = 0$

- Shift index using $k- >k+2$

  $(k+r+4)(a_{k+2}(k+r) - 2a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{k+r}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = \frac{2a_k}{k-2}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{2a_k}{k+2}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{2a_k}{k+2}, a_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(x^2*diff(y(x),x$2)+x*(1-2*x^2)*diff(y(x),x)-4*(1+2*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 e^{x^2} + \frac{c_2 e^{x^2} \left(\text{expIntegral}_1\left(x^2\right) x^4 - e^{-x^2} x^2 + e^{-x^2}\right)}{4x^2}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 46

```
DSolve[x^2*y''[x]+x*(1-2*x^2)*y'[x]-4*(1+2*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{c_2 \left(e^{x^2} x^4 \text{ExpIntegralEi}\left(-x^2\right) + x^2 - 1\right)}{4x^2} + c_1 e^{x^2} x^2$$

2.608 problem 622

2.608.1 Maple step by step solution 5727

Internal problem ID [8098]
Internal file name [`OUTPUT/7031_Sunday_June_05_2022_05_25_52_PM_94151189/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 622.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(-3x^2+1)\,y' - 4(-3x^2+1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (-3x^3+x)\,y' + (12x^2-4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x^3 + x \\ C &= 12x^2 - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^4 - 60x^2 + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1161: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9x^2}{4} - 15 + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3x}{2} - \frac{5}{x} - \frac{85}{12x^3} - \frac{425}{18x^5} - \frac{41225}{432x^7} - \frac{278375}{648x^9} - \frac{1787125}{864x^{11}} - \frac{40534375}{3888x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{3x}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{9x^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^4 - 60x^2 + 15}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{9x^2}{4} - 15\right) + \left(\frac{15}{4x^2}\right) \\
&= \frac{9x^2}{4} - 15 + \frac{15}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is -15. Now b can be found.

$$
\begin{aligned}
b &= (-15) - (0) \\
&= -15
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-15}{\frac{3}{2}} - 1\right) &= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-15}{\frac{3}{2}} - 1\right) &= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^4 - 60x^2 + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{3x}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{9}{2} - \left(\frac{5}{2}\right) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{5}{2x} + (-)\left(\frac{3x}{2}\right) \\ &= \frac{5}{2x} - \frac{3x}{2} \\ &= \frac{5}{2x} - \frac{3x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{5}{2x} - \frac{3x}{2}\right)(2x + a_1) + \left(\left(-\frac{5}{2x^2} - \frac{3}{2}\right) + \left(\frac{5}{2x} - \frac{3x}{2}\right)^2 - \left(\frac{9x^4 - 60x^2 + 15}{4x^2}\right)\right) = 0$$

$$\frac{3x^2 a_1 + 6(2 + a_0)\, x + 5a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2\right) e^{\int \left(\frac{5}{2x} - \frac{3x}{2}\right) dx} \\
&= \left(x^2 - 2\right) e^{-\frac{3x^2}{4} + \frac{5 \ln(x)}{2}} \\
&= \left(x^2 - 2\right) x^{\frac{5}{2}} e^{-\frac{3x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x^3 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{3x^2}{4} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{3x^2}{4}}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2 \left(x^2 - 2\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x^3+x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3x^2}{2} - \ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 \left(x^2-2\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \left(x^2 - 2\right)\right) + c_2 \left(x^2 \left(x^2 - 2\right) \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 \left(x^2-2\right)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(x^2 - 2\right) + c_2 x^2 \left(x^2 - 2\right) \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 \left(x^2-2\right)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(x^2 - 2\right) + c_2 x^2 \left(x^2 - 2\right) \left(\int \frac{\mathrm{e}^{\frac{3x^2}{2}}}{x^5 \left(x^2-2\right)^2} dx\right)$$

Verified OK.

2.608.1 Maple step by step solution

Let's solve

$x^2y'' + (-3x^3 + x)\, y' + (12x^2 - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4(3x^2-1)y}{x^2} + \frac{(3x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3x^2-1)y'}{x} + \frac{4(3x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{3x^2-1}{x}, P_3(x) = \frac{4(3x^2-1)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(3x^2 - 1)\, y' + (12x^2 - 4)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2+r)\,(-2+r)\, x^r + a_1(3+r)\,(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(2+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-2, 2\}$$

- Each term must be 0

  $$a_1(3+r)\,(-1+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+2)\,(k+r-2) - 3a_{k-2}(k-6+r) = 0$$

- Shift index using $k->k+2$

  $$a_{k+2}(k+4+r)\,(k+r) - 3a_k(k+r-4) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{3a_k(k+r-4)}{(k+4+r)(k+r)}$$

- Recursion relation for $r = -2$; series terminates at $k = 6$

  $$a_{k+2} = \frac{3a_k(k-6)}{(k+2)(k-2)}$$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=\frac{3a_k(k-6)}{(k+2)(k-2)}$
- Recursion relation for $r=2$; series terminates at $k=2$

  $a_{k+2}=\frac{3a_k(k-2)}{(k+6)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{3a_k(k-2)}{(k+6)(k+2)}, a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*(1-3*x^2)*diff(y(x),x)-4*(1-3*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 (x^2 - 2) + c_2 x^2 (x^2 - 2) \left(\int \frac{e^{\frac{3x^2}{2}}}{(x^2 - 2)^2 x^5} dx \right)$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 89

```
DSolve[x^2*y''[x]+x*(1-3*x^2)*y'[x]-4*(1-3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{64} \left(27 c_2 (x^2 - 2) x^2 \operatorname{ExpIntegralEi}\left(\frac{3x^2}{2}\right) + 64 c_1 x^4 - 2x^2 \left(9 c_2 e^{\frac{3x^2}{2}} + 64 c_1\right) + 24 c_2 e^{\frac{3x^2}{2}} + \frac{8 c_2 e^{\frac{3x^2}{2}}}{x^2} \right)$$

2.609 problem 623

2.609.1 Maple step by step solution 5737

Internal problem ID [8099]
Internal file name [OUTPUT/7032_Sunday_June_05_2022_05_25_55_PM_5304461/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 623.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x^2+1)\,y''+x(11x^2+5)\,y'+24x^2y=0}$$

The ODE is

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0$$

Or

$$x(x^3y''+11x^2y'+24yx+xy''+5y')=0$$

For $x\neq 0$ the above simplifies to

$$(x^3+x)\,y''+11x^2y'+24yx+5y'=0$$

Writing the ode as

$$(x^4+x^2)\,y''+(11x^3+5x)\,y'+24x^2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x^4+x^2\\B&=11x^3+5x\\C&=24x^2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^4 + 6x^2 + 15 \\ t &= 4\left(x^3 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1163: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{9i}{4(x-i)} - \frac{9i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3x^4 + 6x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (0) \\ &= -\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\ &= -\frac{3}{2x} + \frac{3x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{3}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{11x^3+5x}{x^4+x^2} \, dx} \\ &= z_1 e^{-\frac{5\ln(x)}{2} - \frac{3\ln(x^2+1)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{5}{2}} (x^2+1)^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{11x^3+5x}{x^4+x^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{-5\ln(x)-3\ln(x^2+1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\frac{-2x^2-1}{4\,(x^2+1)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{-2x^2-1}{4\,(x^2+1)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2(-2x^2-1)}{4x^4\,(x^2+1)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2(-2x^2-1)}{4x^4\,(x^2+1)^2}$$

Verified OK. `{x <> 0}`

2.609.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y'' + (11x^3+5x)\,y' + 24x^2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{24y}{x^2+1} - \frac{(11x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(11x^2+5)y'}{x(x^2+1)} + \frac{24y}{x^2+1} = 0$$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{11x^2+5}{x(x^2+1)}, P_3(x) = \frac{24}{x^2+1}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Bigg|_{x=0} = 5$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $24yx + (11x^2 + 5)\, y' + x(x^2 + 1)\, y'' = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(4+r)\, x^{-1+r} + a_1(1+r)(5+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+r+1)(k+5+r) + a_{k-1}(k+5+r)(k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(4+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-4, 0\}$

- Each term must be 0

$a_1(1+r)(5+r) = 0$

- Each term in the series must be 0, giving the recursion relation

$(k+5+r)(a_{k+1}(k+r+1) + a_{k-1}(k+3+r)) = 0$

- Shift index using $k- >k+1$

$(k+r+6)(a_{k+2}(k+2+r) + a_k(k+r+4)) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{a_k(k+r+4)}{k+2+r}$

- Recursion relation for $r = -4$

$a_{k+2} = -\frac{a_k k}{k-2}$

- Series not valid for $r = -4$, division by 0 in the recursion relation at $k = 2$

$a_{k+2} = -\frac{a_k k}{k-2}$

- Recursion relation for $r = 0$

$a_{k+2} = -\frac{a_k(k+4)}{k+2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{k+2}, 5a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+11*x^2)*diff(y(x),x)+24*x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x^2+1)}{(x^2+1)^2 x^4} + \frac{c_2}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 36

```
DSolve[x^2*(1+x^2)*y''[x]+x*(5+11*x^2)*y'[x]+24*x^2*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\frac{-4c_1x^4 + 2c_2x^2 + c_2}{4x^4 (x^2+1)^2}$$

2.610 problem 624

2.610.1 Maple step by step solution 5747

Internal problem ID [8100]
Internal file name [`OUTPUT/7033_Sunday_June_05_2022_05_25_58_PM_65083384/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 624.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+8xy'-(-x^2+35)\,y=0}$$

Writing the ode as

$$(4x^4+4x^2)\,y''+8xy'+(x^2-35)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 8x \\ C &= x^2-35 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 22x^2 + 35}{4(x^3 + x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^4 + 22x^2 + 35$$
$$t = 4(x^3 + x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 22x^2 + 35}{4(x^3 + x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1165: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{35}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{21i}{4(x-i)} - \frac{21i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^4 + 22x^2 + 35}{4(x^3 + x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 22x^2 + 35}{4(x^3 + x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\
&= -\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)} \\
&= -\frac{5}{2x} + \frac{3x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{5}{2x^2} - \frac{3}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{5}{2x} + \frac{3}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^4+4x^2}\,dx} \\
&= z_1 e^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\frac{\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8x}{4x^4+4x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)+\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{1}{x^2+1}+\frac{\ln\left(x^2+1\right)}{2}-\frac{1}{4\left(x^2+1\right)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}\right) + c_2\left(\frac{\left(x^2+1\right)^2}{x^{\frac{7}{2}}}\left(\frac{1}{x^2+1}+\frac{\ln\left(x^2+1\right)}{2}-\frac{1}{4\left(x^2+1\right)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+1\right)^2}{x^{\frac{7}{2}}} + \frac{c_2\left(\ln\left(x^2+1\right)\left(x^2+1\right)^2+2x^2+\frac{3}{2}\right)}{2x^{\frac{7}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 (x^2+1)^2}{x^{\frac{7}{2}}} + \frac{c_2 \left(\ln \left(x^2+1\right) \left(x^2+1\right)^2 + 2x^2 + \frac{3}{2}\right)}{2x^{\frac{7}{2}}}$$

Verified OK.

2.610.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\, y'' + 8xy' + (x^2-35)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-35)y}{4x^2(x^2+1)} - \frac{2y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x(x^2+1)} + \frac{(x^2-35)y}{4x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x(x^2+1)}, P_3(x) = \frac{x^2-35}{4x^2(x^2+1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{35}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2(x^2+1)\, y'' + 8xy' + (x^2-35)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(7+2r)\,(-5+2r)\, x^r + a_1(9+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+7)\,(2k+2r-5) + a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(7+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{7}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

$$a_1(9+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{5}{2}\right)\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{7}{2}\right)\right) = 0$$

- Shift index using $k- >k + 2$

$$4\left(k-\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right)a_k+a_{k+2}\left(k+\frac{11}{2}+r\right)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{(2k+2r-1)a_k}{2k+11+2r}$$

- Recursion relation for $r=-\frac{7}{2}$; series terminates at $k=4$

$$a_{k+2}=-\frac{(2k-8)a_k}{2k+4}$$

- Solution for $r=-\frac{7}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{7}{2}}, a_{k+2}=-\frac{(2k-8)a_k}{2k+4}, a_1=0\right]$$

- Recursion relation for $r=\frac{5}{2}$

$$a_{k+2}=-\frac{(2k+4)a_k}{2k+16}$$

- Solution for $r=\frac{5}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{5}{2}}, a_{k+2}=-\frac{(2k+4)a_k}{2k+16}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{7}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{5}{2}}\right), a_{k+2}=-\frac{(2k-8)a_k}{2k+4}, a_1=0, b_{k+2}=-\frac{(2k+4)b_k}{2k+16}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+8*x*diff(y(x),x)-(35-x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^4+2x^2+1)}{x^{\frac{7}{2}}} + \frac{c_2\left(\frac{\ln(x^2+1)x^4}{2} + \ln(x^2+1)\,x^2 + x^2 + \frac{\ln(x^2+1)}{2} + \frac{3}{4}\right)}{x^{\frac{7}{2}}}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 53

```
DSolve[4*x^2*(1+x^2)*y''[x]+8*x*y'[x]-(35-x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{4c_1(x^2+1)^2 + c_2(4x^2+3) + 2c_2(x^2+1)^2\log(x^2+1)}{4x^{7/2}}$$

2.611 problem 625

2.611.1 Maple step by step solution 5757

Internal problem ID [8101]
Internal file name [OUTPUT/7034_Sunday_June_05_2022_05_26_02_PM_7152326/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 625.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''-x(-x^2+5)\,y'-(25x^2+7)\,y=0}$$

Writing the ode as

$$(x^4+x^2)\,y''+(x^3-5x)\,y'+(-25x^2-7)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4+x^2 \\ B &= x^3-5x \\ C &= -25x^2-7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 99x^4 + 150x^2 + 63$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1167: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{63}{4x^2} + \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} - \frac{15i}{4(x-i)} + \frac{15i}{4(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{99x^4 + 150x^2 + 63}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{11}{2}$	$-\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{9}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= -\frac{9}{2} - \left(-\frac{9}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x - c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} + (-)(0) \\
&= -\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)} \\
&= -\frac{7}{2x} - \frac{x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right)(0) + \left(\left(\frac{7}{2x^2} + \frac{1}{2(x-i)^2} + \frac{1}{2(x+i)^2}\right) + \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{7}{2x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right) dx} \\
&= \frac{1}{x^{\frac{7}{2}} \sqrt{x^2+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3-5x}{x^4+x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}-\frac{3\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{x^{\frac{5}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x\left(x^2+1\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3-5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)-3\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{1}{10}x^{10}+\frac{1}{8}x^8\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x\left(x^2+1\right)^2}\right) + c_2\left(\frac{1}{x\left(x^2+1\right)^2}\left(\frac{1}{10}x^{10}+\frac{1}{8}x^8\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x\left(x^2+1\right)^2} + \frac{c_2 x^7\left(4x^2+5\right)}{40\left(x^2+1\right)^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{x(x^2+1)^2} + \frac{c_2 x^7 (4x^2+5)}{40(x^2+1)^2}$$

Verified OK.

2.611.1 Maple step by step solution

Let's solve

$(x^4 + x^2) y'' + (x^3 - 5x) y' + (-25x^2 - 7) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(25x^2+7)y}{x^2(x^2+1)} - \frac{(x^2-5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-5)y'}{x(x^2+1)} - \frac{(25x^2+7)y}{x^2(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2-5}{x(x^2+1)}, P_3(x) = -\frac{25x^2+7}{x^2(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -5$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -7$

  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2+1) y'' + x(x^2-5) y' + (-25x^2-7) y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-7+r)\, x^r + a_1(2+r)\,(-6+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)\,(k+r-7) + a_{k-2}(k+3-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 7\}$$

- Each term must be 0

$$a_1(2+r)\,(-6+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-7)\left(a_k(k+r+1)+a_{k-2}(k+3+r)\right)=0$$

- Shift index using $k->k+2$

$$(k+r-5)\left(a_{k+2}(k+3+r)+a_k(k+r+5)\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r+5)}{k+3+r}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{a_k(k+4)}{k+2}$$

- Solution for $r=-1$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0\right]$$

- Recursion relation for $r=7$

$$a_{k+2}=-\frac{a_k(k+12)}{k+10}$$

- Solution for $r=7$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+7}, a_{k+2}=-\frac{a_k(k+12)}{k+10}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+7}\right), a_{k+2}=-\frac{a_k(k+4)}{k+2}, a_1=0, b_{k+2}=-\frac{b_k(k+12)}{k+10}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)-x*(5-x^2)*diff(y(x),x)-(7+25*x^2)*y(x)=0,y(x), singsol=all
```

$$y(x) = \frac{c_1}{(x^2+1)^2 x} + \frac{c_2\left(x^{10} + \frac{5}{4}x^8\right)}{(x^2+1)^2 x}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 37

```
DSolve[x^2*(1+x^2)*y''[x]-x*(5-x^2)*y'[x]-(7+25*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{c_2(4x^2+5)\,x^8 + 40c_1}{40x\,(x^2+1)^2}$$

2.612 problem 626

2.612.1 Maple step by step solution 5767

Internal problem ID [8102]
Internal file name [`OUTPUT/7035_Sunday_June_05_2022_05_26_05_PM_39906177/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 626.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^{2}\left(x^{2}+1\right) y''+x\left(2x^{2}+5\right) y'-21y=0}$$

Writing the ode as

$$\left(x^{4}+x^{2}\right) y''+\left(2x^{3}+5x\right) y'-21y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^{4}+x^{2}\\ B&=2x^{3}+5x \\ C&=-21 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 78x^2 + 99$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{78x^2 + 99}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1169: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{99}{4x^2} + \frac{21}{16(x-i)^2} + \frac{21}{16(x+i)^2} + \frac{219i}{16(x-i)} - \frac{219i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{99}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{9}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_{\infty} &= 0 \\
\alpha_{\infty}^{+} &= 0 \\
\alpha_{\infty}^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{78x^2 + 99}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{11}{2}$	$-\frac{9}{2}$
i	2	0	$\frac{7}{4}$	$-\frac{3}{4}$
$-i$	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} + (-)(0) \\
&= -\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)} \\
&= -\frac{9}{2x} + \frac{7x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right)(2x + a_1) + \left(\left(\frac{9}{2x^2} - \frac{7}{4(x-i)^2} - \frac{7}{4(x+i)^2}\right) + \left(-\frac{9}{2x} + \frac{7}{4(x-
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 8, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 8
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 8\right) e^{\int \left(-\frac{9}{2x} + \frac{7}{4(x-i)} + \frac{7}{4(x+i)}\right) dx} \\
&= \left(x^2 + 8\right) e^{-\frac{9\ln(x)}{2} + \frac{7\ln\left(x^2+1\right)}{4}} \\
&= \frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{7}{4}}}{x^{\frac{9}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3+5x}{x^4+x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}+\frac{3\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\frac{\left(x^2+1\right)^{\frac{3}{4}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3+5x}{x^4+x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)+\frac{3\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{-35x^6-140x^4-168x^2-64}{\left(x^2+1\right)^{\frac{5}{2}}\left(35x^2+280\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}\right) + c_2\left(\frac{\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7}\left(\frac{-35x^6-140x^4-168x^2-64}{\left(x^2+1\right)^{\frac{5}{2}}\left(35x^2+280\right)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(x^2+8\right)\left(x^2+1\right)^{\frac{5}{2}}}{x^7} + \frac{c_2\left(-35x^6-140x^4-168x^2-64\right)}{35x^7} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2+8)\,(x^2+1)^{\frac{5}{2}}}{x^7} + \frac{c_2(-35x^6-140x^4-168x^2-64)}{35x^7}$$

Verified OK.

2.612.1 Maple step by step solution

Let's solve

$(x^4+x^2)\,y''+(2x^3+5x)\,y'-21y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{21y}{x^2(x^2+1)}-\frac{(2x^2+5)y'}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x^2+5)y'}{x(x^2+1)}-\frac{21y}{x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x^2+5}{x(x^2+1)}, P_3(x)=-\frac{21}{x^2(x^2+1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=5$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=-21$

  - $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2(x^2+1)\,y''+x(2x^2+5)\,y'-21y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(7+r)\,(-3+r)\, x^r + a_1(8+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(7+r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-7, 3\}$$

- Each term must be 0

  $$a_1(8+r)\,(-2+r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+7)\,(k+r-3) + a_{k-2}(k-2+r)\,(k+r-1) = 0$$

- Shift index using $k- > k+2$

  $$a_{k+2}(k+9+r)\,(k+r-1) + a_k(k+r)\,(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a_k(k+r)(k+r+1)}{(k+9+r)(k+r-1)}$$

- Recursion relation for $r = -7$; series terminates at $k = 6$

$$a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}$$

- Solution for $r = -7$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-7}, a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k(k+3)(k+4)}{(k+12)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-7}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{a_k(k-7)(k-6)}{(k+2)(k-8)}, a_1 = 0, b_{k+2} = -\frac{b_k(k+3)(k+4)}{(k+12)(k+2)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*(5+2*x^2)*diff(y(x),x)-21*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(35x^6 + 140x^4 + 168x^2 + 64)}{x^7} + \frac{c_2(x^2+1)^{\frac{5}{2}}(x^2+8)}{x^7}$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 52

```
DSolve[x^2*(1+x^2)*y''[x]+x*(5+2*x^2)*y'[x]-21*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{35c_1(x^2+1)^{5/2}(x^2+8) - c_2(35x^6+140x^4+168x^2+64)}{35x^7}$$

2.613 problem 627

2.613.1 Maple step by step solution 5777

Internal problem ID [8103]
Internal file name [`OUTPUT/7036_Sunday_June_05_2022_05_26_08_PM_77263945/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 627.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2(x^2+1)\,y''+4x(x^2+2)\,y'-(x^2+15)\,y=0}$$

Writing the ode as

$$(4x^4+4x^2)\,y''+(4x^3+8x)\,y'+(-x^2-15)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^4+4x^2 \\ B &= 4x^3+8x \\ C &= -x^2-15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 10x^2 + 15$$
$$t = 4\left(x^3 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{10x^2 + 15}{4\left(x^3 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1171: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2} + \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{35i}{16(x-i)} - \frac{35i}{16(x+i)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{10x^2 + 15}{4\left(x^3 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+} + \alpha_{c_3}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{+}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} + (-)(0) \\
&= -\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)} \\
&= -\frac{3}{2x} + \frac{5x}{2x^2+2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right)(0) + \left(\left(\frac{3}{2x^2} - \frac{5}{4(x-i)^2} - \frac{5}{4(x+i)^2}\right) + \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{3}{2x} + \frac{5}{4(x-i)} + \frac{5}{4(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{5}{4}}}{x^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^3+8x}{4x^4+4x^2}\, dx} \\
&= z_1 e^{-\ln(x) + \frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1 \left(\frac{(x^2+1)^{\frac{1}{4}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3+8x}{4x^4+4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)+\frac{\ln(x^2+1)}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{-3x^2-2}{3(x^2+1)^{\frac{3}{2}}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}\right) + c_2\left(\frac{(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}\left(\frac{-3x^2-2}{3(x^2+1)^{\frac{3}{2}}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}} + \frac{c_2(-3x^2-2)}{3x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}} + \frac{c_2(-3x^2-2)}{3x^{\frac{5}{2}}}$$

Verified OK.

2.613.1 Maple step by step solution

Let's solve

$(4x^4+4x^2)\,y''+(4x^3+8x)\,y'+(-x^2-15)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2+2)y'}{x(x^2+1)}+\frac{(x^2+15)y}{4x^2(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x^2+2)y'}{x(x^2+1)}-\frac{(x^2+15)y}{4x^2(x^2+1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x^2+2}{x(x^2+1)}, P_3(x)=-\frac{x^2+15}{4x^2(x^2+1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-\frac{15}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2(x^2+1)\,y''+4x(x^2+2)\,y'+(-x^2-15)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+2r)\,(-3+2r)\, x^r + a_1(7+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-3) + a_{k\text{-}}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{3}{2}\right)\left(\left(k+r-\frac{5}{2}\right) a_{k-2} + a_k\left(k+r+\frac{5}{2}\right)\right) = 0$

- Shift index using $k->k+2$

  $4\left(k+\frac{1}{2}+r\right)\left(\left(k-\frac{1}{2}+r\right) a_k + a_{k+2}\left(k+\frac{9}{2}+r\right)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{(2k+2r-1)a_k}{2k+9+2r}$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{(2k-6)a_k}{2k+4}$
- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{(2k+2)a_k}{2k+12}$
- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{(2k+2)a_k}{2k+12}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{(2k-6)a_k}{2k+4}, a_1 = 0, b_{k+2} = -\frac{(2k+2)b_k}{2k+12}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(4*x^2*(1+x^2)*diff(y(x),x$2)+4*x*(2+x^2)*diff(y(x),x)-(15+x^2)*y(x)=0,y(x), singsol=a
```

$$y(x) = \frac{c_1(3x^2+2)}{x^{\frac{5}{2}}} + \frac{c_2(x^2+1)^{\frac{3}{2}}}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 39

```
DSolve[4*x^2*(1+x^2)*y''[x]+4*x*(2+x^2)*y'[x]-(15+x^2)*y[x]==0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{3c_1(x^2+1)^{3/2} - c_2(3x^2+2)}{3x^{5/2}}$$

2.614 problem 628

2.614.1 Maple step by step solution 5787

Internal problem ID [8104]
Internal file name [`OUTPUT/7037_Sunday_June_05_2022_05_26_12_PM_56039791/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 628.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{2(t+1)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0}$$

Writing the ode as

$$y'' + \frac{(-2t-2)\, y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{-2t-2}{t^2+2t-1} \\ C &= \frac{2}{t^2+2t-1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{(t^2 + 2t - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = (t^2 + 2t - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6}{(t^2 + 2t - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 + 2t - 1)^2$. There is a pole at $t = \sqrt{2} - 1$ of order 2. There is a pole at $t = -1 - \sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(t - \sqrt{2} + 1\right)^2} + \frac{3}{4\left(t + 1 + \sqrt{2}\right)^2} - \frac{3\sqrt{2}}{8\left(t - \sqrt{2} + 1\right)} + \frac{3\sqrt{2}}{8\left(t + 1 + \sqrt{2}\right)}$$

For the pole at $t = \sqrt{2} - 1$ let b be the coefficient of $\frac{1}{\left(t-\sqrt{2}+1\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -1 - \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(t+1+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^{+} &= 0\\ \alpha_\infty^{-} &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{(t^2 + 2t - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\sqrt{2} - 1$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-1 - \sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 1 - (1)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} + (-)\,(0) \\
&= -\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)} \\
&= \frac{t+1-2\sqrt{2}}{t^2+2t-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(t-\sqrt{2}+1\right)^2} - \frac{3}{2\left(t+1+\sqrt{2}\right)^2}\right) + \left(-\frac{}{2\,(t -}\right.\right.
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t-\sqrt{2}+1\right)} + \frac{3}{2\left(t+1+\sqrt{2}\right)}\right) dt} \\
&= \frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}}}{\sqrt{t-\sqrt{2}+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+2t-1\right)}{2}} \\
&= z_1\left(\sqrt{t^2+2t-1}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{-2t-2}{t^2+2t-1}}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\ln(t^2+2t-1)}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{-t-1}{\left(t+1+\sqrt{2}\right)^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}}\right) + c_2 \left(\frac{\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} \left(\frac{-t-1}{\left(t+1+\sqrt{2}\right)^2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} + \frac{c_2\sqrt{t^2+2t-1}\,(-t-1)}{\sqrt{t+1+\sqrt{2}}\,\sqrt{t-\sqrt{2}+1}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(t+1+\sqrt{2}\right)^{\frac{3}{2}} \sqrt{t^2+2t-1}}{\sqrt{t-\sqrt{2}+1}} + \frac{c_2\sqrt{t^2+2t-1}\,(-t-1)}{\sqrt{t+1+\sqrt{2}}\,\sqrt{t-\sqrt{2}+1}}$$

Verified OK.

2.614.1 Maple step by step solution

Let's solve

$y'' + \frac{(-2t-2)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2(t+1)y'}{t^2+2t-1} - \frac{2y}{t^2+2t-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(t+1)y'}{t^2+2t-1} + \frac{2y}{t^2+2t-1} = 0$

□ Check to see if t_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = -\frac{2(t+1)}{t^2+2t-1}, P_3(t) = \frac{2}{t^2+2t-1}\right]$

- ◦ $(t+1+\sqrt{2}) \cdot P_2(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2}) \cdot P_2(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

- ◦ $(t+1+\sqrt{2})^2 \cdot P_3(t)$ is analytic at $t = -1-\sqrt{2}$

  $\left((t+1+\sqrt{2})^2 \cdot P_3(t)\right)\Big|_{t=-1-\sqrt{2}} = 0$

- ◦ $t = -1-\sqrt{2}$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1-\sqrt{2}$

- Multiply by denominators

  $y''(t^2+2t-1) + (-2t-2)\,y' + 2y = 0$

- Change variables using $t = u-1-\sqrt{2}$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u\sqrt{2})\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2\sqrt{2})\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2\sqrt{2}\, r(r-2)\, a_0 u^{r-1} + \left(\sum_{k=0}^{\infty} \left(-2\sqrt{2}\,(k+1+r)\,(k+r-1)\, a_{k+1} + a_k(k+r-1)\,(k+r-2)\right) u$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2\sqrt{2}\, r(r-2) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(-2a_{k+1}(k+1+r)\sqrt{2} + a_k(k+r-2)\right)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)\sqrt{2}}{4(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)\sqrt{2}}{4(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0\sqrt{2}}{2}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1\sqrt{2}}{8}$

- Express in terms of a_0

$a_2 = \frac{a_0}{8}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot \left(1 - \frac{u\sqrt{2}}{2} + \frac{u^2}{8}\right)$

- Revert the change of variables $u = t + 1 + \sqrt{2}$

$\left[y = a_0 \left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right)\right]$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}$

- Solution for $r = 2$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$

- Revert the change of variables $u = t + 1 + \sqrt{2}$

$\left[y = \sum_{k=0}^{\infty} a_k \left(t + 1 + \sqrt{2}\right)^{k+2}, a_{k+1} = \frac{a_k k\sqrt{2}}{4(k+3)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0 \left(\frac{(-2t-2)\sqrt{2}}{8} + \frac{t^2}{8} + \frac{t}{4} + \frac{3}{8}\right) + \left(\sum_{k=0}^{\infty} b_k \left(t + 1 + \sqrt{2}\right)^{k+2}\right), b_{k+1} = \frac{b_k k\sqrt{2}}{4(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(t),t$2)-2*(t+1)/(t^2+2*t-1)*diff(y(t),t)+2/(t^2+2*t-1)*y(t)=0,y(t), singsol=al
```

$$y(t) = c_1(t+1) + c_2(t^2+1)$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 64

```
DSolve[y''[t]-2*(t+1)/(t^2+2*t-1)*y'[t]+2/(t^2+2*t-1)*y[t]==0,y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{\sqrt{t^2+2t-1}\left(c_1\left(t^2-2\left(\sqrt{2}-1\right)t-2\sqrt{2}+3\right)+c_2(t+1)\right)}{\sqrt{-t^2-2t+1}}$$

2.615 problem 629

2.615.1 Maple step by step solution 5794

Internal problem ID [8105]
Internal file name [`OUTPUT/7038_Sunday_June_05_2022_05_26_15_PM_50389726/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 629.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4ty' + \left(4t^2 - 2\right) y = 0}$$

Writing the ode as

$$y'' - 4ty' + \left(4t^2 - 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4t \\ C &= 4t^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1175: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4t}{1}\,dt} \\ &= z_1 e^{t^2} \\ &= z_1\left(e^{t^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{t^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t}{1}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{2t^2}}{(y_1)^2}\,dt \\ &= y_1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{t^2}\right) + c_2\left(\mathrm{e}^{t^2}(t)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{t^2} + c_2 \mathrm{e}^{t^2} t \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{t^2} + c_2 \mathrm{e}^{t^2} t
$$

Verified OK.

2.615.1 Maple step by step solution

Let's solve

$y'' - 4ty' + (4t^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..2$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

  ◦ Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

  ◦ Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k k\, t^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, t + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, t^k\right) = 0$$

- The coefficients of each power of t must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(t),t$2)-4*t*diff(y(t),t)+(4*t^2-2)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{t^2} + c_2 e^{t^2} t$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[y''[t]-4*t*y'[t]+(4*t^2-2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{t^2}(c_2 t + c_1)$$

2.616 problem 630

2.616.1 Maple step by step solution 5803

Internal problem ID [8106]
Internal file name [`OUTPUT/7039_Sunday_June_05_2022_05_26_17_PM_80606560/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 630.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-t^2+1)\,y''-2ty'+2y=0}$$

Writing the ode as

$$(-t^2+1)\,y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2t^2 - 3}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2t^2 - 3 \\ t &= (t^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2t^2 - 3}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2-1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4(t+1)} - \frac{1}{4(t+1)^2} + \frac{5}{4(t-1)} - \frac{1}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2t^2 - 3}{(t^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=2\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{2t^2-3}{(t^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= 2-(1)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} + (0) \\
&= \frac{1}{2t-2} + \frac{1}{2t+2} \\
&= \frac{t}{t^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(1) + \left(\left(-\frac{1}{2(t-1)^2} - \frac{1}{2(t+1)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{2t^2-3}{(t^2-1)^2}\right)\right) = -\frac{2a_0}{t^2-1} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t)\, e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= (t)\, e^{\frac{\ln(t-1)}{2} + \frac{\ln(t+1)}{2}} \\
&= t\sqrt{t-1}\,\sqrt{t+1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1}\,dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2}-\frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{1}{t} - \frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\right) + c_2 \left(\frac{t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}}\left(\frac{1}{t} - \frac{\ln(t+1)}{2} + \frac{\ln(t-1)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} + \frac{c_2\sqrt{t^2-1}\,(\ln(t-1)\,t - \ln(t+1)\,t + 2)}{2\sqrt{t-1}\,\sqrt{t+1}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 t\sqrt{t^2-1}}{\sqrt{t-1}\,\sqrt{t+1}} + \frac{c_2\sqrt{t^2-1}\,(\ln(t-1)\,t - \ln(t+1)\,t + 2)}{2\sqrt{t-1}\,\sqrt{t+1}}$$

Verified OK.

2.616.1 Maple step by step solution

Let's solve

$(-t^2+1)\,y'' - 2ty' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2ty'}{t^2-1} + \frac{2y}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2ty'}{t^2-1} - \frac{2y}{t^2-1} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{2}{t^2-1}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

  $\left.((t+1)\cdot P_2(t))\right|_{t=-1} = 1$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

  $\left.((t+1)^2\cdot P_3(t))\right|_{t=-1} = 0$

  ◦ $t=-1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $y''(t^2-1) + 2ty' - 2y = 0$

- Change variables using $t = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u+1)$

- Revert the change of variables $u = t+1$

  $[y = -a_0 t]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((1-t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 \left(-\frac{\ln(t+1)\, t}{2} + \frac{\ln(t-1)\, t}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 33

```
DSolve[(1-t^2)*y''[t]-2*t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 t - \frac{1}{2} c_2 (t \log(1-t) - t \log(t+1) + 2)$$

2.617 problem 631

Internal problem ID [8107]
Internal file name [`OUTPUT/7040_Sunday_June_05_2022_05_26_21_PM_73949353/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 631.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(t^2+1)\,y''-2ty'+2y=0}$$

Writing the ode as

$$(t^2+1)\,y''-2ty'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2+1 \\ B &= -2t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1179: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(t^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)(0) + \left(\left(\frac{1}{2\,(t-i)^2} - \frac{3}{2\,(t+i)^2}\right) + \left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)^2 - \left(-\frac{3}{(t^2 \cdots\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\
&= \frac{(t^2+1)^{\frac{3}{2}}}{(it+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{t^2+1}\, dt} \\
&= z_1 e^{\frac{\ln\left(t^2+1\right)}{2}} \\
&= z_1\left(\sqrt{t^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2t}{t^2+1}\, dt}}{(y_1)^2}\, dt$$

$$= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2}\, dt$$

$$= y_1 \left(-\frac{t}{(t+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2}\left(-\frac{t}{(t+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2\, t}{(t-i)^2\, (t+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2\, t}{(t-i)^2\, (t+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 (t^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 21

```
DSolve[(1+t^2)*y''[t]-2*t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

2.618 problem 632

2.618.1 Maple step by step solution 5819

Internal problem ID [8108]
Internal file name [OUTPUT/7041_Sunday_June_05_2022_05_26_24_PM_54068149/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 632.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-t^2+1)\, y''-2ty'+6y=0}$$

Writing the ode as

$$\begin{align*}(-t^2+1)\, y''-2ty'+6y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=-t^2+1\\ B&=-2t \tag{3}\\ C&=6\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{align*}r&=\frac{s}{t} \tag{5}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6t^2 - 7$$
$$t = (t^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{6t^2 - 7}{(t^2 - 1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2 - 1)^2$. There is a pole at $t = 1$ of order 2. There is a pole at $t = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{13}{4(t+1)} - \frac{1}{4(t+1)^2} + \frac{13}{4(t-1)} - \frac{1}{4(t-1)^2}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6t^2 - 7}{(t^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 3$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2t - 2} + \frac{1}{2t + 2} + (0) \\ &= \frac{1}{2t - 2} + \frac{1}{2t + 2} \\ &= \frac{t}{t^2 - 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)(2t + a_1) + \left(\left(-\frac{1}{2\left(t-1\right)^2} - \frac{1}{2\left(t+1\right)^2}\right) + \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right)^2 - \left(\frac{6t^2-7}{\left(t^2-1\right)^2}\right.\right.$$

$$\frac{-4a_1 t - 6a_0 - 2}{t^2 - 1}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}, a_1 = 0\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\int \left(\frac{1}{2t-2} + \frac{1}{2t+2}\right) dt} \\
&= \left(t^2 - \frac{1}{3}\right) e^{\frac{\ln(t-1)}{2} + \frac{\ln(t+1)}{2}} \\
&= \left(t^2 - \frac{1}{3}\right) \sqrt{t-1}\,\sqrt{t+1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2t}{-t^2+1} \, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{2} - \frac{\ln(t+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t-1}\,\sqrt{t+1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^2 - \frac{1}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2t}{-t^2+1}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-\ln(t-1)-\ln(t+1)}}{(y_1)^2}\, dt \\
&= y_1 \left(-\frac{9\ln(t+1)}{8} + \frac{27t}{12t^2-4} + \frac{9\ln(t-1)}{8}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2 - \frac{1}{3}\right) + c_2 \left(t^2 - \frac{1}{3}\left(-\frac{9\ln(t+1)}{8} + \frac{27t}{12t^2-4} + \frac{9\ln(t-1)}{8}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(t^2 - \frac{1}{3}\right) + c_2 \left(\frac{9\ln(t-1)\, t^2}{8} - \frac{9\ln(t+1)\, t^2}{8} - \frac{3\ln(t-1)}{8} + \frac{3\ln(t+1)}{8} + \frac{9t}{4}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(t^2 - \frac{1}{3}\right) + c_2 \left(\frac{9\ln(t-1)\, t^2}{8} - \frac{9\ln(t+1)\, t^2}{8} - \frac{3\ln(t-1)}{8} + \frac{3\ln(t+1)}{8} + \frac{9t}{4}\right)$$

Verified OK.

2.618.1 Maple step by step solution

Let's solve

$(-t^2+1)\, y'' - 2ty' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2ty'}{t^2-1} + \frac{6y}{t^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2ty'}{t^2-1} - \frac{6y}{t^2-1} = 0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = \frac{2t}{t^2-1}, P_3(t) = -\frac{6}{t^2-1}\right]$

  ◦ $(t+1) \cdot P_2(t)$ is analytic at $t = -1$

  $\left((t+1) \cdot P_2(t)\right)\bigg|_{t=-1} = 1$

  ◦ $(t+1)^2 \cdot P_3(t)$ is analytic at $t = -1$

  $\left((t+1)^2 \cdot P_3(t)\right)\bigg|_{t=-1} = 0$

  ◦ $t = -1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = -1$

- Multiply by denominators

  $y''(t^2-1) + 2ty' - 6y = 0$

- Change variables using $t = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 6y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)^2 + a_k(k+r+3)(k+r-2)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(k+3)(k-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_k(k+3)(k-2)}{2(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -3a_0$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1}{2}$$

- Express in terms of a_0

$$a_2 = \frac{3a_0}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_0 \cdot \left(1 - 3u + \frac{3}{2}u^2\right)$$

- Revert the change of variables $u = t+1$

$$\left[y = a_0\left(\frac{3t^2}{2} - \frac{1}{2}\right)\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve((1-t^2)*diff(y(t),t$2)-2*t*diff(y(t),t)+6*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(-3t^2+1) + c_2\left(-\frac{3\ln(t+1)t^2}{8} + \frac{3\ln(t-1)t^2}{8} + \frac{\ln(t+1)}{8} - \frac{\ln(t-1)}{8} + \frac{3t}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 55

```
DSolve[(1-t^2)*y''[t]-2*t*y'[t]+6*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}c_1(3t^2-1) - \frac{1}{4}c_2\left((3t^2-1)\log(1-t) + (1-3t^2)\log(t+1) + 6t\right)$$

2.619 problem 633

2.619.1 Maple step by step solution 5828

Internal problem ID [8109]
Internal file name [`OUTPUT/7042_Sunday_June_05_2022_05_26_29_PM_56471909/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 633.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2t+1)\,y''-4(t+1)\,y'+4y=0}$$

Writing the ode as

$$(2t+1)\,y''+(-4t-4)\,y'+4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t+1 \\ B &= -4t-4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4t^2 + 2}{(2t+1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4t^2 + 2 \\
t &= (2t+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{4t^2 + 2}{(2t+1)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2t+1)^2$. There is a pole at $t = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4\left(t + \frac{1}{2}\right)^2} - \frac{1}{t + \frac{1}{2}}$$

For the pole at $t = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(t+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{4t^3} + \frac{3}{32t^4} - \frac{3}{64t^5} + \frac{1}{32t^6} - \frac{1}{64t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2 + 2}{4t^2 + 4t + 1} \\ &= Q + \frac{R}{4t^2 + 4t + 1} \\ &= (1) + \left(\frac{-4t + 1}{4t^2 + 4t + 1}\right) \\ &= 1 + \frac{-4t + 1}{4t^2 + 4t + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4t^2 + 2}{(2t + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + (1) \\
&= -\frac{1}{2\left(t+\frac{1}{2}\right)} + 1 \\
&= \frac{2t}{2t+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)(0) + \left(\left(\frac{1}{2\left(t+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right)^2 - \left(\frac{4t^2+2}{(2t+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2\left(t+\frac{1}{2}\right)} + 1\right) dt} \\
&= \frac{\mathrm{e}^t}{\sqrt{2t+1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4t-4}{2t+1} \, dt} \\
&= z_1 e^{t + \frac{\ln(2t+1)}{2}} \\
&= z_1 \left(\sqrt{2t+1}\, \mathrm{e}^t\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4t-4}{2t+1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{2t+\ln(2t+1)}}{(y_1)^2}\, dt \\ &= y_1\left(-(t+1)\,\mathrm{e}^{-2t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2t}\right) + c_2\left(\mathrm{e}^{2t}\left(-(t+1)\,\mathrm{e}^{-2t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2t} + c_2(-t-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2t} + c_2(-t-1)$$

Verified OK.

2.619.1 Maple step by step solution

Let's solve

$(2t+1)\, y'' + (-4t-4)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{4y}{2t+1} + \frac{4(t+1)y'}{2t+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4(t+1)y'}{2t+1} + \frac{4y}{2t+1} = 0$

☐ Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{4(t+1)}{2t+1}, P_3(t) = \frac{4}{2t+1}\right]$

- $\left(t+\frac{1}{2}\right) \cdot P_2(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right) \cdot P_2(t)\right)\bigg|_{t=-\frac{1}{2}} = -1$

- $\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)$ is analytic at $t = -\frac{1}{2}$

  $\left(\left(t+\frac{1}{2}\right)^2 \cdot P_3(t)\right)\bigg|_{t=-\frac{1}{2}} = 0$

- $t = -\frac{1}{2}$is a regular singular point

  Check to see if $t_0 = -\frac{1}{2}$ is a regular singular point

  $t_0 = -\frac{1}{2}$

- Multiply by denominators

  $(2t+1)\, y'' + (-4t-4)\, y' + 4y = 0$

- Change variables using $t = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2} y(u)\right) + (-4u-2)\left(\frac{d}{du} y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) - 4a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$2(a_{k+1}(k+1+r) - 2a_k)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{2a_k}{k+1+r}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{2a_k}{k+1}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k}{k+1}\right]$$

- Revert the change of variables $u = t + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{2a_k}{k+3}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$$

- Revert the change of variables $u = t + \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(t+\frac{1}{2}\right)^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_{k}\left(t+\frac{1}{2}\right)^{k}\right)+\left(\sum_{k=0}^{\infty} b_{k}\left(t+\frac{1}{2}\right)^{k+2}\right), a_{k+1}=\frac{2 a_{k}}{k+1}, b_{k+1}=\frac{2 b_{k}}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2*t+1)*diff(y(t),t$2)-4*(t+1)*diff(y(t),t)+4*y(t)=0,y(t), singsol=all)
```

$$y(t)=c_{1}(t+1)+c_{2} \mathrm{e}^{2 t}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 23

```
DSolve[(2*t+1)*y''[t]-4*(t+1)*y'[t]+4*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow c_{1} e^{2 t+1}-c_{2}(t+1)$$

2.620 problem 634

2.620.1 Maple step by step solution 5835

Internal problem ID [8110]
Internal file name [OUTPUT/7043_Sunday_June_05_2022_05_26_33_PM_64765064/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 634.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0}$$

Writing the ode as

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t \\ C &= t^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t}{t^2}\,dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(t)}{\sqrt{t}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(\tan(t))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(t)}{\sqrt{t}}\right) + c_2\left(\frac{\cos(t)}{\sqrt{t}}(\tan(t))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(t)}{\sqrt{t}} + \frac{c_2 \sin(t)}{\sqrt{t}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(t)}{\sqrt{t}} + \frac{c_2 \sin(t)}{\sqrt{t}}
$$

Verified OK.

2.620.1 Maple step by step solution

Let's solve

$y''t^2 + ty' + \left(t^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4t^2-1)y}{4t^2} - \frac{y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{t} + \frac{(4t^2-1)y}{4t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(t) = \frac{1}{t}, P_3(t) = \frac{4t^2-1}{4t^2}\right]$

- ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = 1$

- ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = -\frac{1}{4}$

- $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $4y''t^2 + 4ty' + (4t^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- Convert $t^m \cdot y$ to series expansion for $m = 0..2$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

- Shift index using $k- >k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

- Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, t^{k+r}$

- Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\, (-1 + 2r)\, t^r + a_1(3 + 2r)\, (1 + 2r)\, t^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\, (2k + 2r - 1) + 4a_{k-2}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\, (-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\, (1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(t^2*diff(y(t),t$2)+t*diff(y(t),t)+(t^2-1/4)*y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1 \sin(t)}{\sqrt{t}} + \frac{c_2 \cos(t)}{\sqrt{t}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 39

```
DSolve[t^2*y''[t]+t*y'[t]+(t^2-1/4)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-it}(2c_1 - ic_2 e^{2it})}{2\sqrt{t}}$$

2.621 problem 635

Internal problem ID [8111]
Internal file name [`OUTPUT/7044_Sunday_June_05_2022_05_26_35_PM_56490024/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 635.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0}$$

Writing the ode as

$$y'' - \frac{2ty'}{t^2+1} + \frac{2y}{t^2+1} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2t}{t^2+1} \\ C &= \frac{2}{t^2+1} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(t^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (t^2+1)^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{3}{(t^2+1)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+1)^2$. There is a pole at $t = i$ of order 2. There is a pole at $t = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(t-i)^2} + \frac{3}{4(t+i)^2} + \frac{3i}{4(t-i)} - \frac{3i}{4(t+i)}$$

For the pole at $t = i$ let b be the coefficient of $\frac{1}{(t-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $t = -i$ let b be the coefficient of $\frac{1}{(t+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(t^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(t-i\right)} + \frac{3}{2\left(t+i\right)} \\
&= \frac{t-2i}{t^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)(0) + \left(\left(\frac{1}{2\,(t-i)^2} - \frac{3}{2\,(t+i)^2}\right) + \left(-\frac{1}{2\,(t-i)} + \frac{3}{2\,(t+i)}\right)^2 - \left(-\frac{3}{(t^2 \cdots}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(t-i)} + \frac{3}{2(t+i)}\right) dt} \\ &= \frac{(t^2+1)^{\frac{3}{2}}}{(it+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2t}{t^2+1}}{1} \, dt} \\ &= z_1 e^{\frac{\ln\left(t^2+1\right)}{2}} \\ &= z_1\left(\sqrt{t^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(t^2+1)^2}{(it+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2t}{t^2+1}}{1} dt}}{(y_1)^2} dt \\
&= y_1 \int \frac{e^{\ln(t^2+1)}}{(y_1)^2} dt \\
&= y_1 \left(-\frac{t}{(t+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(t^2+1)^2}{(it+1)^2}\right) + c_2 \left(\frac{(t^2+1)^2}{(it+1)^2} \left(-\frac{t}{(t+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2 t}{(t-i)^2 (t+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (t^2+1)^2}{(it+1)^2} + \frac{c_2 (t^2+1)^2 t}{(t-i)^2 (t+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(t),t$2)-2*t/(1+t^2)*diff(y(t),t)+2/(1+t^2)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 (t^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 21

```
DSolve[y''[t]-2*t/(1+t^2)*y'[t]+2/(1+t^2)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_2 t - c_1 (t - i)^2$$

2.622 problem 636

2.622.1 Maple step by step solution 5852

Internal problem ID [8112]
Internal file name [OUTPUT/7045_Sunday_June_05_2022_05_26_38_PM_5147488/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 636.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \left(t^2 + 2t + 1\right) y' - (4 + 4t)\, y = 0}$$

Writing the ode as

$$y'' + (t+1)^2 y' + (-4t - 4)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= (t+1)^2 \\ C &= -4t - 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^4 + 4t^3 + 6t^2 + 24t + 21$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(t) = \left(6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1187: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{2} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t^2}{2} + t + \frac{1}{2} + \frac{5}{t} - \frac{5}{t^2} + \frac{5}{t^3} - \frac{30}{t^4} + \frac{105}{t^5} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i t^i \\ &= \frac{1}{2} t^2 + t + \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^1 = t$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = \frac{1}{4} t^4 + t^3 + \frac{3}{2} t^2 + t + \frac{1}{4}$$

This shows that the coefficient of t in the above is 1. Now we need to find the coefficient of t in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of t in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4 + 4t^3 + 6t^2 + 24t + 21}{4} \\
&= Q + \frac{R}{4} \\
&= \left(6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2\right) + (0) \\
&= 6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{t}$ in the quotient is 6. Now b can be found.

$$\begin{aligned}
b &= (6) - (1) \\
&= 5
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}t^2 + t + \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{5}{\frac{1}{2}} - 2\right) &= 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{5}{\frac{1}{2}} - 2\right) &= -6
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = 6t + \frac{21}{4} + \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{1}{2}t^2 + t + \frac{1}{2}$	4	-6

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^+ = 4$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 4
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) \\
&= \frac{1}{2}t^2 + t + \frac{1}{2} \\
&= \frac{(t+1)^2}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)(4t^3 + 3t^2 a_3 + 2ta_2 + a_1) + \left((t+1) + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)^2 - \left(6t + \right.\right.
$$

$$
(-a_3 + 4)\, t^4 + 2(2 - a_2 + a_3)\, t^3 + 3(4 - a_1 + a_3)\, t^2 + 2(-2a_0 - a_1 + a_2 + 3a_3)
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 5, a_1 = 8, a_2 = 6, a_3 = 4\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^4 + 4t^3 + 6t^2 + 8t + 5
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\int \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right) dt} \\
&= \left(t^4 + 4t^3 + 6t^2 + 8t + 5\right) \mathrm{e}^{\frac{(t+1)^3}{6}} \\
&= (t+1)\left(t^3 + 3t^2 + 3t + 5\right) \mathrm{e}^{\frac{(t+1)^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{(t+1)^2}{1} \, dt} \\
&= z_1 e^{-\frac{(t+1)^3}{6}} \\
&= z_1 \left(\mathrm{e}^{-\frac{(t+1)^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (t+1)\left(t^3 + 3t^2 + 3t + 5\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{(t+1)^2}{1} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-\frac{(t+1)^3}{3}}}{(y_1)^2} \, dt \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{(t+1)^3}{3}}}{(t+1)^2 \left(t^3 + 3t^2 + 3t + 5\right)^2} dt \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((t+1)\left(t^3+3t^2+3t+5\right)\big) \\
&\quad + c_2\left((t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{\mathrm{e}^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1(t+1)\left(t^3+3t^2+3t+5\right) \\
&\quad + c_2(t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{\mathrm{e}^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1(t+1)\left(t^3+3t^2+3t+5\right) \\
&\quad + c_2(t+1)\left(t^3+3t^2+3t+5\right)\left(\int \frac{\mathrm{e}^{-\frac{(t+1)^3}{3}}}{(t+1)^2\left(t^3+3t^2+3t+5\right)^2}dt\right)
\end{aligned}
$$

Verified OK.

2.622.1 Maple step by step solution

Let's solve

$y'' + (t+1)^2 y' + (-4t-4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$

  ◦ Shift index using $k- > k-m$

  $t^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$

- Convert $t^m \cdot y'$ to series expansion for $m = 0..2$

$$t^m \cdot y' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, t^{k-1+m}$$

- Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, t^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, t^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^k$$

Rewrite ODE with series expansions

$$2a_2 + a_1 - 4a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) + 2a_k(k-2) + a_{k-1}(k-5)\right) t^k\right) = 0$$

- Each term must be 0

$$2a_2 + a_1 - 4a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (2a_k + a_{k-1} + a_{k+1} + 3a_{k+2})\, k - 4a_k - 5a_{k-1} + a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+1$

$$(k+1)^2\, a_{k+3} + (2a_{k+1} + a_k + a_{k+2} + 3a_{k+3})\,(k+1) - 4a_{k+1} - 5a_k + a_{k+2} + 2a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+3} = -\frac{a_k k + 2a_{k+1} k + k a_{k+2} - 4a_k - 2a_{k+1} + 2a_{k+2}}{k^2+5k+6}, 2a_2 + a_1 - 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
   Special function solution also has integrals. Returning default Liouvillian solution.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve(diff(y(t),t$2)+(t^2+2*t+1)*diff(y(t),t)-(4+4*t)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^4+4t^3+6t^2+8t+5) + c_2(t^4+4t^3+6t^2+8t+5)\left(\int \frac{e^{-\frac{1}{3}t^3-t^2-t}}{(t+1)^2(t^3+3t^2+3t+5)^2}dt\right)$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 132

```
DSolve[y''[t]+(t^2+2*t+1)*y'[t]-(4+4*t)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{36} e^{-\frac{1}{3}t(t^2+3t+3)} \left(-3c_2(t^3+3t^2+3t+4) + 3^{2/3} c_2 e^{\frac{1}{3}(t+1)^3} \sqrt[3]{(t+1)^3}(t^3+3t^2+3t+5)\,\Gamma\left(\frac{2}{3}, \frac{1}{3}(t+1)^3\right) + 36c_1 e^{\frac{t^3}{3}+t^2+t}(t^4+4t^3+6t^2+8t+5)\right)$$

2.623 problem 638

2.623.1 Maple step by step solution 5863

Internal problem ID [8113]
Internal file name [`OUTPUT/7046_Sunday_June_05_2022_05_26_42_PM_84104469/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 638.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{2ty'' + (1-2t)\,y' - y = 0}$$

Writing the ode as

$$2ty'' + (1-2t)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= 1-2t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4t^2 + 4t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4t^2 + 4t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{4t^2 + 4t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1189: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\ &= \sum_{i=0}^{0} a_i t^i \end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4t} - \frac{1}{4t^2} + \frac{1}{8t^3} - \frac{1}{8t^4} + \frac{1}{8t^5} - \frac{9}{64t^6} + \frac{21}{128t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4t^2+4t-3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4t-3}{16t^2}\right) \\ &= \frac{1}{4} + \frac{4t-3}{16t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4t^2 + 4t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{4t} \\
&= \frac{1}{2} + \frac{1}{4t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{4t}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{2} + \frac{1}{4t}\right)^2 - \left(\frac{4t^2 + 4t - 3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(\frac{1}{2} + \frac{1}{4t}\right) dt} \\
&= t^{\frac{1}{4}} e^{\frac{t}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{1-2t}{2t} \, dt} \\
&= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{e^{\frac{t}{2}}}{t^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2t}{2t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t-\frac{\ln(t)}{2}}}{(y_1)^2}\, dt \\ &= y_1\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^t\right) + c_2\left(\mathrm{e}^t\left(\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^t \sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^t \sqrt{\pi}\ \mathrm{erf}\left(\sqrt{t}\right)$$

Verified OK.

2.623.1 Maple step by step solution

Let's solve

$2ty'' + (1-2t)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2t} + \frac{(-1+2t)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(-1+2t)y'}{2t} - \frac{y}{2t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(t) = -\frac{-1+2t}{2t}, P_3(t) = -\frac{1}{2t}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = \frac{1}{2}$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $2ty'' + (1-2t)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- > k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot y''$ to series expansion

$$t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k->k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(a_{k+1}(k+1+r) - a_k)\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{k+\frac{3}{2}}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{k+\frac{3}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+\frac{3}{2}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*t*diff(y(t),t$2)+(1-2*t)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^t + c_2 e^t \left(\int \frac{e^{-t}}{\sqrt{t}} dt\right)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 21

```
DSolve[2*t*y''[t]+(1-2*t)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t \left(c_1 - c_2 \Gamma\left(\frac{1}{2}, t\right)\right)$$

2.624 problem 639

2.624.1 Maple step by step solution 5874

Internal problem ID [8114]
Internal file name [OUTPUT/7047_Sunday_June_05_2022_05_26_46_PM_19160603/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 639.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2ty'' + (t+1)\,y' - 2y = 0}$$

Writing the ode as

$$2ty'' + (t+1)\,y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t \\ B &= t+1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 18t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 + 18t - 3$$
$$t = 16t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 18t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1191: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{9}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{9}{4t} - \frac{21}{2t^2} + \frac{189}{2t^3} - \frac{1071}{t^4} + \frac{13608}{t^5} - \frac{370629}{2t^6} + \frac{5288409}{2t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 + 18t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{18t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{18t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 18. Dividing this by leading coefficient in t which is 16 gives $\frac{9}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{9}{8}\right) - (0) \\ &= \frac{9}{8}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &= \frac{9}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{8}}{\frac{1}{4}} - 0\right) &= -\frac{9}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 18t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{9}{4}$	$-\frac{9}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{9}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{9}{4} - \left(\frac{1}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4t} + \frac{1}{4} \\
&= \frac{t+1}{4t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = t^2 + a_1 t + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{4t} + \frac{1}{4}\right)(2t + a_1) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} + \frac{1}{4}\right)^2 - \left(\frac{t^2 + 18t - 3}{16t^2}\right) \right) = 0
$$

$$
\frac{(-a_1 + 6)\, t - 2a_0 + a_1}{2t} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 3, a_1 = 6\}
$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$
p(t) = t^2 + 6t + 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t^2 + 6t + 3\right) \mathrm{e}^{\int \left(\frac{1}{4t} + \frac{1}{4}\right) dt} \\
&= \left(t^2 + 6t + 3\right) \mathrm{e}^{\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= \left(t^2 + 6t + 3\right) t^{\frac{1}{4}} \mathrm{e}^{\frac{t}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t+1}{2t}\,dt} \\
&= z_1 e^{-\frac{t}{4}-\frac{\ln(t)}{4}} \\
&= z_1 \left(\frac{e^{-\frac{t}{4}}}{t^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = t^2 + 6t + 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t+1}{2t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{t}{2}-\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (t^2+6t+3) + c_2 \left(t^2+6t+3\left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 (t^2+6t+3) + c_2 (t^2+6t+3)\left(\int \frac{e^{-\frac{t}{2}}}{\sqrt{t}\,(t^2+6t+3)^2}dt\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(t^2+6t+3\right)+c_2\left(t^2+6t+3\right)\left(\int \frac{\mathrm{e}^{-\frac{t}{2}}}{\sqrt{t}\,\left(t^2+6t+3\right)^2}dt\right)$$

Verified OK.

2.624.1 Maple step by step solution

Let's solve

$2ty''+(t+1)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(t+1)y'}{2t}+\frac{y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(t+1)y'}{2t}-\frac{y}{t}=0$

☐ Check to see if $t_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(t)=\frac{t+1}{2t}, P_3(t)=-\frac{1}{t}\right]$

  - $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\Big|_{t=0}=\frac{1}{2}$

  - $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\Big|_{t=0}=0$

  - $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $2ty''+(t+1)\,y'-2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

  $$t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot y''$ to series expansion

  $$t \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k- > k+1$

  $$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+2r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+r-2)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r+\frac{1}{2}\right)(k+1+r)\, a_{k+1} + a_k(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-2)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $$a_{k+1} = -\frac{a_k(k-2)}{(2k+1)(k+1)}$$

- Apply recursion relation for $k = 0$

  $$a_1 = 2a_0$$

- Apply recursion relation for $k = 1$

  $$a_2 = \frac{a_1}{6}$$

- Express in terms of a_0

$a_2 = \frac{a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 + 2t + \frac{1}{3}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), b_{k+1} = -\frac{b_k\left(k-\frac{3}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(2*t*diff(y(t),t$2)+(1+t)*diff(y(t),t)-2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^2 + 6t + 3) + c_2(t^2 + 6t + 3)\left(\int \frac{e^{-\frac{t}{2}}}{(t^2 + 6t + 3)^2 \sqrt{t}} dt\right)$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 71

```
DSolve[2*t*y''[t]+(1+t)*y'[t]-2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{24}\left(\sqrt{2\pi}c_2(t^2 + 6t + 3)\operatorname{erf}\left(\frac{\sqrt{t}}{\sqrt{2}}\right) + 24c_1(t^2 + 6t + 3) + 2c_2e^{-t/2}\sqrt{t}(t + 5)\right)$$

2.625 problem 640

2.625.1 Maple step by step solution 5883

Internal problem ID [8115]
Internal file name [`OUTPUT/7048_Sunday_June_05_2022_05_26_49_PM_26424986/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 640.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2t^2y'' - ty' + (t+1)\,y = 0}$$

Writing the ode as

$$2t^2y'' - ty' + (t+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= -t \\ C &= t+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 8t}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 - 8t \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-3 - 8t}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(t)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{t-c}\\ &= \frac{1}{2}\left(\frac{1}{(t-(0))}\right)\\ &= \frac{1}{2t}\end{aligned}$$

Now we search for a monic polynomial $p(t)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(t)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2t}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2t} + \frac{1+8t}{16t^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{-t}}{4t}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{-t}}{4t} dt} \\ &= t^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t}{2t^2} \, dt} \\ &= z_1 e^{\frac{\ln(t)}{4}} \\ &= z_1 \left(t^{\frac{1}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t}{2t^2} \, dt}}{(y_1)^2} \, dt \\ &= y_1 \int \frac{e^{\frac{\ln(t)}{2}}}{(y_1)^2} \, dt \\ &= y_1 \left(\frac{\sqrt{2}\,\sqrt{-t}\left(-1 + \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}}\right) + c_2 \left(\sqrt{t}\,\mathrm{e}^{\sqrt{2}\,\sqrt{-t}} \left(\frac{\sqrt{2}\,\sqrt{-t}\left(-1 + \mathrm{e}^{-2\sqrt{2}\,\sqrt{-t}}\right)}{2\sqrt{t}} \right) \right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \sqrt{t}\, e^{\sqrt{2}\, \sqrt{-t}} - \frac{c_2 \sqrt{2}\, \sqrt{-t}\, \left(e^{\sqrt{2}\, \sqrt{-t}} - e^{-\sqrt{2}\, \sqrt{-t}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{t}\, e^{\sqrt{2}\, \sqrt{-t}} - \frac{c_2 \sqrt{2}\, \sqrt{-t}\, \left(e^{\sqrt{2}\, \sqrt{-t}} - e^{-\sqrt{2}\, \sqrt{-t}}\right)}{2}$$

Verified OK.

2.625.1 Maple step by step solution

Let's solve

$2y''t^2 - ty' + (t+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2t} - \frac{(t+1)y}{2t^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2t} + \frac{(t+1)y}{2t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{1}{2t}, P_3(t) = \frac{t+1}{2t^2}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -\frac{1}{2}$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = \frac{1}{2}$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

$2y''t^2 - ty' + (t+1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y$ to series expansion for $m = 0..1$

  $t^m \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+m}$

  ◦ Shift index using $k- > k - m$

  $t^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} t^{k+r}$

  ◦ Convert $t \cdot y'$ to series expansion

  $t \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, t^{k+r}$

  ◦ Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\,t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}\right) t^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+2r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{1, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$2\left(k+r-\frac{1}{2}\right)(k+r-1)\,a_k + a_{k-1} = 0$

- Shift index using $k- > k + 1$

$2\left(k+\frac{1}{2}+r\right)(k+r)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{a_k}{(2k+1+2r)(k+r)}$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{(2k+3)(k+1)}, b_{k+1} = -\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(2*t^2*diff(y(t),t$2)-t*diff(y(t),t)+(1+t)*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 \sin\left(\sqrt{2}\,\sqrt{t}\right)\sqrt{t} + c_2\sqrt{t}\,\cos\left(\sqrt{2}\,\sqrt{t}\right)$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 62

```
DSolve[2*t^2*y''[t]-t*y'[t]+(1+t)*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} e^{-i\sqrt{2}\sqrt{t}} \sqrt{t} \left(2c_1 e^{2i\sqrt{2}\sqrt{t}} + i\sqrt{2} c_2\right)$$

2.626 problem 641

2.626.1 Maple step by step solution 5894

Internal problem ID [8116]
Internal file name [`OUTPUT/7049_Sunday_June_05_2022_05_26_52_PM_96017688/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 641.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2t^2y'' + (t^2 - t)\, y' + y = 0}$$

Writing the ode as

$$2t^2y'' + (t^2 - t)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2t^2 \\ B &= t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t - 3}{16t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 - 2t - 3 \\ t &= 16t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t - 3}{16t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1195: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} - \frac{1}{8t} - \frac{3}{16t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{1}{4t} - \frac{1}{2t^2} - \frac{1}{2t^3} - \frac{1}{t^4} - \frac{2}{t^5} - \frac{9}{2t^6} - \frac{21}{2t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{t^2 - 2t - 3}{16t^2} \\ &= Q + \frac{R}{16t^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-2t - 3}{16t^2}\right) \\ &= \frac{1}{16} + \frac{-2t - 3}{16t^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 16 gives $-\frac{1}{8}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{8}\right) - (0) \\ &= -\frac{1}{8}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= -\frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{8}}{\frac{1}{4}} - 0\right) &= \frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t - 3}{16t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4t} + (-)\left(\frac{1}{4}\right) \\
&= \frac{1}{4t} - \frac{1}{4} \\
&= -\frac{t-1}{4t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4t} - \frac{1}{4}\right)(0) + \left(\left(-\frac{1}{4t^2}\right) + \left(\frac{1}{4t} - \frac{1}{4}\right)^2 - \left(\frac{t^2 - 2t - 3}{16t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{4t} - \frac{1}{4}\right) dt} \\
&= t^{\frac{1}{4}} \mathrm{e}^{-\frac{t}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t^2 - t}{2t^2} \, dt} \\
&= z_1 e^{-\frac{t}{4} + \frac{\ln(t)}{4}} \\
&= z_1 \left(t^{\frac{1}{4}} \mathrm{e}^{-\frac{t}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-t}{2t^2}\,dt}}{(y_1)^2}\,dt \\ &= y_1 \int \frac{e^{-\frac{t}{2}+\frac{\ln(t)}{2}}}{(y_1)^2}\,dt \\ &= y_1 \left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\right) + c_2\left(\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} - ic_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} - ic_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,\sqrt{t}}{2}\right)$$

Verified OK.

2.626.1 Maple step by step solution

Let's solve

$2y''t^2+(t^2-t)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{2t^2}-\frac{(t-1)y'}{2t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(t-1)y'}{2t}+\frac{y}{2t^2}=0$

□ Check to see if $t_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t)=\frac{t-1}{2t},P_3(t)=\frac{1}{2t^2}\right]$

- ◦ $t\cdot P_2(t)$ is analytic at $t=0$

  $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=-\frac{1}{2}$

- ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

  $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=\frac{1}{2}$

- ◦ $t=0$is a regular singular point

  Check to see if $t_0=0$ is a regular singular point

  $t_0=0$

- Multiply by denominators

  $2y''t^2+t(t-1)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $t^m\cdot y'$ to series expansion for $m=1..2$

  $t^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,t^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(\left(k+r-\frac{1}{2}\right) a_k + \frac{a_{k-1}}{2}\right)(k+r-1) = 0$$

- Shift index using $k- > k+1$

$$2\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + \frac{a_k}{2}\right)(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{2k+1+2r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{2k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{2k+3}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2k+3}, b_{k+1} = -\frac{b_k}{2k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(2*t^2*diff(y(t),t$2)+(t^2-t)*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}} + c_2\sqrt{t}\,\mathrm{e}^{-\frac{t}{2}}\left(\int \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}dt\right)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 46

```
DSolve[2*t^2*y''[t]+(t^2-t)*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t/2}\left(c_2\sqrt{t} + \sqrt{2}c_1\sqrt{-t}\Gamma\left(\frac{1}{2}, -\frac{t}{2}\right)\right)$$

2.627 problem 642

2.627.1 Maple step by step solution 5905

Internal problem ID [8117]
Internal file name [OUTPUT/7050_Sunday_June_05_2022_05_26_56_PM_91831813/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 642.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2y'' + \left(-t^2 + t\right) y' - y = 0}$$

Writing the ode as

$$t^2y'' + \left(-t^2 + t\right) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 + t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 2t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1197: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2t} + \frac{1}{2t^2} + \frac{1}{2t^3} + \frac{1}{4t^4} - \frac{1}{4t^5} - \frac{3}{4t^6} - \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 - 2t + 3}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2t+3}{4t^2}\right) \\ &= \frac{1}{4} + \frac{-2t+3}{4t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2t} \\
&= \frac{t-1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2t}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(\frac{1}{2} - \frac{1}{2t}\right)^2 - \left(\frac{t^2-2t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2t}\right) dt} \\
&= \frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+t}{t^2} \, dt} \\
&= z_1 e^{\frac{t}{2} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^t}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1\left(-(t+1)\,\mathrm{e}^{-t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^t}{t}\right) + c_2\left(\frac{\mathrm{e}^t}{t}\left(-(t+1)\,\mathrm{e}^{-t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^t}{t} + \frac{c_2(-t-1)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^t}{t} + \frac{c_2(-t-1)}{t}$$

Verified OK.

2.627.1 Maple step by step solution

Let's solve

$y''t^2 + (-t^2 + t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{t^2} + \frac{(t-1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t-1)y'}{t} - \frac{y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t-1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\bigg|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\bigg|_{t=0} = -1$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 - t(t-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+2+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(t^2*diff(y(t),t$2)+(t-t^2)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1(t+1)}{t} + \frac{c_2 e^t}{t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[t^2*y''[t]+(t-t^2)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_2 e^t - c_1(t+1)}{t}$$

2.628 problem 643

2.628.1 Maple step by step solution 5915

Internal problem ID [8118]
Internal file name [`OUTPUT/7051_Sunday_June_05_2022_05_26_59_PM_41187558/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 643.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{ty'' - (t^2+2)\,y' + yt = 0}$$

Writing the ode as

$$\begin{aligned} ty'' + (-t^2-2)\,y' + yt &= 0 \qquad (1)\\ Ay'' + By' + Cy &= 0 \qquad (2) \end{aligned}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t\\ B &= -t^2-2 \qquad (3)\\ C &= t \end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \qquad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \qquad (5)\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 2t^2 + 8}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^4 - 2t^2 + 8 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1199: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{1}{2t} + \frac{7}{4t^3} + \frac{7}{4t^5} - \frac{21}{16t^7} - \frac{119}{16t^9} - \frac{189}{32t^{11}} + \frac{791}{32t^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\ &= \frac{t}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^4 - 2t^2 + 8}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{t^2}{4} - \frac{1}{2}\right) + \left(\frac{2}{t^2}\right) \\ &= \frac{t^2}{4} - \frac{1}{2} + \frac{2}{t^2} \end{aligned}$$

We see that the coefficient of the term t in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{t}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 1\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 2t^2 + 8}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	-1	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{t} + \left(\frac{t}{2}\right) \\ &= -\frac{1}{t} + \frac{t}{2} \\ &= -\frac{1}{t} + \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{t} + \frac{t}{2}\right)(0) + \left(\left(\frac{1}{t^2} + \frac{1}{2}\right) + \left(-\frac{1}{t} + \frac{t}{2}\right)^2 - \left(\frac{t^4 - 2t^2 + 8}{4t^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega\, dt} \\ &= e^{\int \left(-\frac{1}{t} + \frac{t}{2}\right) dt} \\ &= \frac{e^{\frac{t^2}{4}}}{t} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-t^2-2}{t}\, dt} \\ &= z_1 e^{\frac{t^2}{4} + \ln(t)} \\ &= z_1 \left(t\, e^{\frac{t^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{t^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-t^2-2}{t} dt}}{(y_1)^2} dt$$

$$= y_1 \int \frac{e^{\frac{t^2}{2}+2\ln(t)}}{(y_1)^2} dt$$

$$= y_1 \left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(\mathrm{e}^{\frac{t^2}{2}}\right) + c_2\left(\mathrm{e}^{\frac{t^2}{2}}\left(-t\,\mathrm{e}^{-\frac{t^2}{2}} + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{t^2}{2}} + c_2\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)\mathrm{e}^{\frac{t^2}{2}}}{2} - t\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{t^2}{2}} + c_2\left(\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{t\sqrt{2}}{2}\right)\mathrm{e}^{\frac{t^2}{2}}}{2} - t\right)$$

Verified OK.

2.628.1 Maple step by step solution

Let's solve

$ty'' + (-t^2 - 2)\, y' + yt = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(t^2+2)y'}{t} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t^2+2)y'}{t} + y = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t^2+2}{t}, P_3(t) = 1\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\Big|_{t=0} = -2$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t^2 - 2)\, y' + yt = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t \cdot y$ to series expansion

    $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- Shift index using $k->k-1$

  $t \cdot y=\sum_{k=1}^{\infty} a_{k-1} t^{k+r}$
- Convert $t^{m} \cdot y'$ to series expansion for $m=0..2$

  $t^{m} \cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, t^{k+r-1+m}$
- Shift index using $k->k+1-m$

  $t^{m} \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$
- Convert $t \cdot y''$ to series expansion

  $t \cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r-1}$
- Shift index using $k->k+1$

  $t \cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_{0} r(-3+r)\, t^{-1+r}+a_{1}(1+r)\,(-2+r)\, t^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k-2+r)-a_{k-1}(k-2+r)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r)=0$
- Values of r that satisfy the indicial equation

  $r \in \{0,3\}$
- Each term must be 0

  $a_{1}(1+r)\,(-2+r)=0$
- Each term in the series must be 0, giving the recursion relation

  $(k-2+r)\left(a_{k+1}(k+r+1)-a_{k-1}\right)=0$
- Shift index using $k->k+1$

  $(k+r-1)\left(a_{k+2}(k+2+r)-a_{k}\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}}{k+2+r}$
- Recursion relation for $r=0$

  $a_{k+2}=\frac{a_{k}}{k+2}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

  $$a_{k+2} = \frac{a_k}{k+5}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{k+5}, 4b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(t*diff(y(t),t$2)-(t^2+2)*diff(y(t),t)+t*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{\frac{t^2}{2}} + \frac{c_2 e^{\frac{t^2}{2}} \left(-\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,t}{2}\right) + 2t\,e^{-\frac{t^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 52

```
DSolve[t*y''[t]-(t^2+2)*y'[t]+t*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{\frac{\pi}{2}} c_2 e^{\frac{t^2}{2}} \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right) + c_1 e^{\frac{t^2}{2}} - c_2 t$$

2.629 problem 644

2.629.1 Maple step by step solution 5926

Internal problem ID [8119]
Internal file name [`OUTPUT/7052_Sunday_June_05_2022_05_27_03_PM_55149392/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 644.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2y'' + t(t+1)\,y' - y = 0}$$

Writing the ode as

$$t^2y'' + \left(t^2+t\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2+t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 2t + 3 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1201: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} + \frac{1}{2t^2} - \frac{1}{2t^3} + \frac{1}{4t^4} + \frac{1}{4t^5} - \frac{3}{4t^6} + \frac{3}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 + 2t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{2t + 3}{4t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 2t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2t} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{2t} - \frac{1}{2} \\
&= -\frac{t+1}{2t}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2t} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2t^2}\right) + \left(-\frac{1}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+2t+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{2t} - \frac{1}{2}\right) dt} \\
&= \frac{e^{-\frac{t}{2}}}{\sqrt{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{t^2+t}{t^2}\, dt} \\
&= z_1 e^{-\frac{t}{2} - \frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{t}{2}}}{\sqrt{t}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^{-t}}{t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t^2+t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-t-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left((t-1)\, e^{t}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{e^{-t}}{t}\right) + c_2 \left(\frac{e^{-t}}{t} \left((t-1)\, e^{t}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-t}}{t} + \frac{c_2(t-1)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{-t}}{t} + \frac{c_2(t-1)}{t}$$

Verified OK.

2.629.1 Maple step by step solution

Let's solve

$y''t^2 + (t^2 + t)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{t^2} - \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t+1)y'}{t} - \frac{y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(t) = \frac{t+1}{t}, P_3(t) = -\frac{1}{t^2}\right]$

  ○ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = 1$

  ○ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = -1$

  ○ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 + t(t + 1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, t^{k+r-1+m}$

  ○ Shift index using $k- >k + 1 - m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- ○ Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-1+r)\, t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r+1) + a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = -\frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k t^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(t^2*diff(y(t),t$2)+t*(t+1)*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = \frac{c_1(t-1)}{t} + \frac{c_2 \mathrm{e}^{-t}}{t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[t^2*y''[t]+t*(t+1)*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{e^{-t}(c_1 e^t (t-1) + c_2)}{t}$$

2.630 problem 645

2.630.1 Maple step by step solution 5936

Internal problem ID [8120]
Internal file name [OUTPUT/7053_Sunday_June_05_2022_05_27_06_PM_4908321/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 645.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{ty'' - (t+4)\,y' + 2y = 0}$$

Writing the ode as

$$ty'' + (-4-t)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= -4-t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 24}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 24 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 24}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1203: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{6}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{6}{t^2} - \frac{36}{t^4} + \frac{432}{t^6} - \frac{6480}{t^8} + \frac{108864}{t^{10}} - \frac{1959552}{t^{12}} + \frac{36951552}{t^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t^2 + 24}{4t^2} \\ &= Q + \frac{R}{4t^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6}{t^2}\right) \\ &= \frac{1}{4} + \frac{6}{t^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 24}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{2}{t} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{2}{t} - \frac{1}{2} \\ &= -\frac{t+4}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 2$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(2) + 2\left(-\frac{2}{t} - \frac{1}{2}\right)(2t + a_1) + \left(\left(\frac{2}{t^2}\right) + \left(-\frac{2}{t} - \frac{1}{2}\right)^2 - \left(\frac{t^2+24}{4t^2}\right)\right) = 0 \\ \frac{(a_1 - 6)\,t + 2a_0 - 4a_1}{t} = 0\end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 12, a_1 = 6\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^2 + 6t + 12$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega\, dt} \\ &= \left(t^2 + 6t + 12\right) e^{\int \left(-\frac{2}{t} - \frac{1}{2}\right) dt} \\ &= \left(t^2 + 6t + 12\right) e^{-\frac{t}{2} - 2\ln(t)} \\ &= \frac{\left(t^2 + 6t + 12\right) e^{-\frac{t}{2}}}{t^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4-t}{t}\,dt} \\
&= z_1 e^{\frac{t}{2}+2\ln(t)} \\
&= z_1 \left(t^2 \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^2 + 6t + 12
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4-t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{t+4\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{(t^2 - 6t + 12)\,\mathrm{e}^t}{t^2 + 6t + 12}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^2 + 6t + 12\right) + c_2 \left(t^2 + 6t + 12 \left(\frac{(t^2 - 6t + 12)\,\mathrm{e}^t}{t^2 + 6t + 12}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \left(t^2 + 6t + 12\right) + c_2 \left(t^2 - 6t + 12\right) \mathrm{e}^t \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(t^2 + 6t + 12\right) + c_2 \left(t^2 - 6t + 12\right) \mathrm{e}^t
$$

Verified OK.

2.630.1 Maple step by step solution

Let's solve

$ty'' + (-4 - t)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{t} + \frac{(t+4)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t+4)y'}{t} + \frac{2y}{t} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(t) = -\frac{t+4}{t}, P_3(t) = \frac{2}{t}\right]$

  - $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\bigg|_{t=0} = -4$

  - $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\bigg|_{t=0} = 0$

  - $t = 0$is a regular singular point

    Check to see if $t_0 = 0$ is a regular singular point

    $t_0 = 0$

- Multiply by denominators

  $ty'' + (-4 - t)\, y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $t^m \cdot y'$ to series expansion for $m = 0..1$

    $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, t^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$$t^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t \cdot y''$ to series expansion

$$t \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r-1}$$

- Shift index using $k->k+1$

$$t \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}r(-5+r)\, t^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-4+r) - a_{k}(k+r-2)\right) t^{k+r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r(-5+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 5\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-4+r) - a_{k}(k+r-2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_{k}(k+r-2)}{(k+1+r)(k-4+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_{k}(k-2)}{(k+1)(k-4)}$$

- Apply recursion relation for $k = 0$

$$a_{1} = \frac{a_{0}}{2}$$

- Apply recursion relation for $k = 1$

$$a_{2} = \frac{a_{1}}{6}$$

- Express in terms of a_{0}

$$a_{2} = \frac{a_{0}}{12}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y = a_{0} \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^{2}\right)$$

- Recursion relation for $r = 5$

$$a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}$$

- Solution for $r = 5$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+5}, a_{k+1} = \frac{a_k(k+3)}{(k+6)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 + \frac{1}{2}t + \frac{1}{12}t^2\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+5}\right), b_{k+1} = \frac{b_k(k+3)}{(k+6)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(t*diff(y(t),t$2)-(4+t)*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^2 + 6t + 12) + c_2 e^t (t^2 - 6t + 12)$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 85

```
DSolve[t*y''[t]-(4+t)*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{2e^{t/2}\sqrt{t}\left((c_2t^2 - 6ic_1t + 12c_2)\cosh\left(\frac{t}{2}\right) + i(c_1(t^2 + 12) + 6ic_2t)\sinh\left(\frac{t}{2}\right)\right)}{\sqrt{\pi}\sqrt{-it}}$$

2.631 problem 646

2.631.1 Maple step by step solution 5946

Internal problem ID [8121]
Internal file name [`OUTPUT/7054_Sunday_June_05_2022_05_27_09_PM_29080740/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 646.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{t^2y'' + (t^2 - 3t)\,y' + 3y = 0}$$

Writing the ode as

$$t^2y'' + (t^2 - 3t)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= t^2 - 3t \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 - 6t + 3}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^2 - 6t + 3$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 - 6t + 3}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1205: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2t} + \frac{3}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2t} - \frac{3}{2t^2} - \frac{9}{2t^3} - \frac{63}{4t^4} - \frac{243}{4t^5} - \frac{999}{4t^6} - \frac{4293}{4t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 - 6t + 3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-6t + 3}{4t^2}\right) \\
&= \frac{1}{4} + \frac{-6t + 3}{4t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 - 6t + 3}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty}\\ &= \frac{3}{2t} + (-)\left(\frac{1}{2}\right)\\ &= \frac{3}{2t} - \frac{1}{2}\\ &= -\frac{t-3}{2t}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{3}{2t} - \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2t^2}\right) + \left(\frac{3}{2t} - \frac{1}{2}\right)^2 - \left(\frac{t^2-6t+3}{4t^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(t) &= p e^{\int \omega \, dt}\\ &= \mathrm{e}^{\int \left(\frac{3}{2t} - \frac{1}{2}\right) dt}\\ &= t^{\frac{3}{2}} \mathrm{e}^{-\frac{t}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}\\ &= z_1 e^{-\int \frac{1}{2} \frac{t^2-3t}{t^2} \, dt}\\ &= z_1 e^{-\frac{t}{2} + \frac{3\ln(t)}{2}}\\ &= z_1 \left(t^{\frac{3}{2}} \mathrm{e}^{-\frac{t}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = t^3 \mathrm{e}^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t^2-3t}{t^2}\, dt}}{(y_1)^2}\, dt \\
&= y_1 \int \frac{e^{-t+3\ln(t)}}{(y_1)^2}\, dt \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-t)\, t^2 - (t+1)\, \mathrm{e}^t}{2t^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^3 \mathrm{e}^{-t}\right) + c_2 \left(t^3 \mathrm{e}^{-t} \left(\frac{-\operatorname{expIntegral}_1(-t)\, t^2 - (t+1)\, \mathrm{e}^t}{2t^2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 t^3 \mathrm{e}^{-t} - \frac{c_2 t \left(\operatorname{expIntegral}_1(-t)\, t^2 \mathrm{e}^{-t} + t + 1\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 t^3 \mathrm{e}^{-t} - \frac{c_2 t \left(\operatorname{expIntegral}_1(-t)\, t^2 \mathrm{e}^{-t} + t + 1\right)}{2}$$

Verified OK.

2.631.1 Maple step by step solution

Let's solve

$y''t^2 + (t^2 - 3t)\, y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y}{t^2} - \frac{(t-3)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(t-3)y'}{t} + \frac{3y}{t^2} = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(t) = \frac{t-3}{t}, P_3(t) = \frac{3}{t^2}\right]$

- ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\Big|_{t=0} = -3$

- ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\Big|_{t=0} = 3$

- ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $y''t^2 + t(t - 3)\, y' + 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, t^{k+r-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

$$t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$$

- Convert $t^2 \cdot y''$ to series expansion

$$t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-3+r)\, t^r + \left(\sum_{k=1}^{\infty} (a_k(k+r-1)\,(k+r-3) + a_{k-1}(k+r-1))\, t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+r)\,(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r-3) + a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k-2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k-2+r}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{k-1}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k}{k-1}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k t^{k+3}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(t^2*diff(y(t),t$2)+(t^2-3*t)*diff(y(t),t)+3*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t^3 \mathrm{e}^{-t} + \frac{c_2 t\,\mathrm{e}^{-t}(\mathrm{expIntegral}_1\left(-t\right) t^2 + \mathrm{e}^t t + \mathrm{e}^t)}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 41

```
DSolve[t^2*y''[t]+(t^2-3*t)*y'[t]+3*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2} e^{-t} (c_1 t^3 \,\mathrm{ExpIntegralEi}(t) + 2 c_2 t^3 - c_1 e^t (t+1) t)$$

2.632 problem 647

2.632.1 Maple step by step solution 5955

Internal problem ID [8122]
Internal file name [`OUTPUT/7055_Sunday_June_05_2022_05_27_13_PM_89287734/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 647.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + ty' + 2y = 0}$$

Writing the ode as

$$ty'' + ty' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= t \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t-8}{4t} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t-8 \\ t &= 4t \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t-8}{4t}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1207: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t$. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\ &= \sum_{i=0}^{0} a_i t^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{2}{t} - \frac{4}{t^2} - \frac{16}{t^3} - \frac{80}{t^4} - \frac{448}{t^5} - \frac{2688}{t^6} - \frac{16896}{t^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i\\ &= \frac{1}{2} \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{t-8}{4t} \\ &= Q + \frac{R}{4t} \\ &= \left(\frac{1}{4}\right) + \left(-\frac{2}{t}\right) \\ &= \frac{1}{4} - \frac{2}{t} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -8. Dividing this by leading coefficient in t which is 4 gives -2. Now b can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 0\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 0\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t-8}{4t}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{t} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{t} - \frac{1}{2} \\
&= \frac{1}{t} - \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{t} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} - \frac{1}{2}\right)^2 - \left(\frac{t-8}{4t}\right)\right) = 0$$

$$\frac{2 + a_0}{t} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -2\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= (t-2)\, e^{\int \left(\frac{1}{t} - \frac{1}{2}\right) dt} \\
&= (t-2)\, e^{-\frac{t}{2} + \ln(t)} \\
&= (t-2)\, t\, e^{-\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{t} \, dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(e^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (t-2)\, t\, e^{-t}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{t}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-t}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-t)\,t^2 - \mathrm{e}^t t + 2\operatorname{expIntegral}_1(-t)\,t + \mathrm{e}^t}{2t\,(t-2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((t-2)\,t\,\mathrm{e}^{-t}\right) \\
&\quad + c_2\left((t-2)\,t\,\mathrm{e}^{-t}\left(\frac{-\operatorname{expIntegral}_1(-t)\,t^2 - \mathrm{e}^t t + 2\operatorname{expIntegral}_1(-t)\,t + \mathrm{e}^t}{2t\,(t-2)}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1(t-2)\,t\,\mathrm{e}^{-t} + c_2\left(-\frac{(t-2)\,t\,\mathrm{e}^{-t}\operatorname{expIntegral}_1(-t)}{2} - \frac{t}{2} + \frac{1}{2}\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1(t-2)\,t\,\mathrm{e}^{-t} + c_2\left(-\frac{(t-2)\,t\,\mathrm{e}^{-t}\operatorname{expIntegral}_1(-t)}{2} - \frac{t}{2} + \frac{1}{2}\right)
$$

Verified OK.

2.632.1 Maple step by step solution

Let's solve

$ty'' + ty' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{2y}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{2y}{t}=0$

☐ Check to see if $t_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t)=1, P_3(t)=\frac{2}{t}\right]$

  ◦ $t\cdot P_2(t)$ is analytic at $t=0$

    $\left(t\cdot P_2(t)\right)\bigg|_{t=0}=0$

  ◦ $t^2\cdot P_3(t)$ is analytic at $t=0$

    $\left(t^2\cdot P_3(t)\right)\bigg|_{t=0}=0$

  ◦ $t=0$is a regular singular point

    Check to see if $t_0=0$ is a regular singular point

    $t_0=0$

- Multiply by denominators

  $ty''+ty'+2y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $t\cdot y'$ to series expansion

    $t\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r}$

  ◦ Convert $t\cdot y''$ to series expansion

    $t\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, t^{k+r-1}$

  ◦ Shift index using $k->k+1$

    $t\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+r)\, t^{-1+r}+\left(\sum_{k=0}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r)+a_k(k+r+2)\right) t^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r)+a_k(k+r+2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r+2)}{(k+1+r)(k+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{a_k(k+2)}{(k+1)k}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k t^k, a_{k+1}=-\frac{a_k(k+2)}{(k+1)k}\right]$

- Recursion relation for $r=1$

  $a_{k+1}=-\frac{a_k(k+3)}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1}=-\frac{a_k(k+3)}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k t^k\right)+\left(\sum_{k=0}^{\infty} b_k t^{k+1}\right), a_{k+1}=-\frac{a_k(k+2)}{(k+1)k}, b_{k+1}=-\frac{b_k(k+3)}{(k+2)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(t*diff(y(t),t$2)+t*diff(y(t),t)+2*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^{-t}(t-2)\,t + \frac{c_2(\mathrm{expIntegral}_1\,(-t)\,t^2 + e^t t - 2\,\mathrm{expIntegral}_1\,(-t)\,t - e^t)\,e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 51

```
DSolve[t*y''[t]+t*y'[t]+2*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t}\left(c_2(t-2)t\,\mathrm{ExpIntegralEi}(t) + 2c_1t^2 - t\left(c_2e^t + 4c_1\right) + c_2e^t\right)$$

2.633 problem 648

2.633.1 Maple step by step solution 5966

Internal problem ID [8123]
Internal file name [OUTPUT/7056_Sunday_June_05_2022_05_27_16_PM_25936037/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 648.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{ty'' + \left(-t^2 + 1\right) y' + 4yt = 0}$$

Writing the ode as

$$\begin{align*}
ty'' + \left(-t^2 + 1\right) y' + 4yt &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}
A &= t \\
B &= -t^2 + 1 \tag{3}\\
C &= 4t
\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^4 - 20t^2 - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = t^4 - 20t^2 - 1$$
$$t = 4t^2$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^4 - 20t^2 - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{t^2}{4} - 5 - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $t^v = t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{t}{2} - \frac{5}{t} - \frac{101}{4t^3} - \frac{505}{2t^5} - \frac{50601}{16t^7} - \frac{355015}{8t^9} - \frac{21351501}{32t^{11}} - \frac{168167525}{16t^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i \\
&= \frac{t}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $t^{v-1}=t^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{t^2}{4}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^4 - 20t^2 - 1}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{t^2}{4} - 5\right) + \left(-\frac{1}{4t^2}\right) \\
&= \frac{t^2}{4} - 5 - \frac{1}{4t^2}
\end{aligned}
$$

We see that the coefficient of the term t in the quotient is -5. Now b can be found.

$$
\begin{aligned}
b &= (-5) - (0) \\
&= -5
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{t}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-5}{\frac{1}{2}} - 1\right) &= -\frac{11}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-5}{\frac{1}{2}} - 1\right) &= \frac{9}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{t^4 - 20t^2 - 1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{t}{2}$	$-\frac{11}{2}$	$\frac{9}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{9}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{9}{2} - \left(\frac{1}{2}\right) \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\left(\frac{t}{2}\right) \\ &= \frac{1}{2t} - \frac{t}{2} \\ &= \frac{1}{2t} - \frac{t}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 4$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(12t^2 + 6ta_3 + 2a_2) + 2\left(\frac{1}{2t} - \frac{t}{2}\right)(4t^3 + 3a_3t^2 + 2a_2t + a_1) + \left(\left(-\frac{1}{2t^2} - \frac{1}{2}\right) + \left(\frac{1}{2t} - \frac{t}{2}\right)^2 - \left(\frac{t^4 - 20t}{4t^2}\right.\right.$$

$$\frac{t^4 a_3 + 2(8 + a_2)\, t^3 + 3(a_1 + 3a_3)\, t^2 + 4(a_0 + a_2)\, t}{t}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 8, a_1 = 0, a_2 = -8, a_3 = 0\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t^4 - 8t^2 + 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= pe^{\int \omega\, dt} \\
&= (t^4 - 8t^2 + 8)\, e^{\int \left(\frac{1}{2t} - \frac{t}{2}\right) dt} \\
&= (t^4 - 8t^2 + 8)\, e^{\frac{\ln(t)}{2} - \frac{t^2}{4}} \\
&= (t^4 - 8t^2 + 8)\, \sqrt{t}\, e^{-\frac{t^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-t^2+1}{t}\, dt} \\
&= z_1 e^{-\frac{\ln(t)}{2} + \frac{t^2}{4}} \\
&= z_1 \left(\frac{e^{\frac{t^2}{4}}}{\sqrt{t}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = t^4 - 8t^2 + 8$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2+1}{t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{\frac{t^2}{2}-\ln(t)}}{(y_1)^2}\, dt \\ &= y_1 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t\,(t^4-8t^2+8)^2} dt \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8 \left(\int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t\,(t^4-8t^2+8)^2} dt \right) \right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8\right) \left(\int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t\,(t^4-8t^2+8)^2} dt \right) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \left(t^4 - 8t^2 + 8\right) + c_2 \left(t^4 - 8t^2 + 8\right) \left(\int \frac{\mathrm{e}^{\frac{t^2}{2}}}{t\,(t^4-8t^2+8)^2} dt \right)$$

Verified OK.

2.633.1 Maple step by step solution

Let's solve

$ty'' + (-t^2 + 1) y' + 4yt = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(t^2-1)y'}{t} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(t^2-1)y'}{t} + 4y = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(t) = -\frac{t^2-1}{t}, P_3(t) = 4\right]$

  ◦ $t \cdot P_2(t)$ is analytic at $t = 0$

    $(t \cdot P_2(t))\Big|_{t=0} = 1$

  ◦ $t^2 \cdot P_3(t)$ is analytic at $t = 0$

    $(t^2 \cdot P_3(t))\Big|_{t=0} = 0$

  ◦ $t = 0$is a regular singular point

  Check to see if $t_0 = 0$ is a regular singular point

  $t_0 = 0$

- Multiply by denominators

  $ty'' + (-t^2 + 1) y' + 4yt = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k t^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $t \cdot y$ to series expansion

    $t \cdot y = \sum_{k=0}^{\infty} a_k t^{k+r+1}$

- Shift index using $k->k-1$

  $t\cdot y=\sum_{k=1}^{\infty} a_{k-1} t^{k+r}$

- Convert $t^{m}\cdot y'$ to series expansion for $m=0..2$

  $t^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, t^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $t^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, t^{k+r}$

- Convert $t\cdot y''$ to series expansion

  $t\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, t^{k+r-1}$

- Shift index using $k->k+1$

  $t\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r^{2}t^{-1+r}+a_{1}(1+r)^{2}\, t^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)^{2}-a_{k-1}(k-5+r)\right) t^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r^{2}=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term must be 0

  $a_{1}(1+r)^{2}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^{2}-a_{k-1}(k-5)=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2)^{2}-a_{k}(k-4)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_{k}(k-4)}{(k+2)^{2}}$

- Recursion relation for $r=0$; series terminates at $k=4$

$a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k t^k, a_{k+2} = \frac{a_k(k-4)}{(k+2)^2}, a_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve(t*diff(y(t),t$2)+(1-t^2)*diff(y(t),t)+4*t*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1(t^4 - 8t^2 + 8) + c_2(t^4 - 8t^2 + 8)\left(\int \frac{e^{\frac{t^2}{2}}}{(t^4 - 8t^2 + 8)^2 t} dt\right)$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 61

```
DSolve[t*y''[t]+(1-t^2)*y'[t]+4*t*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{128} c_2\left((t^4 - 8t^2 + 8)\,\text{ExpIntegralEi}\left(\frac{t^2}{2}\right) - 2e^{\frac{t^2}{2}}(t^2 - 6)\right) + c_1(t^4 - 8t^2 + 8)$$

2.634 problem 649

2.634.1 Maple step by step solution 5976

Internal problem ID [8124]
Internal file name [OUTPUT/7057_Sunday_June_05_2022_05_27_20_PM_94316704/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 649.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{t^2y'' - t(t+1)\,y' + y = 0}$$

Writing the ode as

$$t^2y'' + \left(-t^2 - t\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -t^2 - t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{t^2 + 2t - 1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= t^2 + 2t - 1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{t^2 + 2t - 1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2t} - \frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i \\
&= \sum_{i=0}^{0} a_i t^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $t^v = t^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2t} - \frac{1}{2t^2} + \frac{1}{2t^3} - \frac{3}{4t^4} + \frac{5}{4t^5} - \frac{9}{4t^6} + \frac{17}{4t^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i t^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $t^{v-1} = t^{-1} = \frac{1}{t}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{t}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{t}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{t}$ in r will be the coefficient in R of the term in t of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{t^2 + 2t - 1}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-1 + 2t}{4t^2}\right) \\
&= \frac{1}{4} + \frac{-1 + 2t}{4t^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term t in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{t^2 + 2t - 1}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{t - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2t} \\
&= \frac{t+1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} + \frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2} + \frac{1}{2t}\right)^2 - \left(\frac{t^2 + 2t - 1}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{2t}\right) dt} \\
&= \sqrt{t}\, \mathrm{e}^{\frac{t}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-t^2 - t}{t^2}\, dt} \\
&= z_1 e^{\frac{t}{2} + \frac{\ln(t)}{2}} \\
&= z_1 \left(\sqrt{t}\, \mathrm{e}^{\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^t t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-t^2-t}{t^2}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{t+\ln(t)}}{(y_1)^2}\, dt \\ &= y_1(-\operatorname{expIntegral}_1(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^t t\right) + c_2 \left(\mathrm{e}^t t(-\operatorname{expIntegral}_1(t))\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^t t - c_2 \mathrm{e}^t t \operatorname{expIntegral}_1(t) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^t t - c_2 \mathrm{e}^t t \operatorname{expIntegral}_1(t)$$

Verified OK.

2.634.1 Maple step by step solution

Let's solve

$y'' t^2 + (-t^2 - t)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{t^2} + \frac{(t+1)y'}{t}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y}{t^2} - \frac{(t+1)y'}{t} = 0$

☐ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(t) = -\frac{t+1}{t}, P_3(t) = \frac{1}{t^2}\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

  $\left(t \cdot P_2(t)\right)\bigg|_{t=0} = -1$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

  $\left(t^2 \cdot P_3(t)\right)\bigg|_{t=0} = 1$

- $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$y''t^2 - t(t+1)\,y' + y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k t^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $t^m \cdot y'$ to series expansion for $m = 1..2$

  $t^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,t^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $t^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,t^{k+r}$

- Convert $t^2 \cdot y''$ to series expansion

  $t^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,t^{k+r}$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 t^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 - a_{k-1}(k+r-1)\right) t^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 1$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_k(k+r-1) - a_{k-1}\right) = 0$
- Shift index using $k->k+1$

  $(k+r)\left(a_{k+1}(k+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+r}$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k t^{k+1}, a_{k+1} = \frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(t^2*diff(y(t),t$2)-t*(1+t)*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 e^t t + c_2 e^t t \operatorname{expIntegral}_1(t)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[t^2*y''[t]-t*(1+t)*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^t t(c_1 \operatorname{ExpIntegralEi}(-t) + c_2)$$

2.635 problem 650

2.635.1 Maple step by step solution 5983

Internal problem ID [8125]
Internal file name [`OUTPUT/7058_Sunday_June_05_2022_05_27_23_PM_61139878/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 650.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + (4x^2 + 6)\, y = 0}$$

Writing the ode as

$$y'' + 4xy' + (4x^2 + 6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(\mathrm{e}^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)\,\mathrm{e}^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\cos(2x)\,\mathrm{e}^{-x^2}\right) + c_2\left(\cos(2x)\,\mathrm{e}^{-x^2}\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\cos(2x)\,\mathrm{e}^{-x^2} + \frac{c_2\sin(2x)\,\mathrm{e}^{-x^2}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1\cos(2x)\,\mathrm{e}^{-x^2} + \frac{c_2\sin(2x)\,\mathrm{e}^{-x^2}}{2}
$$

Verified OK.

2.635.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2+6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + 6a_0 + (6a_3 + 10a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+3) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 + 6a_0 = 0, 6a_3 + 10a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 6a_k + 4a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 6a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2k a_{k+2} + 2a_k + 7a_{k+2})}{k^2 + 7k + 12}, a_2 = -3a_0, a_3 = -\frac{5a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x^2} \cos(2x) + c_2 \mathrm{e}^{-x^2} \sin(2x)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 37

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-x(x+2i)} \left(4c_1 - ic_2 e^{4ix}\right)$$

2.636 problem 651

2.636.1 Maple step by step solution 5992

Internal problem ID [8126]
Internal file name [OUTPUT/7059_Sunday_June_05_2022_05_27_25_PM_69572051/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 651.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(-z^2+1)\,y''-3zy'+y=0}$$

Writing the ode as

$$(-z^2+1)\,y''-3zy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -z^2+1 \\ B &= -3z \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z)=ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z)=rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7z^2 - 10}{4\left(z^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7z^2 - 10 \\ t &= 4\left(z^2 - 1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{7z^2 - 10}{4\left(z^2 - 1\right)^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(z^2 - 1)^2$. There is a pole at $z = 1$ of order 2. There is a pole at $z = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{17}{16(z-1)} - \frac{17}{16(z+1)} - \frac{3}{16(z+1)^2} - \frac{3}{16(z-1)^2}$$

For the pole at $z = 1$ let b be the coefficient of $\frac{1}{(z-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $z = -1$ let b be the coefficient of $\frac{1}{(z+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7z^2 - 10}{4(z^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1,2,3\}$
-1	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(z)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (1))) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{z-c} \\ &= \frac{1}{2}\left(\frac{1}{(z-(1))} + \frac{1}{(z-(-1))}\right) \\ &= \frac{1}{2z-2} + \frac{1}{2z+2} \end{aligned}$$

Now we search for a monic polynomial $p(z)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(z)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2z-2} + \frac{1}{2z+2}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2z-2} + \frac{1}{2z+2}\right) w + \frac{-7z^2+8}{4\left(z^2-1\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{z + 2\sqrt{2z^2-2}}{2\left(z-1\right)\left(z+1\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(z) &= e^{\int \omega \, dz} \\ &= \mathrm{e}^{\int \frac{z+2\sqrt{2z^2-2}}{2(z-1)(z+1)} dz} \\ &= \left(z^2-1\right)^{\frac{1}{4}} 2^{\frac{\sqrt{2}}{2}} \left(z + \sqrt{z^2-1}\right)^{\sqrt{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3z}{-z^2+1} \, dz} \\ &= z_1 e^{-\frac{3\ln(z-1)}{4} - \frac{3\ln(z+1)}{4}} \\ &= z_1 \left(\frac{1}{\left(z-1\right)^{\frac{3}{4}} \left(z+1\right)^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(z^2-1\right)^{\frac{1}{4}} 2^{\frac{\sqrt{2}}{2}} \left(z + \sqrt{z^2-1}\right)^{\sqrt{2}}}{\left(z-1\right)^{\frac{3}{4}} \left(z+1\right)^{\frac{3}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3z}{-z^2+1}\, dz}}{(y_1)^2}\, dz \\
&= y_1 \int \frac{e^{-\frac{3\ln(z-1)}{2}-\frac{3\ln(z+1)}{2}}}{(y_1)^2}\, dz \\
&= y_1 \left(-\frac{2^{\frac{1}{2}-\sqrt{2}}\left(z+\sqrt{z^2-1}\right)^{-2\sqrt{2}}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(z^2-1)^{\frac{1}{4}}\, 2^{\frac{\sqrt{2}}{2}}\left(z+\sqrt{z^2-1}\right)^{\sqrt{2}}}{(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}}\right) \\
&\quad + c_2 \left(\frac{(z^2-1)^{\frac{1}{4}}\, 2^{\frac{\sqrt{2}}{2}}\left(z+\sqrt{z^2-1}\right)^{\sqrt{2}}}{(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}}\left(-\frac{2^{\frac{1}{2}-\sqrt{2}}\left(z+\sqrt{z^2-1}\right)^{-2\sqrt{2}}}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (z^2-1)^{\frac{1}{4}}\, 2^{\frac{\sqrt{2}}{2}}\left(z+\sqrt{z^2-1}\right)^{\sqrt{2}}}{(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}} - \frac{c_2 (z^2-1)^{\frac{1}{4}}\, 2^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\left(z+\sqrt{z^2-1}\right)^{-\sqrt{2}}}{4\,(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (z^2-1)^{\frac{1}{4}}\, 2^{\frac{\sqrt{2}}{2}}\left(z+\sqrt{z^2-1}\right)^{\sqrt{2}}}{(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}} - \frac{c_2 (z^2-1)^{\frac{1}{4}}\, 2^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\left(z+\sqrt{z^2-1}\right)^{-\sqrt{2}}}{4\,(z-1)^{\frac{3}{4}}\,(z+1)^{\frac{3}{4}}}$$

Verified OK.

2.636.1 Maple step by step solution

Let's solve

$(-z^2+1)\, y'' - 3zy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3zy'}{z^2-1} + \frac{y}{z^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3zy'}{z^2-1} - \frac{y}{z^2-1} = 0$

□ Check to see if z_0 is a regular singular point

- ○ Define functions

  $\left[P_2(z) = \frac{3z}{z^2-1}, P_3(z) = -\frac{1}{z^2-1}\right]$

- ○ $(z+1)\cdot P_2(z)$ is analytic at $z=-1$

  $\left((z+1)\cdot P_2(z)\right)\Big|_{z=-1} = \frac{3}{2}$

- ○ $(z+1)^2\cdot P_3(z)$ is analytic at $z=-1$

  $\left((z+1)^2\cdot P_3(z)\right)\Big|_{z=-1} = 0$

- ○ $z=-1$is a regular singular point

  Check to see if z_0 is a regular singular point

  $z_0 = -1$

- Multiply by denominators

  $y''(z^2-1) + 3zy' - y = 0$

- Change variables using $z=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u-3)\left(\frac{d}{du}y(u)\right) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) u^{k +r -1+m}$$

- Shift index using $k->k +1-m$

$$u^{m}\cdot \left(\frac{d}{d u}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion for $m =1..2$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -2+m}$$

- Shift index using $k->k +2-m$

$$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$-a_{0} r \left(1+2 r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-a_{k +1} \left(k +1+r \right) \left(2 k +3+2 r \right)+a_{k} \left(k^{2}+2 k r +r^{2}+2 k +2 r -1\right)\right)$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-r \left(1+2 r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2 \left(k +\frac{3}{2}+r \right) \left(k +1+r \right) a_{k +1}+\left(k^{2}+\left(2 r +2\right) k +r^{2}+2 r -1\right) a_{k}=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=\frac{\left(k^{2}+2 k r +r^{2}+2 k +2 r -1\right) a_{k}}{\left(2 k +3+2 r \right) \left(k +1+r \right)}$$

- Recursion relation for $r =0$

$$a_{k +1}=\frac{\left(k^{2}+2 k -1\right) a_{k}}{\left(2 k +3\right) \left(k +1\right)}$$

- Solution for $r =0$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k}, a_{k +1}=\frac{\left(k^{2}+2 k -1\right) a_{k}}{\left(2 k +3\right) \left(k +1\right)}\right]$$

- Revert the change of variables $u =z +1$

$$\left[y =\sum_{k =0}^{\infty}a_{k} \left(z +1\right)^{k}, a_{k +1}=\frac{\left(k^{2}+2 k -1\right) a_{k}}{\left(2 k +3\right) \left(k +1\right)}\right]$$

- Recursion relation for $r =-\frac{1}{2}$

$$a_{k+1} = \frac{\left(k^2+k-\frac{7}{4}\right)a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1} = \frac{\left(k^2+k-\frac{7}{4}\right)a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Revert the change of variables $u = z+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (z+1)^{k-\frac{1}{2}}, a_{k+1} = \frac{\left(k^2+k-\frac{7}{4}\right)a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (z+1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (z+1)^{k-\frac{1}{2}}\right), a_{k+1} = \frac{\left(k^2+2k-1\right)a_k}{(2k+3)(k+1)}, b_{k+1} = \frac{\left(k^2+k-\frac{7}{4}\right)b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve((1-z^2)*diff(y(z),z$2)-3*z*diff(y(z),z)+y(z)=0,y(z), singsol=all)
```

$$y(z) = \frac{c_1\left(z+\sqrt{z^2-1}\right)^{\sqrt{2}}}{\sqrt{z^2-1}} + \frac{c_2\left(z+\sqrt{z^2-1}\right)^{-\sqrt{2}}}{\sqrt{z^2-1}}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 90

```
DSolve[(1-z^2)*y''[z]-3*z*y'[z]+y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to \frac{\sqrt{2}c_1 \cos \left(2\sqrt{2} \arcsin \left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right) + \sqrt{\pi} c_2 \sqrt[4]{1-z^2} Q_{-\frac{1}{2}+\sqrt{2}}^{\frac{1}{2}}(z)}{\sqrt{\pi} \sqrt[4]{-\left(z^2-1\right)^2}}$$

2.637 problem 652

2.637.1 Maple step by step solution 6003

Internal problem ID [8127]
Internal file name [`OUTPUT/7060_Sunday_June_05_2022_05_27_30_PM_23684911/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 652.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4zy'' + 2(1-z)\,y' - y = 0}$$

Writing the ode as

$$4zy'' + (-2z+2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4z \\ B &= -2z+2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 + 2z - 3}{16z^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = z^2 + 2z - 3$$
$$t = 16z^2$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{z^2 + 2z - 3}{16z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{8z} - \frac{3}{16z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{4z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{z^4} + \frac{2}{z^5} - \frac{9}{2z^6} + \frac{21}{2z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= \frac{1}{4} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{z^2+2z-3}{16z^2} \\ &= Q + \frac{R}{16z^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{2z-3}{16z^2}\right) \\ &= \frac{1}{16} + \frac{2z-3}{16z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 2. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{8}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{8}\right) - (0) \\ &= \frac{1}{8} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{4} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{8}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{z^2 + 2z - 3}{16z^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z-c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{4z} + \left(\frac{1}{4}\right) \\
&= \frac{1}{4} + \frac{1}{4z} \\
&= \frac{z+1}{4z}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d=0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(z) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4} + \frac{1}{4z}\right)(0) + \left(\left(-\frac{1}{4z^2}\right) + \left(\frac{1}{4} + \frac{1}{4z}\right)^2 - \left(\frac{z^2+2z-3}{16z^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \mathrm{e}^{\int \left(\frac{1}{4} + \frac{1}{4z}\right) dz} \\
&= z^{\frac{1}{4}} \mathrm{e}^{\frac{z}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2z+2}{4z}\, dz} \\
&= z_1 e^{\frac{z}{4} - \frac{\ln(z)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{z}{4}}}{z^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{z}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2z+2}{4z}\, dz}}{(y_1)^2}\, dz \\ &= y_1 \int \frac{e^{\frac{z}{2}-\frac{\ln(z)}{2}}}{(y_1)^2}\, dz \\ &= y_1 \left(\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, \sqrt{z}}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\frac{z}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{z}{2}} \left(\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, \sqrt{z}}{2}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{z}{2}} + c_2 \mathrm{e}^{\frac{z}{2}} \sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, \sqrt{z}}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{z}{2}} + c_2 \mathrm{e}^{\frac{z}{2}} \sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, \sqrt{z}}{2}\right)$$

Verified OK.

2.637.1 Maple step by step solution

Let's solve

$4zy'' + (-2z + 2) y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{4z} + \frac{(z-1)y'}{2z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(z-1)y'}{2z} - \frac{y}{4z} = 0$

□ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(z) = -\frac{z-1}{2z}, P_3(z) = -\frac{1}{4z}\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

  $\left(z \cdot P_2(z)\right)\bigg|_{z=0} = \frac{1}{2}$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

  $\left(z^2 \cdot P_3(z)\right)\bigg|_{z=0} = 0$

  ◦ $z = 0$is a regular singular point

  Check to see if $z_0 = 0$ is a regular singular point

  $z_0 = 0$

- Multiply by denominators

  $4zy'' + (-2z + 2) y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $z^m \cdot y'$ to series expansion for $m = 0..1$

  $z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r) z^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$$

◦ Convert $z \cdot y''$ to series expansion

$$z \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$$

◦ Shift index using $k- >k+1$

$$z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+2r)\, z^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(a_{k+1}(k+1+r) - \frac{a_k}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{2(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{2(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+1} = \frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k z^k\right)+\left(\sum_{k=0}^{\infty}b_k z^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{2(k+1)}, b_{k+1}=\frac{b_k}{2(k+\frac{3}{2})}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(4*z*diff(y(z),z$2)+2*(1-z)*diff(y(z),z)-y(z)=0,y(z), singsol=all)
```

$$y(z)=c_1 e^{\frac{z}{2}}+c_2 e^{\frac{z}{2}}\left(\int \frac{e^{-\frac{z}{2}}}{\sqrt{z}}dz\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 34

```
DSolve[4*z*y''[z]+2*(1-z)*y'[z]-y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \to e^{z/2}\left(c_1 - \sqrt{2}c_2\Gamma\left(\frac{1}{2}, \frac{z}{2}\right)\right)$$

2.638 problem 653

2.638.1 Maple step by step solution 6013

Internal problem ID [8128]
Internal file name [`OUTPUT/7061_Sunday_June_05_2022_05_27_33_PM_79552254/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 653.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{f'' + 2(z-1) f' + 4f = 0}$$

Writing the ode as

$$f'' + (2z-2) f' + 4f = 0 \tag{1}$$
$$Af'' + Bf' + Cf = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2z-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = fe^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{z^2 - 2z - 2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= z^2 - 2z - 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = (z^2 - 2z - 2)\, z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then f is found using the inverse transformation

$$f = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{1} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx z - 1 - \frac{3}{2z} - \frac{3}{2z^2} - \frac{21}{8z^3} - \frac{39}{8z^4} - \frac{159}{16z^5} - \frac{339}{16z^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i z^i \\ &= z - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = z^2 - 2z + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{z^2 - 2z - 2}{1} \\
&= Q + \frac{R}{1} \\
&= \left(z^2 - 2z - 2\right) + (0) \\
&= z^2 - 2z - 2
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{z}$ in the quotient is -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (1) \\
&= -3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= z - 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = z^2 - 2z - 2$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$z-1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\,(z-1) \\
&= 1-z \\
&= 1-z
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(1-z)\,(1) + \left((-1) + (1-z)^2 - \left(z^2 - 2z - 2\right)\right) &= 0 \\
2 + 2a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= (z-1)\, e^{\int (1-z) dz} \\
&= (z-1)\, e^{z - \frac{1}{2}z^2} \\
&= (z-1)\, e^{-\frac{z(z-2)}{2}}
\end{aligned}$$

The first solution to the original ode in f is found from

$$
\begin{aligned}
f_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-2}{1}\, dz} \\
&= z_1 e^{z-\frac{1}{2}z^2} \\
&= z_1 \left(\mathrm{e}^{-\frac{z(z-2)}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
f_1 = (z-1)\,\mathrm{e}^{-z(z-2)}
$$

The second solution f_2 to the original ode is found using reduction of order

$$
f_2 = f_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{f_1^2}\, dz
$$

Substituting gives

$$
\begin{aligned}
f_2 &= f_1 \int \frac{e^{\int -\frac{2z-2}{1}\, dz}}{(f_1)^2}\, dz \\
&= f_1 \int \frac{e^{-z^2+2z}}{(f_1)^2}\, dz \\
&= f_1 \left(\frac{-i\sqrt{\pi}\,(z-1)\,\mathrm{e}^{-1}\operatorname{erf}\left(i(z-1)\right) - \mathrm{e}^{z(z-2)}}{z-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
f &= c_1 f_1 + c_2 f_2 \\
&= c_1\left((z-1)\,\mathrm{e}^{-z(z-2)}\right) + c_2\left((z-1)\,\mathrm{e}^{-z(z-2)}\left(\frac{-i\sqrt{\pi}\,(z-1)\,\mathrm{e}^{-1}\operatorname{erf}\left(i(z-1)\right) - \mathrm{e}^{z(z-2)}}{z-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
f = c_1(z-1)\,\mathrm{e}^{-z(z-2)} + c_2\left(-1 - i(z-1)\sqrt{\pi}\,\operatorname{erf}\left(i(z-1)\right)\mathrm{e}^{-(z-1)^2}\right) \tag{1}
$$

Verification of solutions

$$
f = c_1(z-1)\,\mathrm{e}^{-z(z-2)} + c_2\left(-1 - i(z-1)\sqrt{\pi}\,\operatorname{erf}\left(i(z-1)\right)\mathrm{e}^{-(z-1)^2}\right)
$$

Verified OK.

2.638.1 Maple step by step solution

Let's solve

$f'' + (2z-2) f' + 4f = 0$

- Highest derivative means the order of the ODE is 2

f''

- Assume series solution for f

$f = \sum_{k=0}^{\infty} a_k z^k$

□ Rewrite DE with series expansions

  ◦ Convert $z^m \cdot f'$ to series expansion for $m = 0..1$

  $z^m \cdot f' = \sum_{k=\max(0,1-m)}^{\infty} a_k k \, z^{k-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $z^m \cdot f' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m) \, z^k$

  ◦ Convert f'' to series expansion

  $f'' = \sum_{k=2}^{\infty} a_k k(k-1) \, z^{k-2}$

  ◦ Shift index using $k- >k+2$

  $f'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, z^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_{k+1}(k+1) + 2a_k(k+2)) \, z^k = 0$

- Each term in the series must be 0, giving the recursion relation

$k^2 a_{k+2} + (2a_k - 2a_{k+1} + 3a_{k+2}) k + 4a_k - 2a_{k+1} + 2a_{k+2} = 0$

- Recursion relation that defines the series solution to the ODE

$\left[f = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{2(a_k k - a_{k+1} k + 2a_k - a_{k+1})}{k^2+3k+2}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(diff(f(z),z$2)+2*(z-1)*diff(f(z),z)+4*f(z)=0,f(z), singsol=all)
```

$$f(z) = c_1 \mathrm{e}^{-z^2+2z}(z-1) + c_2 \mathrm{e}^{-z^2+2z}\left(\sqrt{\pi}\,\mathrm{e}^{-1}\,\mathrm{erf}\,(iz-i)\,z - \sqrt{\pi}\,\mathrm{e}^{-1}\,\mathrm{erf}\,(iz-i) - i\mathrm{e}^{z^2-2z}\right)$$

✓ Solution by Mathematica

Time used: 0.174 (sec). Leaf size: 72

```
DSolve[f''[z]+2*(z-a)*f'[z]+4*f[z]==0,f[z],z,IncludeSingularSolutions -> True]
```

$$f(z) \to e^{z(2a-z)}\left(-\sqrt{\pi}c_2\sqrt{(a-z)^2}\mathrm{erfi}\left(\sqrt{(a-z)^2}\right) + c_2 e^{(a-z)^2} - 2ac_1 + 2c_1 z\right)$$

2.639 problem 654

2.639.1 Maple step by step solution 6021

Internal problem ID [8129]
Internal file name [OUTPUT/7062_Sunday_June_05_2022_05_27_37_PM_3481982/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 654.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{zy'' - 2y' + zy = 0}$$

Writing the ode as

$$zy'' - 2y' + zy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= -2 \\ C &= z \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2}{z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -z^2 + 2 \\ t &= z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2}{z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\
&= \sum_{i=0}^{0} a_i z^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{z^2} - \frac{i}{2z^4} - \frac{i}{2z^6} - \frac{5i}{8z^8} - \frac{7i}{8z^{10}} - \frac{21i}{16z^{12}} - \frac{33i}{16z^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\
&= i
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = -1
$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-z^2+2}{z^2} \\
&= Q + \frac{R}{z^2} \\
&= (-1) + \left(\frac{2}{z^2}\right) \\
&= -1 + \frac{2}{z^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2}{z^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{z} + (-)(i) \\ &= -\frac{1}{z} - i \\ &= -\frac{1}{z} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{z} - i\right)(1) + \left(\left(\frac{1}{z^2}\right) + \left(-\frac{1}{z} - i\right)^2 - \left(\frac{-z^2 + 2}{z^2}\right)\right) = 0$$
$$\frac{2ia_0 - 2}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= p e^{\int \omega \, dz} \\ &= (z - i)\, \mathrm{e}^{\int \left(-\frac{1}{z} - i\right) dz} \\ &= (z - i)\, \mathrm{e}^{-iz - \ln(z)} \\ &= \frac{(z - i)\, \mathrm{e}^{-iz}}{z} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{z} \, dz} \\ &= z_1 e^{\ln(z)} \\ &= z_1(z) \end{aligned}$$

Which simplifies to

$$y_1 = (z - i)\, \mathrm{e}^{-iz}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dz}}{y_1^2}\, dz$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{z}\, dz}}{(y_1)^2}\, dz \\
&= y_1 \int \frac{e^{2\ln(z)}}{(y_1)^2}\, dz \\
&= y_1 \left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((z-i)\,\mathrm{e}^{-iz}\right) + c_2\left((z-i)\,\mathrm{e}^{-iz}\left(\frac{(iz-1)\,\mathrm{e}^{2iz}}{-2z+2i}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(z-i)\,\mathrm{e}^{-iz} - \frac{c_2(iz-1)\,\mathrm{e}^{iz}}{2} \tag{1}$$

Verification of solutions

$$y = c_1(z-i)\,\mathrm{e}^{-iz} - \frac{c_2(iz-1)\,\mathrm{e}^{iz}}{2}$$

Verified OK.

2.639.1 Maple step by step solution

Let's solve

$zy'' - 2y' + zy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{2y'}{z} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2y'}{z} + y = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

◦ Define functions

$\left[P_2(z) = -\frac{2}{z}, P_3(z) = 1\right]$

◦ $z \cdot P_2(z)$ is analytic at $z = 0$

$\left(z \cdot P_2(z)\right)\Big|_{z=0} = -2$

◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

$\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 0$

◦ $z = 0$is a regular singular point

Check to see if $z_0 = 0$ is a regular singular point

$z_0 = 0$

- Multiply by denominators

$zy'' - 2y' + zy = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k z^{k+r}$

☐ Rewrite ODE with series expansions

◦ Convert $z \cdot y$ to series expansion

$z \cdot y = \sum_{k=0}^{\infty} a_k z^{k+r+1}$

◦ Shift index using $k- > k - 1$

$z \cdot y = \sum_{k=1}^{\infty} a_{k-1} z^{k+r}$

◦ Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k (k+r)\, z^{k+r-1}$

◦ Shift index using $k- > k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, z^{k+r}$$

- Convert $z \cdot y''$ to series expansion

$$z \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r-1}$$

- Shift index using $k->k+1$

$$z \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, z^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r)\, z^{-1+r} + a_1(1+r)\,(-2+r)\, z^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1}\right) z^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term must be 0

$$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k-2+r) + a_{k-1} = 0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(k+r-1) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k z^k, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[y=\sum_{k=0}^{\infty} a_k z^{k+3}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, 4a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k z^k\right)+\left(\sum_{k=0}^{\infty} b_k z^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, -2a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, 4b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(z*diff(y(z),z$2)-2*diff(y(z),z)+z*y(z)=0,y(z), singsol=all)
```

$$y(z)=c_1(\cos(z)\,z-\sin(z))+c_2(\cos(z)+\sin(z)\,z)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 39

```
DSolve[z*y''[z]-2*y'[z]+z*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z)\to -\sqrt{\frac{2}{\pi}}((c_1z+c_2)\cos(z)+(c_2z-c_1)\sin(z))$$

2.640 problem 655

2.640.1 Maple step by step solution 6032

Internal problem ID [8130]
Internal file name [OUTPUT/7063_Sunday_June_05_2022_05_27_40_PM_99190427/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 655.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{zy'' + (2z-3)\,y' + \frac{4y}{z} = 0}$$

Writing the ode as

$$zy'' + (2z-3)\,y' + \frac{4y}{z} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z \\ B &= 2z-3 \\ C &= \frac{4}{z} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ye^{\int \frac{B}{2A}\,dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4z^2 - 12z - 1}{4z^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4z^2 - 12z - 1 \\ t &= 4z^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{4z^2 - 12z - 1}{4z^2}\right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then y is found using the inverse transformation

$$y = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1223: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^2$. There is a pole at $z = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{3}{z} - \frac{1}{4z^2}$$

For the pole at $z = 0$ let b be the coefficient of $\frac{1}{z^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving z^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i z^i \\ &= \sum_{i=0}^{0} a_i z^i \end{aligned} \tag{8}$$

Let a be the coefficient of $z^v = z^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{3}{2z} - \frac{5}{4z^2} - \frac{15}{8z^3} - \frac{115}{32z^4} - \frac{495}{64z^5} - \frac{2285}{128z^6} - \frac{11055}{256z^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i z^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $z^{v-1} = z^{-1} = \frac{1}{z}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{z}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{z}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{z}$ in r will be the coefficient in R of the term in z of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4z^2 - 12z - 1}{4z^2} \\ &= Q + \frac{R}{4z^2} \\ &= (1) + \left(\frac{-12z - 1}{4z^2}\right) \\ &= 1 + \frac{-12z - 1}{4z^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term z in the remainder R is -12. Dividing this by leading coefficient in t which is 4 gives -3. Now b can be

found.

$$
\begin{aligned}
b &= (-3) - (0)\\
&= -3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 0\right) &&= -\frac{3}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 0\right) &&= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4z^2 - 12z - 1}{4z^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right)\\
&= \frac{3}{2} - \left(\frac{1}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{z-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2z} + (-)\,(1) \\
&= \frac{1}{2z} - 1 \\
&= \frac{1}{2z} - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 1$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = z + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2z} - 1\right)(1) + \left(\left(-\frac{1}{2z^2}\right) + \left(\frac{1}{2z} - 1\right)^2 - \left(\frac{4z^2 - 12z - 1}{4z^2}\right)\right) = 0$$

$$\frac{1 + 2a_0}{z} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{2}\right\}$$

Substituting these coefficients in $p(z)$ in eq. (2A) results in

$$p(z) = z - \frac{1}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(z) &= p e^{\int \omega \, dz} \\
&= \left(z - \frac{1}{2}\right) e^{\int \left(\frac{1}{2z} - 1\right) dz} \\
&= \left(z - \frac{1}{2}\right) e^{-z + \frac{\ln(z)}{2}} \\
&= \frac{(-1 + 2z)\sqrt{z}\, e^{-z}}{2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dz} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2z-3}{z}\,dz} \\
&= z_1 e^{-z+\frac{3\ln(z)}{2}} \\
&= z_1\left(z^{\frac{3}{2}}\mathrm{e}^{-z}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dz}}{y_1^2}\,dz
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2z-3}{z}\,dz}}{(y_1)^2}\,dz \\
&= y_1 \int \frac{e^{-2z+3\ln(z)}}{(y_1)^2}\,dz \\
&= y_1\left(\frac{(-8z+4)\,\mathrm{expIntegral}_1(-2z) - 4\,\mathrm{e}^{2z}}{-1+2z}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}\right) + c_2\left(\frac{(-1+2z)\,z^2\mathrm{e}^{-2z}}{2}\left(\frac{(-8z+4)\,\mathrm{expIntegral}_1(-2z) - 4\,\mathrm{e}^{2z}}{-1+2z}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(-1+2z)\,z^2\mathrm{e}^{-2z}}{2} - 4c_2\left(\frac{1}{2} + \left(z-\frac{1}{2}\right)\mathrm{expIntegral}_1(-2z)\,\mathrm{e}^{-2z}\right)z^2 \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(-1+2z)\, z^2 \mathrm{e}^{-2z}}{2} - 4c_2\left(\frac{1}{2} + \left(z - \frac{1}{2}\right) \mathrm{expIntegral}_1\left(-2z\right) \mathrm{e}^{-2z}\right) z^2$$

Verified OK.

2.640.1 Maple step by step solution

Let's solve

$zy'' + (2z-3)\, y' + \frac{4y}{z} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{z^2} - \frac{(2z-3)y'}{z}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2z-3)y'}{z} + \frac{4y}{z^2} = 0$

☐ Check to see if $z_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(z) = \frac{2z-3}{z}, P_3(z) = \frac{4}{z^2}\right]$

  ◦ $z \cdot P_2(z)$ is analytic at $z = 0$

    $\left(z \cdot P_2(z)\right)\Big|_{z=0} = -3$

  ◦ $z^2 \cdot P_3(z)$ is analytic at $z = 0$

    $\left(z^2 \cdot P_3(z)\right)\Big|_{z=0} = 4$

  ◦ $z = 0$is a regular singular point

    Check to see if $z_0 = 0$ is a regular singular point

    $z_0 = 0$

- Multiply by denominators

  $y''z^2 + (2z-3)\, zy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k z^{k+r}$

- Rewrite ODE with series expansions
  - Convert $z^m \cdot y'$ to series expansion for $m = 1..2$

    $$z^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, z^{k+r-1+m}$$
  - Shift index using $k->k+1-m$

    $$z^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, z^{k+r}$$
  - Convert $z^2 \cdot y''$ to series expansion

    $$z^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, z^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2 z^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + 2a_{k-1}(k+r-1)\right) z^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$
- Values of r that satisfy the indicial equation

  $$r = 2$$
- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-2)^2 + 2a_{k-1}(k+r-1) = 0$$
- Shift index using $k->k+1$

  $$a_{k+1}(k+r-1)^2 + 2a_k(k+r) = 0$$
- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k(k+r)}{(k+r-1)^2}$$
- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}$$
- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k z^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(z*diff(y(z),z$2)+(2*z-3)*diff(y(z),z)+4/z*y(z)=0,y(z), singsol=all)
```

$$y(z) = c_1 z^2 \mathrm{e}^{-2z}(2z-1) + c_2 z^2 \left(2 \operatorname{expIntegral}_1(-2z)\, z - \operatorname{expIntegral}_1(-2z) + \mathrm{e}^{2z}\right) \mathrm{e}^{-2z}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 47

```
DSolve[z*y''[z]+(2*z-3)*y'[z]+4/z*y[z]==0,y[z],z,IncludeSingularSolutions -> True]
```

$$y(z) \rightarrow -\frac{1}{2} e^{-2z} z^2 \left(4c_2(1-2z) \operatorname{ExpIntegralEi}(2z) - 2c_1 z + 4c_2 e^{2z} + c_1\right)$$

2.641 problem 656

2.641.1 Maple step by step solution 6040

Internal problem ID [8131]
Internal file name [`OUTPUT/7064_Sunday_June_05_2022_05_27_44_PM_66551282/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 656.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\,y' + (x-1)\,y = 0}$$

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x - \ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \ln(x)
$$

Verified OK.

2.641.1 Maple step by step solution

Let's solve

$xy'' + (1-2x)\, y' + (x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (1-2x)\, y' + (x-1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

 ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

 $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

 ◦ Shift index using $k- > k - m$

 $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

 ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

 $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

 ◦ Shift index using $k- > k + 1 - m$

 $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

 ◦ Convert $x \cdot y''$ to series expansion

 $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

 ◦ Shift index using $k- > k + 1$

 $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1 (1+r)^2 - a_0 (1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k+1+r)^2 - a_k (2k+2r+1) + a_{k-1}\right) x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2k a_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x \ln(x)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

2.642 problem 657

2.642.1 Maple step by step solution 6047

Internal problem ID [8132]
Internal file name [`OUTPUT/7065_Sunday_June_05_2022_05_27_47_PM_12626098/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 657.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + (x^2 + 2)\, y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + (x^2 + 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x) \end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x \cos(x)) + c_2(x \cos(x) (\tan(x)))
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x \cos(x) + c_2 x \sin(x) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x \cos(x) + c_2 x \sin(x)$$

Verified OK.

2.642.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (x^2+2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - 2xy' + (x^2+2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\, x^r + a_1 r(-1+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-1)\,(k+r-2) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{1, 2\}$

- Each term must be 0

  $a_1 r(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$
- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$
- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.643 problem 658

2.643.1 Maple step by step solution 6057

Internal problem ID [8133]
Internal file name [`OUTPUT/7066_Sunday_June_05_2022_05_27_49_PM_48524557/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 658.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\, y''-2xy'+2y=0}$$

Writing the ode as

$$(1-x^2)\, y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{5}{4(x-1)} - \frac{5}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\
&= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\
&= \frac{x}{x^2 - 1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x - 2} + \frac{1}{2x + 2}\right)(1) + \left(\left(-\frac{1}{2(x - 1)^2} - \frac{1}{2(1 + x)^2}\right) + \left(\frac{1}{2x - 2} + \frac{1}{2x + 2}\right)^2 - \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) \right.
$$

$$
- \frac{2a_0}{x^2 - 1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x) \, e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= x\sqrt{x - 1}\,\sqrt{1 + x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\right) + c_2 \left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2\sqrt{x^2-1}\,(\ln(x-1)\,x-\ln(1+x)\,x+2)}{2\sqrt{x-1}\,\sqrt{1+x}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{x^2 - 1}}{\sqrt{x - 1}\,\sqrt{1 + x}} + \frac{c_2 \sqrt{x^2 - 1}\,(\ln(x - 1)\, x - \ln(1 + x)\, x + 2)}{2\sqrt{x - 1}\,\sqrt{1 + x}}$$

Verified OK.

2.643.1 Maple step by step solution

Let's solve

$(1 - x^2)\, y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{2}{x^2-1}\right]$

  ◦ $(1 + x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1 + x) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1 + x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1 + x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2 - 1)\, y'' + 2xy' - 2y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2} y(u)\right) + (2u - 2)\left(\frac{d}{du} y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  ◦ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  ◦ Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = 1 + x$

  $[y = -a_0 x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(-\frac{\ln (x+1)\, x}{2} + \frac{\ln (x-1)\, x}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.644 problem 659

2.644.1 Maple step by step solution 6063

Internal problem ID [8134]
Internal file name [`OUTPUT/7067_Sunday_June_05_2022_05_27_52_PM_73939856/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 659.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2 - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{4x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.644.1 Maple step by step solution

Let's solve

$4x^2 y'' + 4xy' + (4x^2 - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 39

```
DSolve[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

2.645 problem 660

2.645.1 Maple step by step solution 6074

Internal problem ID [8135]
Internal file name [OUTPUT/7068_Sunday_June_05_2022_05_27_54_PM_11681148/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 660.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - (2x+1)\,y' + 2y = 0}$$

Writing the ode as

$$xy'' + (-2x-1)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 4x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^x}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x-1}{x} \, dx} \\
&= z_1 e^{x + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^x\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x} \left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \left(-\frac{x}{2} - \frac{1}{4}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + c_2 \left(-\frac{x}{2} - \frac{1}{4}\right)$$

Verified OK.

2.645.1 Maple step by step solution

Let's solve

$xy'' + (-2x-1)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} + \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x-1)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1) - 2a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_{k+1}(k+1+r) - 2a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{2a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{2a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{2a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{2a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{2a_k}{k+1}, b_{k+1} = \frac{2b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x$2)-(2*x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(2x + 1) + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(2*x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2x} - \frac{1}{4} c_2(2x + 1)$$

2.646 problem 661

2.646.1 Maple step by step solution 6083

Internal problem ID [8136]
Internal file name [OUTPUT/7069_Sunday_June_05_2022_05_27_58_PM_26164223/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 661.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_erf]`

$$\boxed{y'' + 2xy' + 4y = 0}$$

Writing the ode as

$$y'' + 2xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1235: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{3}{2x} - \frac{9}{8x^3} - \frac{27}{16x^5} - \frac{405}{128x^7} - \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} - \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{x^2-3}{1}\\ &= Q + \frac{R}{1}\\ &= (x^2-3) + (0)\\ &= x^2-3\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -3. Now b can be found.

$$\begin{aligned}b &= (-3) - (0)\\ &= -3\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= x\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-3}{1} - 1\right) &&= -2\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-3}{1} - 1\right) &&= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^-\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\,(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(-x)\,(1) + \left((-1) + (-x)^2 - (x^2 - 3)\right) &= 0 \\ 2a_0 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, e^{\int -x dx} \\ &= (x)\, e^{-\frac{x^2}{2}} \\ &= x\, e^{-\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\,dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\,\mathrm{e}^{-x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{\pi}\,\operatorname{erfi}(x)\,x - \mathrm{e}^{x^2}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{-x^2}\right) + c_2\left(x\,\mathrm{e}^{-x^2}\left(\frac{\sqrt{\pi}\,\operatorname{erfi}(x)\,x - \mathrm{e}^{x^2}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x^2} + c_2\left(\sqrt{\pi}\,\operatorname{erfi}(x)\,x\,\mathrm{e}^{-x^2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{-x^2} + c_2\left(\sqrt{\pi}\,\operatorname{erfi}(x)\,x\,\mathrm{e}^{-x^2} - 1\right)
$$

Verified OK.

2.646.1 Maple step by step solution

Let's solve

$y'' + 2xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(k+2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + 2a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \,\mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2}\left(\sqrt{\pi}\, \mathrm{erfi}\left(x\right) x - \mathrm{e}^{x^2}\right)$$

✓ Solution by Mathematica
Time used: 0.041 (sec). Leaf size: 51

```
DSolve[y''[x]+2*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}\left(-\sqrt{\pi} c_2 \sqrt{x^2} \mathrm{erfi}\left(\sqrt{x^2}\right) + c_2 e^{x^2} + 2 c_1 x\right)$$

2.647 problem 662

2.647.1 Maple step by step solution 6091

Internal problem ID [8137]
Internal file name [`OUTPUT/7070_Sunday_June_05_2022_05_28_01_PM_95946425/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 662.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 3y = 0}$$

Writing the ode as

$$y'' + xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^- \\ &= 2\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1x + 2a_0 + 2 &= 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-1\right) e^{\int -\frac{x}{2} dx} \\
&= \left(x^2-1\right) e^{-\frac{x^2}{4}} \\
&= \left(x^2-1\right) e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x^2-1\right) e^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2-1\right) e^{-\frac{x^2}{2}}\right) + c_2 \left(\left(x^2-1\right) e^{-\frac{x^2}{2}} \left(\int \frac{e^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}} + c_2(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}} + c_2(x^2-1)\,\mathrm{e}^{-\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.647.1 Maple step by step solution

Let's solve

$y'' + xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+3))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right)a_{k+2}+a_k(k+3)=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1 \mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)+c_2 \mathrm{e}^{-\frac{x^2}{2}}\left(x^2-1\right)\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x-1)^2(x+1)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 65

```
DSolve[y''[x]+x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-\frac{x^2}{2}} \left(\sqrt{2\pi} c_2 (x^2 - 1) \operatorname{erfi}\left(\frac{x}{\sqrt{2}} \right) + 4c_1 (x^2 - 1) - 2c_2 e^{\frac{x^2}{2}} x \right)$$

2.648 problem 663

2.648.1 Maple step by step solution 6100

Internal problem ID [8138]
Internal file name [`OUTPUT/7071_Sunday_June_05_2022_05_28_06_PM_76540582/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 663.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - x^2 y' - 3yx = 0}$$

Writing the ode as

$$y'' - x^2 y' - 3yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= -3x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x(x^3+8)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x(x^3+8)$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3+8)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \frac{80}{x^{10}} + \frac{448}{x^{13}} - \frac{2688}{x^{16}} + \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3+8)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 + 2x\right) + (0) \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3+8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	1	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2} \\ &= \frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{x^2}{2}\right)(1) + \left((x) + \left(\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3+8)}{4}\right)\right) = 0$$
$$-xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int \frac{x^2}{2} dx} \\
&= (x) \, e^{\frac{x^3}{6}} \\
&= x \, e^{\frac{x^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{1} \, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1 \left(e^{\frac{x^3}{6}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = x \, e^{\frac{x^3}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{-\frac{x^3}{3}}}{x^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\,\mathrm{e}^{\frac{x^3}{3}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{\frac{x^3}{3}} + c_2 x\,\mathrm{e}^{\frac{x^3}{3}}\left(\int \frac{\mathrm{e}^{-\frac{x^3}{3}}}{x^2}dx\right)
$$

Verified OK.

2.648.1 Maple step by step solution

Let's solve

$y'' - x^2 y' - 3yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^{k+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}(k+2)\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)\,(k a_{k+2} - a_{k-1} + a_{k+2}) = 0$$

- Shift index using $k- >k+1$

$$(k+3)\,((k+1)\,a_{k+3} - a_k + a_{k+3}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k+2}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 76

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)-3*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^3}{3}} x - \frac{9c_2 e^{\frac{x^3}{3}} 3^{\frac{2}{3}} e^{-\frac{x^3}{6}} \left(x^6 \operatorname{WhittakerM}\left(\frac{1}{3}, \frac{5}{6}, \frac{x^3}{3}\right) + 5 \operatorname{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right) x^3 + 10 \operatorname{WhittakerM}\left(\frac{4}{3}, \frac{5}{6}, \frac{x^3}{3}\right)\right)}{10x^3 \left(x^3\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 51

```
DSolve[y''[x]-x^2*y'[x]-3*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9} e^{\frac{x^3}{3}} \left(9c_1 x - 3^{2/3} c_2 \sqrt[3]{x^3} \Gamma\left(-\frac{1}{3}, \frac{x^3}{3}\right)\right)$$

2.649 problem 664

2.649.1 Maple step by step solution 6109

Internal problem ID [8139]
Internal file name [OUTPUT/7072_Sunday_June_05_2022_05_28_10_PM_53281286/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 664.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[_Gegenbauer]
```

$$\boxed{(-4x^2+1)\,y''-20xy'-16y=0}$$

Writing the ode as

$$(-4x^2+1)\,y''-20xy'-16y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^2+1\\ B &= -20x\\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 6}{(4x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^2 + 6 \\ t &= (4x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 6}{(4x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x^2 - 1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16\left(x + \frac{1}{2}\right)^2} + \frac{7}{8\left(x + \frac{1}{2}\right)} + \frac{5}{16\left(x - \frac{1}{2}\right)^2} - \frac{7}{8\left(x - \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 6}{(4x^2 - 1)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4x^2+6}{(4x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-\frac{1}{2}$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} + (-)\left(0\right) \\ &= -\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)} \\ &= -\frac{2x}{4x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)(1) + \left(\left(\frac{1}{4\left(x-\frac{1}{2}\right)^2} + \frac{1}{4\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right)^2 -$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x)\, \mathrm{e}^{\int \left(-\frac{1}{4\left(x-\frac{1}{2}\right)} - \frac{1}{4\left(x+\frac{1}{2}\right)}\right) dx} \\ &= (x)\, \mathrm{e}^{-\frac{\ln(2x-1)}{4} - \frac{\ln(2x+1)}{4}} \\ &= \frac{x}{\left(2x-1\right)^{\frac{1}{4}} \left(2x+1\right)^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-20x}{-4x^2+1}\,dx} \\
&= z_1 e^{-\frac{5\ln\left(4x^2-1\right)}{4}} \\
&= z_1\left(\frac{1}{\left(4x^2-1\right)^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-20x}{-4x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln\left(4x^2-1\right)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}\right) + c_2\left(\frac{x}{\left(4x^2-1\right)^{\frac{3}{2}}}\left(\frac{2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}}$$

Verified OK.

2.649.1 Maple step by step solution

Let's solve

$(-4x^2+1)\,y''-20xy'-16y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{20xy'}{4x^2-1}-\frac{16y}{4x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{20xy'}{4x^2-1}+\frac{16y}{4x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{20x}{4x^2-1}, P_3(x)=\frac{16}{4x^2-1}\right]$

  - $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}}=\frac{5}{2}$

  - $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}}=0$

  - $x=-\frac{1}{2}$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-\frac{1}{2}$

- Multiply by denominators

  $y''(4x^2-1)+20xy'+16y=0$
- Change variables using $x=u-\frac{1}{2}$ so that the regular singular point is at $u=0$

  $(4u^2-4u)\left(\frac{d^2}{du^2}y(u)\right)+(20u-10)\left(\frac{d}{du}y(u)\right)+16y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$
  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$
  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(3+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)(2k+5+2r)+4a_k(k+r+2)^2\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(3+2r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,-\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4a_k(k+r+2)^2-4\left(k+\frac{5}{2}+r\right)a_{k+1}(k+1+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r+2)^2}{(2k+5+2r)(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k, a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}\right]$
- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}$
- Solution for $r = -\frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k\right) + \left(\sum_{k=0}^{\infty} b_k \left(x + \frac{1}{2}\right)^{k-\frac{3}{2}}\right), a_{k+1} = \frac{2a_k(k+2)^2}{(2k+5)(k+1)}, b_{k+1} = \frac{2b_k\left(k+\frac{1}{2}\right)^2}{(2k+2)\left(k-\frac{1}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve((1-4*x^2)*diff(y(x),x$2)-20*x*diff(y(x),x)-16*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(4x^2-1)^{\frac{3}{2}}} + \frac{c_2\left(2\ln\left(2x+\sqrt{4x^2-1}\right)x-\sqrt{4x^2-1}\right)}{(4x^2-1)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 73

```
DSolve[(1-4*x^2)*y''[x]-20*x*y'[x]-16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_2 x \arctan\left(\frac{\sqrt{1-4x^2}}{2x+1}\right) - c_2\sqrt{1-4x^2} + c_1 x}{\sqrt[4]{1-4x^2}\,(4x^2-1)^{5/4}}$$

2.650 problem 665

2.650.1 Maple step by step solution 6118

Internal problem ID [8140]
Internal file name [`OUTPUT/7073_Sunday_June_05_2022_05_28_13_PM_13526471/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 665.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''-6xy'+12y=0}$$

Writing the ode as

$$(x^2-1)\,y''-6xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-1 \\ B &= -6x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(1+x)^2} + \frac{15}{4(x-1)^2} - \frac{15}{4(x-1)} + \frac{15}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$
-1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} + (-)(0) \\
&= -\frac{3}{2(x-1)} + \frac{5}{2(1+x)} \\
&= \frac{x-4}{x^2-1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)}\right)(0) + \left(\left(\frac{3}{2\,(x-1)^2} - \frac{5}{2\,(1+x)^2}\right) + \left(-\frac{3}{2\,(x-1)} + \frac{5}{2\,(1+x)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{3}{2(x-1)} + \frac{5}{2(1+x)}\right) dx} \\
&= \frac{(1+x)^{\frac{5}{2}}}{(x-1)^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x}{x^2-1} \, dx} \\
&= z_1 e^{\frac{3\ln(x-1)}{2} + \frac{3\ln(1+x)}{2}} \\
&= z_1 \left((x-1)^{\frac{3}{2}} (1+x)^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (1+x)^4$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3\ln(x-1) + 3\ln(1+x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x(x^2+1)}{(1+x)^4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((1+x)^4\right) + c_2\left((1+x)^4\left(-\frac{x(x^2+1)}{(1+x)^4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)^4 - c_2x(x^2+1) \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)^4 - c_2x(x^2+1)
$$

Verified OK.

2.650.1 Maple step by step solution

Let's solve

$(x^2-1)\,y'' - 6xy' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6xy'}{x^2-1} - \frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6xy'}{x^2-1} + \frac{12y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{6x}{x^2-1}, P_3(x) = \frac{12}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = -3$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''-6xy'+12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(-6u+6)\left(\frac{d}{du}y(u)\right)+12y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-4+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)\,(k+r-3)+a_k(k+r-3)\,(k+r-4)\right)u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,4\}$

- Each term in the series must be 0, giving the recursion relation

$((-2k-2r-2)\,a_{k+1}+a_k(k+r-4))\,(k+r-3)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+r-4)}{2(k+1+r)}$

- Recursion relation for $r=0$; series terminates at $k=4$

  $a_{k+1}=\frac{a_k(k-4)}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-2a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{3a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{3a_0}{2}$

- Apply recursion relation for $k=2$

  $a_3=-\frac{a_2}{3}$

- Express in terms of a_0

  $a_3=-\frac{a_0}{2}$

- Apply recursion relation for $k=3$

  $a_4=-\frac{a_3}{8}$

- Express in terms of a_0

  $a_4=\frac{a_0}{16}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1-2u+\frac{3}{2}u^2-\frac{1}{2}u^3+\frac{1}{16}u^4\right)$

- Revert the change of variables $u=1+x$

  $\left[y=\frac{a_0(x-1)^4}{16}\right]$

- Recursion relation for $r=4$

  $a_{k+1}=\frac{a_k k}{2(k+5)}$

- Solution for $r=4$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k (1+x)^{k+4}, a_{k+1}=\frac{a_k k}{2(k+5)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\frac{a_0 (x-1)^4}{16}+\left(\sum_{k=0}^{\infty} b_k (1+x)^{k+4}\right), b_{k+1}=\frac{b_k k}{2(k+5)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x^2-1)*diff(y(x),x$2)-6*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1 (x^3+x)+c_2 (x^4+6x^2+1)$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 45

```
DSolve[(x^2-1)*y''[x]-6*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{x^2-1}(c_2 x(x^2+1)+c_1 (x-1)^4)}{\sqrt{1-x^2}}$$

2.651 problem 666

2.651.1 Maple step by step solution 6128

Internal problem ID [8141]
Internal file name [`OUTPUT/7074_Sunday_June_05_2022_05_28_16_PM_7474623/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 666.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + (x+2)\, y = 0}$$

Writing the ode as

$$y'' + xy' + (x+2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - 1 - \frac{5}{2x} - \frac{5}{x^2} - \frac{65}{4x^3} - \frac{115}{2x^4} - \frac{885}{4x^5} - \frac{1785}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} - 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 - x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x - 6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right) + (0) \\ &= \frac{1}{4}x^2 - x - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (1) \\ &= -\frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} - 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= -3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &= 2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 - x - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} - 1$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} - 1\right) \\ &= 1 - \frac{x}{2} \\ &= 1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(1 - \frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 - x - \frac{3}{2}\right)\right) &= 0 \\ (x+2)\, a_1 + 4x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = -4\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 4x + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 3\right) e^{\int \left(1 - \frac{x}{2}\right) dx} \\
&= \left(x^2 - 4x + 3\right) e^{x - \frac{1}{4}x^2} \\
&= \left(x^2 - 4x + 3\right) e^{-\frac{x(x-4)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{e^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} \right) + c_2 \left(\left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} \left(\int \frac{e^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} + c_2 \left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} \left(\int \frac{e^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 \left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} + c_2 \left(x^2 - 4x + 3\right) e^{-\frac{x(x-2)}{2}} \left(\int \frac{e^{\frac{x(x-4)}{2}}}{\left(x^2 - 4x + 3\right)^2} dx\right)$$

Verified OK.

2.651.1 Maple step by step solution

Let's solve

$y'' + xy' + (x + 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) + a_k(k+2) + a_{k-1})x^k\right) = 0$$

- Each term must be 0

$$2a_2 + 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)a_{k+2} + a_k k + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right)a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{1}{2}x^2+x}(x^2-4x+3) + c_2 e^{-\frac{1}{2}x^2+x}(x^2-4x+3)\left(\int \frac{e^{\frac{1}{2}x^2-2x}}{(x-1)^2(x-3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.273 (sec). Leaf size: 94

```
DSolve[y''[x]+x*y'[x]+(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-\frac{x^2}{2}+x-\frac{9}{2}} \left(e^{5/2} \sqrt{2\pi} c_2 (x^2 - 4x + 3) \operatorname{erfi}\left(\frac{x-2}{\sqrt{2}} \right) + 4 e^{9/2} c_1 (x^2 - 4x + 3) - 2 c_2 e^{\frac{1}{2}(x-3)^2 + x} (x-2) \right)$$

2.652 problem 667

Internal problem ID [8142]
Internal file name [`OUTPUT/7075_Sunday_June_05_2022_05_28_19_PM_20291651/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 667.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+1)\,y''+7xy'+2y=0}$$

Writing the ode as

$$(2x^2+1)\,y''+7xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=2x^2+1\\ B&=7x\\ C&=2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 6$$
$$t = 4\left(2x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 6}{4\left(2x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x^2+1)^2$. There is a pole at $x = \frac{i\sqrt{2}}{2}$ of order 2. There is a pole at $x = -\frac{i\sqrt{2}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{7}{64\left(x-\frac{i\sqrt{2}}{2}\right)^2} - \frac{7}{64\left(x+\frac{i\sqrt{2}}{2}\right)^2} - \frac{17i\sqrt{2}}{64\left(x-\frac{i\sqrt{2}}{2}\right)} + \frac{17i\sqrt{2}}{64\left(x+\frac{i\sqrt{2}}{2}\right)}$$

For the pole at $x = \frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

For the pole at $x = -\frac{i\sqrt{2}}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{i\sqrt{2}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{8}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{5x^2+6}{4\left(2x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$
$-\frac{i\sqrt{2}}{2}$	2	0	$\frac{7}{8}$	$\frac{1}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\
&= \frac{5}{4}-\left(\frac{1}{4}\right)\\
&= 1
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} + (0) \\
&= \frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}} \\
&= \frac{x}{4x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right)(1) + \left(\left(-\frac{1}{8\left(x - \frac{i\sqrt{2}}{2}\right)^2} - \frac{1}{8\left(x + \frac{i\sqrt{2}}{2}\right)^2}\right) + \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{8x - 4i\sqrt{2}} + \frac{1}{8x + 4i\sqrt{2}}\right) dx} \\
&= (x)\left(4x^2 + 2\right)^{\frac{1}{8}} \\
&= x\left(4x^2 + 2\right)^{\frac{1}{8}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{2x^2+1}\,dx} \\
&= z_1 e^{-\frac{7\ln\left(2x^2+1\right)}{8}} \\
&= z_1 \left(\frac{1}{\left(2x^2+1\right)^{\frac{7}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{2x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{7\ln\left(2x^2+1\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\right) + c_2 \left(\frac{x 2^{\frac{1}{8}}}{\left(2x^2+1\right)^{\frac{3}{4}}}\left(\int \frac{2^{\frac{3}{4}}}{2\left(2x^2+1\right)^{\frac{1}{4}} x^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2 (2x^2+1)^{\frac{3}{4}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x 2^{\frac{1}{8}}}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x 2^{\frac{7}{8}} \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{2 (2x^2+1)^{\frac{3}{4}}}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Legendre successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve((1+2*x^2)*diff(y(x),x$2)+7*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(2x^2+1)^{\frac{3}{4}}} + \frac{c_2 x \left(\int \frac{1}{(2x^2+1)^{\frac{1}{4}} x^2} dx\right)}{(2x^2+1)^{\frac{3}{4}}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 66

```
DSolve[(1+2*x^2)*y''[x]+7*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 Q_{\frac{1}{4}}^{\frac{3}{4}}\left(i\sqrt{2}x\right)}{(2x^2+1)^{3/8}} + \frac{2i\sqrt[4]{2}c_1 x}{(2x^2+1)^{3/4}\,\text{Gamma}\left(\frac{1}{4}\right)}$$

2.653 problem 668

2.653.1 Maple step by step solution 6146

Internal problem ID [8143]
Internal file name [`OUTPUT/7076_Sunday_June_05_2022_05_28_22_PM_69681933/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 668.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{4y'' + xy' + 4y = 0}$$

Writing the ode as

$$4y'' + xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 56}{64} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 56 \\ t &= 64 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{64} - \frac{7}{8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{8} - \frac{7}{2x} - \frac{49}{x^3} - \frac{1372}{x^5} - \frac{48020}{x^7} - \frac{1882384}{x^9} - \frac{79060128}{x^{11}} - \frac{3478645632}{x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{64}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 56}{64} \\
&= Q + \frac{R}{64} \\
&= \left(\frac{x^2}{64} - \frac{7}{8}\right) + (0) \\
&= \frac{x^2}{64} - \frac{7}{8}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{7}{8}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{7}{8}\right) - (0) \\
&= -\frac{7}{8}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{8} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= -4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{8}}{\frac{1}{8}} - 1\right) &&= 3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{64} - \frac{7}{8}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{8}$	-4	3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 3$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{8}\right) \\ &= -\frac{x}{8} \\ &= -\frac{x}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (6x + 2a_2) + 2\left(-\frac{x}{8}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{8}\right) + \left(-\frac{x}{8}\right)^2 - \left(\frac{x^2}{64} - \frac{7}{8}\right)\right) &= 0 \\ 6x + 2a_2 + \frac{1}{4}a_2 x^2 + \frac{1}{2}a_1 x + \frac{3}{4}a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = -12, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - 12x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 - 12x\right) e^{\int -\frac{x}{8} dx} \\
&= \left(x^3 - 12x\right) e^{-\frac{x^2}{16}} \\
&= x\left(x^2 - 12\right) e^{-\frac{x^2}{16}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{4} dx} \\
&= z_1 e^{-\frac{x^2}{16}} \\
&= z_1 \left(e^{-\frac{x^2}{16}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x\left(x^2 - 12\right) e^{-\frac{x^2}{8}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{4} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{8}}}{(y_1)^2} dx \\
&= y_1 \left(\int \frac{e^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x\left(x^2 - 12\right) e^{-\frac{x^2}{8}}\right) + c_2 \left(x\left(x^2 - 12\right) e^{-\frac{x^2}{8}} \left(\int \frac{e^{\frac{x^2}{8}}}{x^2 \left(x^2 - 12\right)^2} dx\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 x(x^2-12)\, \mathrm{e}^{-\frac{x^2}{8}} + c_2 x(x^2-12)\, \mathrm{e}^{-\frac{x^2}{8}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 (x^2-12)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 x(x^2-12)\, \mathrm{e}^{-\frac{x^2}{8}} + c_2 x(x^2-12)\, \mathrm{e}^{-\frac{x^2}{8}} \left(\int \frac{\mathrm{e}^{\frac{x^2}{8}}}{x^2 (x^2-12)^2} dx\right)$$

Verified OK.

2.653.1 Maple step by step solution

Let's solve

$4y'' + xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{xy'}{4} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{xy'}{4} + y = 0$

- Multiply by denominators

  $4y'' + xy' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(4a_{k+2}(k+2)(k+1) + a_k(k+4)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2+3k+2)a_{k+2} + a_k(k+4) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k(k+4)}{4(k^2+3k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(4*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{x^2}{8}} x(x^2 - 12) + c_2 e^{-\frac{x^2}{8}} x(x^2 - 12) \left(\int \frac{e^{\frac{x^2}{8}}}{(x^2 - 12)^2 x^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 122

```
DSolve[4*y''[x]+x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{x^2}{8}} \left(\sqrt{2\pi} c_2 (x^2 - 12) x^2 \operatorname{erfi}\left(\frac{\sqrt{x^2}}{2\sqrt{2}}\right) + 4\sqrt{x^2} \left(2\sqrt{2} c_1 x^3 - c_2 e^{\frac{x^2}{8}} x^2 + 8 c_2 e^{\frac{x^2}{8}} - 24\sqrt{2} c_1 x\right)\right)}{32\sqrt{x^2}}$$

2.654 problem 669

Internal problem ID [8144]
Internal file name [`OUTPUT/7077_Sunday_June_05_2022_05_28_26_PM_43685074/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 669.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' - 4y = 0}$$

Writing the ode as

$$y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 18}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 18 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{9}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{9}{2x} - \frac{81}{4x^3} + \frac{729}{4x^5} - \frac{32805}{16x^7} + \frac{413343}{16x^9} - \frac{11160261}{32x^{11}} + \frac{157837977}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+18}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4}+\frac{9}{2}\right) + (0) \\
&= \frac{x^2}{4}+\frac{9}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{9}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(\frac{9}{2}\right) - (0) \\
&= \frac{9}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &= \frac{1}{2}\left(\frac{\frac{9}{2}}{\frac{1}{2}}-1\right) &= 4 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{2}}{\frac{1}{2}}-1\right) &= -5
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2}{4}+\frac{9}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	4	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 4$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(12x^2 + 6xa_3 + 2a_2\right) + 2\left(\frac{x}{2}\right)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{9}{2}\right)\right) = 0$$
$$-a_3x^3 + (-2a_2 + 12)\,x^2 + (-3a_1 + 6a_3)\,x - 4a_0 + 2a_2 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 3, a_1 = 0, a_2 = 6, a_3 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 + 6x^2 + 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^4 + 6x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 + 6x^2 + 3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 + 6x^2 + 3\right) + c_2 \left(x^4 + 6x^2 + 3 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^4 + 6x^2 + 3\right)^2} dx \right) \right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1(x^4+6x^2+3)+c_2(x^4+6x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4+6x^2+3)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^4+6x^2+3)+c_2(x^4+6x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4+6x^2+3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^4 + 6x^2 + 3) + c_2(x^4 + 6x^2 + 3)\left(\int \frac{e^{-\frac{x^2}{2}}}{(x^4 + 6x^2 + 3)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 43

```
DSolve[y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}} \text{HermiteH}\left(-5, \frac{x}{\sqrt{2}}\right) + \frac{1}{3}c_2(x^4 + 6x^2 + 3)$$

2.655 problem 670

Internal problem ID [8145]
Internal file name [OUTPUT/7078_Sunday_June_05_2022_05_28_29_PM_42058507/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 670.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

[[_2nd_order, _with_linear_symmetries]]

$$\boxed{4xy'' - xy' + 2y = 0}$$

Writing the ode as

$$4xy'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32 + x}{64x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32 + x \\ t &= 64x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32 + x}{64x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1251: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{8} - \frac{2}{x} - \frac{16}{x^2} - \frac{256}{x^3} - \frac{5120}{x^4} - \frac{114688}{x^5} - \frac{2752512}{x^6} - \frac{69206016}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{8}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{8} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{64}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-32 + x}{64x} \\
&= Q + \frac{R}{64x} \\
&= \left(\frac{1}{64}\right) + \left(-\frac{1}{2x}\right) \\
&= \frac{1}{64} - \frac{1}{2x}
\end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is -32. Dividing this by leading coefficient in t which is 64 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{8} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{8}} - 0\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-32 + x}{64x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{8}$	-2	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 2$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (-)\left(\frac{1}{8}\right) \\ &= \frac{1}{x} - \frac{1}{8} \\ &= \frac{1}{x} - \frac{1}{8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{8}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} - \frac{1}{8}\right)^2 - \left(\frac{-32 + x}{64x}\right)\right) = 0$$

$$\frac{8 + a_0}{4x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -8\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 8$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - 8)\, e^{\int \left(\frac{1}{x} - \frac{1}{8}\right) dx} \\
&= (x - 8)\, e^{-\frac{x}{8} + \ln(x)} \\
&= (x - 8)\, x\, e^{-\frac{x}{8}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{4x} \, dx} \\
&= z_1 e^{\frac{x}{8}} \\
&= z_1 \left(e^{\frac{x}{8}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x - 8)\, x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{4x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(-x^2 + 8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right) - 4\,\mathrm{e}^{\frac{x}{4}}(x-4)}{128\,(x-8)\,x} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1((x-8)\,x) + c_2 \left((x-8)\,x \left(\frac{(-x^2 + 8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right) - 4\,\mathrm{e}^{\frac{x}{4}}(x-4)}{128\,(x-8)\,x} \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(x-8)\,x + c_2 \left(\frac{(-x^2 + 8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}(x-4)}{32} \right) \tag{1}$$

Verification of solutions

$$y = c_1(x-8)\,x + c_2 \left(\frac{(-x^2 + 8x)\,\mathrm{expIntegral}_1\left(-\frac{x}{4}\right)}{128} - \frac{\mathrm{e}^{\frac{x}{4}}(x-4)}{32} \right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 48

```
dsolve(4*x*diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 - 8x) + c_2\left(\frac{\text{expIntegral}_1\left(-\frac{x}{4}\right)x^2}{128} + \frac{e^{\frac{x}{4}}x}{32} - \frac{\text{expIntegral}_1\left(-\frac{x}{4}\right)x}{16} - \frac{e^{\frac{x}{4}}}{8}\right)$$

✓ Solution by Mathematica
Time used: 0.06 (sec). Leaf size: 43

```
DSolve[4*x*y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{128}c_2\left((x-8)x\,\text{ExpIntegralEi}\left(\frac{x}{4}\right) - 4e^{x/4}(x-4)\right) + c_1(x-8)x$$

2.656 problem 671

2.656.1 Maple step by step solution 6172

Internal problem ID [8146]
Internal file name [OUTPUT/7079_Sunday_June_05_2022_05_28_32_PM_50211130/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 671.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{6x^2y'' + x(1+18x)\,y' + (12x+1)\,y = 0}$$

Writing the ode as

$$6x^2y'' + \left(18x^2+x\right)y' + (12x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6x^2 \\ B &= 18x^2 + x \\ C &= 12x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{324x^2 - 252x - 35}{144x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 324x^2 - 252x - 35 \\ t &= 144x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{324x^2 - 252x - 35}{144x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{7}{4x} - \frac{35}{144x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{35}{144}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{5}{12} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{7}{12x} - \frac{7}{36x^2} - \frac{49}{648x^3} - \frac{245}{5832x^4} - \frac{343}{13122x^5} - \frac{66199}{3779136x^6} - \frac{837949}{68024448x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{324x^2 - 252x - 35}{144x^2} \\
&= Q + \frac{R}{144x^2} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-252x - 35}{144x^2}\right) \\
&= \frac{9}{4} + \frac{-252x - 35}{144x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -252. Dividing this by leading coefficient in t which is 144 gives $-\frac{7}{4}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{7}{4}\right) - (0) \\
&= -\frac{7}{4}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= -\frac{7}{12} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{7}{4}}{\frac{3}{2}} - 0\right) &&= \frac{7}{12}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{324x^2 - 252x - 35}{144x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{12}$	$\frac{5}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{7}{12}$	$\frac{7}{12}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{7}{12}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{7}{12} - \left(\frac{7}{12}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{7}{12x} + (-)\left(\frac{3}{2}\right) \\ &= \frac{7}{12x} - \frac{3}{2} \\ &= \frac{7}{12x} - \frac{3}{2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{7}{12x} - \frac{3}{2}\right)(0) + \left(\left(-\frac{7}{12x^2}\right) + \left(\frac{7}{12x} - \frac{3}{2}\right)^2 - \left(\frac{324x^2 - 252x - 35}{144x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(\frac{7}{12x} - \frac{3}{2}\right) dx} \\ &= x^{\frac{7}{12}} \mathrm{e}^{-\frac{3x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{18x^2+x}{6x^2}\, dx} \\ &= z_1 e^{-\frac{3x}{2} - \frac{\ln(x)}{12}} \\ &= z_1\left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x^{\frac{1}{12}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{18x^2+x}{6x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x-\frac{\ln(x)}{6}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\sqrt{x}\,\mathrm{e}^{-3x}\right) + c_2 \left(\sqrt{x}\,\mathrm{e}^{-3x} \left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-3x} + c_2 \sqrt{x}\,\mathrm{e}^{-3x} \left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\,\mathrm{e}^{-3x} + c_2 \sqrt{x}\,\mathrm{e}^{-3x} \left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}} dx\right)$$

Verified OK.

2.656.1 Maple step by step solution

Let's solve

$6x^2y'' + (18x^2 + x)\, y' + (12x + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(12x+1)y}{6x^2} - \frac{(1+18x)y'}{6x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+18x)y'}{6x} + \frac{(12x+1)y}{6x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+18x}{6x}, P_3(x) = \frac{12x+1}{6x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{6}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{6}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $6x^2y'' + x(1 + 18x)\, y' + (12x + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ○ Shift index using $k- > k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+3r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-1)\,(2k+2r-1) + 6a_{k-1}(3k+3r-1)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+3r)\,(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$6\left(k+r-\frac{1}{3}\right)\left(\left(k+r-\frac{1}{2}\right) a_k + 3a_{k-1}\right) = 0$$

- Shift index using $k- > k+1$

$$6\left(k+\frac{2}{3}+r\right)\left(\left(k+\frac{1}{2}+r\right) a_{k+1} + 3a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{6a_k}{2k+1+2r}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{6a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{6a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{6a_k}{2k+\frac{5}{3}}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+1} = -\frac{6a_k}{2k+2}, b_{k+1} = -\frac{6b_k}{2k+\frac{5}{3}}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(6*x^2*diff(y(x),x$2)+x*(1+18*x)*diff(y(x),x)+(1+12*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,\mathrm{e}^{-3x} + c_2\sqrt{x}\,\mathrm{e}^{-3x}\left(\int \frac{\mathrm{e}^{3x}}{x^{\frac{7}{6}}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 47

```
DSolve[6*x^2*y''[x]+x*(1+18*x)*y'[x]+(1+12*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-3x}\left(\frac{\sqrt[6]{3}c_2x^{4/3}\Gamma\left(-\frac{1}{6},-3x\right)}{(-x)^{5/6}}+c_1\sqrt{x}\right)$$

2.657 problem 672

2.657.1 Maple step by step solution 6183

Internal problem ID [8147]
Internal file name [OUTPUT/7080_Sunday_June_05_2022_05_28_36_PM_98535222/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 672.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3x^2y'' - x(x+8)\,y' + 6y = 0}$$

Writing the ode as

$$3x^2y'' + \left(-x^2 - 8x\right)y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x^2 \\ B &= -x^2 - 8x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 16x + 40}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 16x + 40 \\ t &= 36x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 16x + 40}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{36} + \frac{4}{9x} + \frac{10}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{10}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{3} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{2}{3}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{6} + \frac{4}{3x} - \frac{2}{x^2} + \frac{16}{x^3} - \frac{140}{x^4} + \frac{1312}{x^5} - \frac{12944}{x^6} + \frac{132736}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{6}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{6} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{36}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 16x + 40}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{36}\right) + \left(\frac{16x + 40}{36x^2}\right) \\ &= \frac{1}{36} + \frac{16x + 40}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 16. Dividing this by leading coefficient in t which is 36 gives $\frac{4}{9}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{4}{9}\right) - (0) \\ &= \frac{4}{9} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{6} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= \frac{4}{3} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{4}{9}}{\frac{1}{6}} - 0\right) &&= -\frac{4}{3}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 16x + 40}{36x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{3}$	$-\frac{2}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{6}$	$\frac{4}{3}$	$-\frac{4}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{4}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{4}{3} - \left(-\frac{2}{3}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{3x} + \left(\frac{1}{6}\right) \\
&= -\frac{2}{3x} + \frac{1}{6} \\
&= \frac{x - 4}{6x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(-\frac{2}{3x} + \frac{1}{6}\right)(2x + a_1) + \left(\left(\frac{2}{3x^2}\right) + \left(-\frac{2}{3x} + \frac{1}{6}\right)^2 - \left(\frac{x^2 + 16x + 40}{36x^2}\right)\right) &= 0 \\
\frac{(-a_1 - 2)\, x - 2a_0 - 4a_1}{3x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4, a_1 = -2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 2x + 4
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 2x + 4\right) e^{\int \left(-\frac{2}{3x} + \frac{1}{6}\right) dx} \\
&= \left(x^2 - 2x + 4\right) e^{\frac{x}{6} - \frac{2\ln(x)}{3}} \\
&= \frac{\left(x^2 - 2x + 4\right) e^{\frac{x}{6}}}{x^{\frac{2}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-8x}{3x^2}\,dx} \\
&= z_1 e^{\frac{x}{6}+\frac{4\ln(x)}{3}} \\
&= z_1\left(x^{\frac{4}{3}}\mathrm{e}^{\frac{x}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-8x}{3x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{3}+\frac{8\ln(x)}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}}\right) + c_2\left(\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}}\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}} + c_2\left(x^2-2x+4\right)\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}}\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{\left(x^2-2x+4\right)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1(x^2-2x+4)\,\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}} + c_2(x^2-2x+4)\,\mathrm{e}^{\frac{x}{3}}x^{\frac{2}{3}}\left(\int \frac{x^{\frac{4}{3}}\mathrm{e}^{-\frac{x}{3}}}{(x^2-2x+4)^2}dx\right)$$

Verified OK.

2.657.1 Maple step by step solution

Let's solve

$3x^2y'' + (-x^2-8x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2} + \frac{(x+8)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+8)y'}{3x} + \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+8}{3x}, P_3(x) = \frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -\frac{8}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' - x(x+8)\,y' + 6y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+3r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(3k+3r-2)\,(k+r-3) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+3r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{3, \frac{2}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+r-\frac{2}{3}\right)(k+r-3)\, a_k - a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

  $3\left(k+\frac{1}{3}+r\right)(k-2+r)\, a_{k+1} - a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(3k+1+3r)(k-2+r)}$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k(k+3)}{(3k+10)(k+1)}\right]$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+1} = \frac{a_k\left(k+\frac{2}{3}\right)}{(3k+3)\left(k-\frac{4}{3}\right)}$

- Solution for $r = \frac{2}{3}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k\left(k+\frac{2}{3}\right)}{\left(3k+3\right)\left(k-\frac{4}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+3}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{2}{3}}\right), a_{k+1}=\frac{a_k\left(k+3\right)}{\left(3k+10\right)\left(k+1\right)}, b_{k+1}=\frac{b_k\left(k+\frac{2}{3}\right)}{\left(3k+3\right)\left(k-\frac{4}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(3*x^2*diff(y(x),x$2)-x*(x+8)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{2}{3}} e^{\frac{x}{3}} (x^2 - 2x + 4) + c_2 x^{\frac{2}{3}} e^{\frac{x}{3}} (x^2 - 2x + 4) \left(\int \frac{x^{\frac{4}{3}} e^{-\frac{x}{3}}}{(x^2 - 2x + 4)^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 79

```
DSolve[3*x^2*y''[x]-x*(x+8)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x/3} x^{2/3} (x^2 - 2x + 4) - \frac{c_2 e^{x/3} x^{2/3} (x^2 - 2x + 4) \Gamma\left(\frac{1}{3}, \frac{x}{3}\right)}{6\ 3^{2/3}} + \frac{1}{6} c_2 (x - 4) x$$

2.658 problem 673

2.658.1 Maple step by step solution 6194

Internal problem ID [8148]
Internal file name [OUTPUT/7081_Sunday_June_05_2022_05_28_39_PM_43447699/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 673.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2y'' - x(2x+1)\,y' + 2(4x-1)\,y = 0}$$

Writing the ode as

$$2x^2y'' + \left(-2x^2 - x\right)y' + (8x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -2x^2 - x \\ C &= 8x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 60x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 60x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 60x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{15}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{15}{4x} - \frac{51}{4x^2} - \frac{765}{8x^3} - \frac{3519}{4x^4} - \frac{144585}{16x^5} - \frac{6358527}{64x^6} - \frac{146409525}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{4x^2 - 60x + 21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-60x + 21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{-60x + 21}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) &= -\frac{15}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{1}{2}} - 0\right) = \frac{15}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 - 60x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{15}{4}$	$\frac{15}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{15}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{15}{4} - \left(\frac{7}{4}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{7}{4x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{7}{4x} - \frac{1}{2} \\
&= \frac{7}{4x} - \frac{1}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{7}{4x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{7}{4x^2}\right) + \left(\frac{7}{4x} - \frac{1}{2}\right)^2 - \left(\frac{4x^2 - 60x + 21}{16x^2}\right)\right) = 0
$$

$$
\frac{2(9 + a_1)x + 4a_0 + 7a_1}{2x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{63}{4}, a_1 = -9\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 9x + \frac{63}{4}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{\int \left(\frac{7}{4x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 9x + \frac{63}{4}\right) e^{-\frac{x}{2} + \frac{7\ln(x)}{4}} \\
&= \frac{(4x^2 - 36x + 63)\, x^{\frac{7}{4}} e^{-\frac{x}{2}}}{4}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{2x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{\ln(x)}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}} x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 9x^3 + \frac{63}{4}x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + c_2\left(x^4 - 9x^3 + \frac{63}{4}x^2\left(\int \frac{16\,\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + 16c_2\left(\int \frac{\mathrm{e}^x}{x^{\frac{7}{2}}\left(4x^2-36x+63\right)^2}dx\right)\left(x^2 - 9x + \frac{63}{4}\right)x^2 \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 9x^3 + \frac{63}{4}x^2\right) + 16c_2\left(\int \frac{e^x}{x^{\frac{7}{2}}\left(4x^2 - 36x + 63\right)^2}dx\right)\left(x^2 - 9x + \frac{63}{4}\right)x^2$$

Verified OK.

2.658.1 Maple step by step solution

Let's solve

$2x^2y'' + (-2x^2 - x)\,y' + (8x - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-1)y}{x^2} + \frac{(2x+1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{2x} + \frac{(4x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x+1}{2x}, P_3(x) = \frac{4x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' - x(2x + 1)\,y' + (8x - 2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(k+r-2) - 2a_{k-1}(k-5+r)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{2, -\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+r-2)\, a_k - 2a_{k-1}(k-5+r) = 0$
- Shift index using $k- > k + 1$

  $2\left(k+\frac{3}{2}+r\right)(k+r-1)\, a_{k+1} - 2a_k(k+r-4) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-4)}{(2k+3+2r)(k+r-1)}$
- Recursion relation for $r = 2$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k-2)}{(2k+7)(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{4a_0}{7}$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{9}$
- Express in terms of a_0

  $a_2 = \frac{4a_0}{63}$
- Terminating series solution of the ODE for $r = 2$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right)$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{4}{7}x + \frac{4}{63}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), b_{k+1} = \frac{2b_k\left(k-\frac{9}{2}\right)}{(2k+2)\left(k-\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(2*x^2*diff(y(x),x$2)-x*(1+2*x)*diff(y(x),x)+2*(4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 (4x^2 - 36x + 63) + c_2 x^2 (4x^2 - 36x + 63) \left(\int \frac{e^x}{(4x^2 - 36x + 63)^2 x^{\frac{7}{2}}} dx \right)$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 89

```
DSolve[2*x^2*y''[x]-x*(1+2*x)*y'[x]+2*(4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to c_1 \left(x^4 - 9x^3 + \frac{63x^2}{4}\right) - \frac{4c_2 \left(\sqrt{\pi}(-4x^2 + 36x - 63)\, x^{5/2} \operatorname{erfi}\left(\sqrt{x}\right) + 2e^x(2x^4 - 17x^3 + 24x^2 + 6x + 3)\right)}{945\sqrt{x}}$$

2.659 problem 674

2.659.1 Maple step by step solution 6205

Internal problem ID [8149]
Internal file name [`OUTPUT/7082_Sunday_June_05_2022_05_28_42_PM_97656851/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 674.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4x^2y' + (2x+1)\,y = 0}$$

Writing the ode as

$$4x^2y'' - 4x^2y' + (2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2 \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-2x-1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^2}{4x^2}\, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2}{4x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{expIntegral}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(-\operatorname{expIntegral}_1(-x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} - c_2\sqrt{x}\operatorname{expIntegral}_1(-x) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} - c_2\sqrt{x}\operatorname{expIntegral}_1(-x)$$

Verified OK.

2.659.1 Maple step by step solution

Let's solve

$4x^2y'' - 4x^2y' + (2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = y' - \frac{(2x+1)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - y' + \frac{(2x+1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = -1, P_3(x) = \frac{2x+1}{4x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left.(x^2 \cdot P_3(x))\right|_{x=0} = \frac{1}{4}$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$4x^2y'' - 4x^2y' + (2x+1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x^2 \cdot y'$ to series expansion

$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$

- ○ Shift index using $k- >k - 1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r-1)^2 - 2a_{k-1}(2k-3+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k+2r-1)^2 + (-4k+6-4r)\, a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(2k+1+2r)^2 + a_k(-4k-4r+2) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k(2k+2r-1)}{(2k+1+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{4a_k k}{(2k+2)^2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{4a_k k}{(2k+2)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(4*x^2*diff(y(x),x$2)-4*x^2*diff(y(x),x)+(1+2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\ \mathrm{expIntegral}_1\left(-x\right)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 19

```
DSolve[4*x^2*y''[x]-4*x^2*y'[x]+(1+2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}(c_2\ \mathrm{ExpIntegralEi}(x) + c_1)$$

2.660 problem 675

2.660.1 Maple step by step solution 6216

Internal problem ID [8150]
Internal file name [OUTPUT/7083_Sunday_June_05_2022_05_28_45_PM_73576135/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 675.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-2x+3)\,y' + (1-2x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(-2x^2+3x\right)y' + (1-2x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2+3x \\ C &= 1-2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 - \frac{1}{x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} - \frac{1}{4x^2} - \frac{1}{8x^3} - \frac{3}{32x^4} - \frac{5}{64x^5} - \frac{9}{128x^6} - \frac{17}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-1 - 4x}{4x^2}\right) \\ &= 1 + \frac{-1 - 4x}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(1) \\
&= \frac{1}{2x} - 1 \\
&= \frac{1}{2x} - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - 1\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - 1\right)^2 - \left(\frac{4x^2 - 4x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - 1\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 + 3x}{x^2} \, dx} \\
&= z_1 e^{x - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{x}}{x^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x-3\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(-\operatorname{expIntegral}_1(-2x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(-\operatorname{expIntegral}_1(-2x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} - \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} - \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x}$$

Verified OK.

2.660.1 Maple step by step solution

Let's solve

$x^2y'' + (-2x^2 + 3x)\, y' + (1 - 2x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x-1)y}{x^2} + \frac{(2x-3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-3)y'}{x} - \frac{(2x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x-3}{x}, P_3(x) = -\frac{2x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(2x - 3)\, y' + (1 - 2x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) \, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) \, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) \, (k+r-1) \, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+r)^2 \, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)^2 - 2a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -1$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r+1)^2 - 2a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1} (k+2+r)^2 - 2a_k (k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k (k+r+1)}{(k+2+r)^2}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{2a_k k}{(k+1)^2}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{2a_k k}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*(3-2*x)*diff(y(x),x)+(1-2*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + \frac{c_2 \operatorname{expIntegral}_1(-2x)}{x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]+x*(3-2*x)*y'[x]+(1-2*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \operatorname{ExpIntegralEi}(2x) + c_1}{x}$$

2.661 problem 676

2.661.1 Maple step by step solution 6226

Internal problem ID [8151]
Internal file name [`OUTPUT/7084_Sunday_June_05_2022_05_28_49_PM_70301779/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 676.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(x+3)\,y' + (-x+4)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2-3x\right)y' + (-x+4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2-3x \\ C &= -x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1262: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{2x} - \frac{13}{2x^2} + \frac{65}{2x^3} - \frac{819}{4x^4} + \frac{5785}{4x^5} - \frac{43797}{4x^6} + \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 10x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{10x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{10x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 10. Dividing this by leading coefficient in t which is 4 gives $\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 10x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(2) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 10x - 1}{4x^2}\right) \right) &= 0 \\
\frac{(-a_1 + 4)\, x - 2a_0 + a_1}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2, a_1 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 4x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + 4x + 2\right) \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 + 4x + 2\right) \mathrm{e}^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 + 4x + 2\right) \sqrt{x}\, \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+4x+2\right) x^2 \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(x+3)\,\mathrm{e}^{-x} - \left(x^2+4x+2\right)\mathrm{expIntegral}_1\left(x\right)}{4x^2+16x+8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^2+4x+2\right) x^2 \mathrm{e}^{x}\right) \\
&\quad + c_2\left(\left(x^2+4x+2\right) x^2 \mathrm{e}^{x}\left(\frac{(x+3)\,\mathrm{e}^{-x} - \left(x^2+4x+2\right)\mathrm{expIntegral}_1\left(x\right)}{4x^2+16x+8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^2+4x+2\right) x^2 \mathrm{e}^{x} - \frac{c_2 x^2\left(\left(x^2+4x+2\right)\mathrm{e}^{x}\,\mathrm{expIntegral}_1\left(x\right) - x - 3\right)}{4} \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^2+4x+2\right)x^2\mathrm{e}^x - \frac{c_2x^2\left(\left(x^2+4x+2\right)\mathrm{e}^x \operatorname{expIntegral}_1(x) - x - 3\right)}{4}$$

Verified OK.

2.661.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2-3x)\,y' + (-x+4)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} - \frac{(x-4)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x+3)\,y' + (-x+4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$
  - Shift index using $k- >k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-2 + r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k + r - 2)^2 - a_{k-1}(k + r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2 + r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k + r - 2)^2 - a_{k-1}(k + r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1}(k + r - 1)^2 - a_k (k + r + 1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k (k+r+1)}{(k+r-1)^2}$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k (k+3)}{(k+1)^2}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+3)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(x^2*diff(y(x),x$2)-x*(3+x)*diff(y(x),x)+(4-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 (x^2 + 4x + 2) - \frac{c_2 x^2 (-x^2 \operatorname{expIntegral}_1(x) + e^{-x} x - 4 \operatorname{expIntegral}_1(x) x + 3 e^{-x} - 2 \operatorname{expIntegral}_1(x)) e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 52

```
DSolve[x^2*y''[x]-x*(3+x)*y'[x]+(4-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} x^2 \left(c_2 e^x (x^2 + 4x + 2) \operatorname{ExpIntegralEi}(-x) + 4 c_1 e^x (x^2 + 4x + 2) + c_2 (x + 3)\right)$$

2.662 problem 677

2.662.1 Maple step by step solution 6236

Internal problem ID [8152]
Internal file name [`OUTPUT/7085_Sunday_June_05_2022_05_28_52_PM_95957648/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 677.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(3-x)\,y' + y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 + 3x\right)y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 + 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 6x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1264: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 6x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-6x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{1}{2x} - \frac{1}{2} \\ &= -\frac{x-1}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 6x - 1}{4x^2}\right)\right) = 0$$

$$\frac{1 + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (x-1)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\ &= (x-1)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= (x-1)\sqrt{x}\, e^{-\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x-1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x+\operatorname{expIntegral}_1(-x)-\mathrm{e}^x}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x-1}{x}\right) + c_2\left(\frac{x-1}{x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x+\operatorname{expIntegral}_1(-x)-\mathrm{e}^x}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-1)}{x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x+\operatorname{expIntegral}_1(-x)-\mathrm{e}^x)}{x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x-1)}{x} + \frac{c_2(-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - \mathrm{e}^x)}{x}$$

Verified OK.

2.662.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2+3x)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2} + \frac{(-3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(-3+x)y'}{x} + \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{-3+x}{x}, P_3(x) = \frac{1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(-3+x)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = -1$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)^2 - a_{k-1}(k+r-1) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)^2 - a_k(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r)}{(k+2+r)^2}$
- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)}{(k+1)^2}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (1-x)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x$2)+x*(3-x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x-1)}{x} + \frac{c_2(\text{expIntegral}_1(-x)\,x - \text{expIntegral}_1(-x) + e^x)}{x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]+x*(3-x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2e^x}{x}$$

2.663 problem 678

2.663.1 Maple step by step solution 6242

Internal problem ID [8153]
Internal file name [`OUTPUT/7086_Sunday_June_05_2022_05_28_56_PM_54023490/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 678.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - \left(2\sqrt{5}-1\right)xy' + \left(\frac{19}{4}-3x^2\right)y = 0}$$

Writing the ode as

$$x^2y'' + \left(-2\sqrt{5}\,x + x\right)y' + \left(\frac{19}{4}-3x^2\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2\sqrt{5}\,x + x \\ C &= \frac{19}{4} - 3x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1266: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2\sqrt{5}\,x+x}{x^2}\,dx}\\ &= z_1 e^{\ln(x)\sqrt{5}-\frac{\ln(x)}{2}}\\ &= z_1\left(x^{\sqrt{5}-\frac{1}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2\sqrt{5}\,x+x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{\left(2\sqrt{5}-1\right)\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}}\right) + c_2\left(\mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}}\left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}} + \frac{c_2\sqrt{3}\,x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{\sqrt{3}\,x}}{6} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\sqrt{3}\,x} x^{\sqrt{5}-\frac{1}{2}} + \frac{c_2\sqrt{3}\,x^{\sqrt{5}-\frac{1}{2}}\mathrm{e}^{\sqrt{3}\,x}}{6}
$$

Verified OK.

2.663.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(-2\sqrt{5}\,x + x\right) y' + \left(\frac{19}{4} - 3x^2\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(12x^2-19)y}{4x^2} + \frac{\left(2\sqrt{5}-1\right)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{\left(2\sqrt{5}-1\right)y'}{x} - \frac{(12x^2-19)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{2\sqrt{5}-1}{x}, P_3(x) = -\frac{12x^2-19}{4x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1 - 2\sqrt{5}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{19}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4(2\sqrt{5} - 1)\, xy' + (-12x^2 + 19)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $(1 + 2\sqrt{5} - 2r)\, (-1 + 2\sqrt{5} - 2r)\, a_0 x^r + (-1 + 2\sqrt{5} - 2r)\, (-3 + 2\sqrt{5} - 2r)\, a_1 x^{1+r} + \left(\sum_{k=2}^{\infty} ((-\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2\sqrt{5} - 2r)\, (-1 + 2\sqrt{5} - 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{2} + \sqrt{5}, \sqrt{5} - \frac{1}{2}\right\}$

- Each term must be 0

$$\left(-1+2\sqrt{5}-2r\right)\left(-3+2\sqrt{5}-2r\right)a_1 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-8a_k(k+r)\sqrt{5}+\left(4k^2+8kr+4r^2+19\right)a_k-12a_{k-2}=0$$

- Shift index using $k->k+2$

$$-8a_{k+2}(k+2+r)\sqrt{5}+\left(4(k+2)^2+8(k+2)\,r+4r^2+19\right)a_{k+2}-12a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{12a_k}{-35+8k\sqrt{5}+8\sqrt{5}\,r-4k^2-8kr-4r^2+16\sqrt{5}-16k-16r}$$

- Recursion relation for $r=\frac{1}{2}+\sqrt{5}$

$$a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}$$

- Solution for $r=\frac{1}{2}+\sqrt{5}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}, a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right)^2-16k}, a_1=0\right]$$

- Recursion relation for $r=\sqrt{5}-\frac{1}{2}$

$$a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}$$

- Solution for $r=\sqrt{5}-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\sqrt{5}-\frac{1}{2}}, a_{k+2} = -\frac{12a_k}{-27+8k\sqrt{5}+8\sqrt{5}\left(\sqrt{5}-\frac{1}{2}\right)-4k^2-8k\left(\sqrt{5}-\frac{1}{2}\right)-4\left(\sqrt{5}-\frac{1}{2}\right)^2-16k}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}+\sqrt{5}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\sqrt{5}-\frac{1}{2}}\right), a_{k+2} = -\frac{12a_k}{-43+8k\sqrt{5}+8\sqrt{5}\left(\frac{1}{2}+\sqrt{5}\right)-4k^2-8k\left(\frac{1}{2}+\sqrt{5}\right)-4\left(\frac{1}{2}+\sqrt{5}\right.}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(x^2*diff(y(x),x$2)-(2*sqrt(5)-1)*x*diff(y(x),x)+(19/4-3*x^2)*y(x)=0,y(x), singsol=all
```

$$y(x) = \frac{c_1 x^{\sqrt{5}} \sinh\left(\sqrt{3}\, x\right)}{\sqrt{x}} + \frac{c_2 x^{\sqrt{5}} \cosh\left(\sqrt{3}\, x\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 53

```
DSolve[x^2*y''[x]-(2*Sqrt[5]-1)*x*y'[x]+(19/4-3*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{6} e^{-\sqrt{3}x} x^{\sqrt{5}-\frac{1}{2}} \left(\sqrt{3} c_2 e^{2\sqrt{3}x} + 6c_1\right)$$

2.664 problem 679

2.664.1 Maple step by step solution 6252

Internal problem ID [8154]
Internal file name [OUTPUT/7087_Sunday_June_05_2022_05_28_58_PM_85193299/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 679.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(-3+x)\,y' + (-x+4)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2-3x\right)y' + (-x+4)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2-3x \\ C &= -x+4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1268: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} - \frac{1}{2x^2} - \frac{1}{2x^3} - \frac{3}{4x^4} - \frac{5}{4x^5} - \frac{9}{4x^6} - \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x - 1}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2}\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x - 1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \sqrt{x}\, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2 - 3x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} + \frac{3 \ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x+3\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(-\operatorname{expIntegral}_1(-x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x} x^2\right) + c_2\left(\mathrm{e}^{-x} x^2 (-\operatorname{expIntegral}_1(-x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x} c_1 x^2 - c_2 \mathrm{e}^{-x} x^2 \operatorname{expIntegral}_1(-x) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x} c_1 x^2 - c_2 \mathrm{e}^{-x} x^2 \operatorname{expIntegral}_1(-x)$$

Verified OK.

2.664.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - 3x)\, y' + (-x + 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-4)y}{x^2} - \frac{(-3+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-3+x)y'}{x} - \frac{(x-4)y}{x^2} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{-3+x}{x}, P_3(x) = -\frac{x-4}{x^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -3$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 4$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $x^2 y'' + x(-3+x)\, y' + (-x+4)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k(k+r-2) + a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r-1)\,(a_{k+1}(k+r-1) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{k+r-1}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x^2*diff(y(x),x$2)+x*(x-3)*diff(y(x),x)+(4-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1x^2 + c_2x^2e^{-x}\,\mathrm{expIntegral}_1\left(-x\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*(x-3)*y'[x]+(4-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}x^2(c_2\,\mathrm{ExpIntegralEi}(x) + c_1)$$

2.665 problem 680

2.665.1 Maple step by step solution 6263

Internal problem ID [8155]
Internal file name [`OUTPUT/7088_Sunday_June_05_2022_05_29_02_PM_84937841/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 680.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x^2y' - (x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + x^2y' + (-x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= -x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1270: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2} + \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{2}{x^5} - \frac{6}{x^6} + \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{x+2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\ &= y_1 \left(\left(x^2-2x+2\right) \mathrm{e}^{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x}\left(\left(x^2-2x+2\right) \mathrm{e}^{x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2 (x^2-2x+2)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-x}}{x} + \frac{c_2 (x^2-2x+2)}{x}$$

Verified OK.

2.665.1 Maple step by step solution

Let's solve

$x^2 y'' + x^2 y' + (-x-2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{(x+2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{(x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 1, P_3(x) = -\frac{x+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x^2 y' + (-x-2) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

  $$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 (1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)\,(k+r-2) + a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-2)\,(a_k (k+r+1) + a_{k-1}) = 0$$

- Shift index using $k->k+1$

  $$(k-1+r)\,(a_{k+1}(k+2+r) + a_k) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{k+2+r}$$

- Recursion relation for $r=-1$

  $$a_{k+1} = -\frac{a_k}{k+1}$$

- Solution for $r=-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

  $$a_{k+1} = -\frac{a_k}{k+4}$$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1}=-\frac{a_k}{k+1}, b_{k+1}=-\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-(2+x)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x^2-2x+2)}{x}+\frac{c_2 \mathrm{e}^{-x}}{x}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]+x^2*y'[x]-(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-x}(c_2 e^x(x^2-2x+2)+c_1)}{x}$$

2.666 problem 681

2.666.1 Maple step by step solution 6273

Internal problem ID [8156]
Internal file name [OUTPUT/7089_Sunday_June_05_2022_05_29_05_PM_95357748/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 681.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 \\ C &= x - \frac{3}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 - 4x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1272: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = 1 - \frac{1}{x} + \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be

found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x^2}{x^2} \, dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}}\left(-\frac{(2x+1)\,\mathrm{e}^{-2x}}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x}} - \frac{c_2(2x+1)\,\mathrm{e}^{-2x}}{4\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} - \frac{c_2(2x+1)\,\mathrm{e}^{-2x}}{4\sqrt{x}}$$

Verified OK.

2.666.1 Maple step by step solution

Let's solve

$x^2y'' + 2x^2y' + \left(x - \frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-3)y}{4x^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{(4x-3)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 2, P_3(x) = \frac{4x-3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 8x^2y' + (4x - 3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}(2k-1+2r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(k+r+\frac{1}{2}\right) a_k + 8a_{k-1}\left(k-\frac{1}{2}+r\right) = 0$$

- Shift index using $k->k+1$

$$4\left(k-\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+1} + 8a_k\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{4a_k(2k+2r+1)}{(2k-1+2r)(2k+3+2r)}$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = -\frac{8a_k k}{(2k-2)(2k+2)}$$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{8a_k k}{(2k-2)(2k+2)}$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = -\frac{4a_k(2k+4)}{(2k+2)(2k+6)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4a_k(2k+4)}{(2k+2)(2k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x-3/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 e^{-2x}(2x+1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]+2*x^2*y'[x]+(x-3/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 - c_2 e^{-2x}(2x+1)}{4\sqrt{x}}$$

2.667 problem 682

2.667.1 Maple step by step solution 6282

Internal problem ID [8157]
Internal file name [`OUTPUT/7090_Sunday_June_05_2022_05_29_09_PM_42092271/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 682.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1+x)\,y''+x^2y'-2y=0}$$

Writing the ode as

$$\begin{align*} x^2(1+x)\,y''+x^2y'-2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A&=x^2(1+x)\\ B&=x^2 \tag{3}\\ C&=-2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r&=\frac{s}{t} \tag{5}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 8x + 8 \\ t &= 4\left(x^2 + x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} - \frac{1}{4(1+x)^2} + \frac{2}{x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2+8x+8}{4(x^2+x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 8x + 8}{4\left(x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x + 2} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{2x + 2} - \frac{1}{x} \\
&= -\frac{x + 2}{2x\,(1 + x)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+2} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{2\,(1+x)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2x+2} - \frac{1}{x}\right)^2 - \left(\frac{-x^2+8x+8}{4\,(x^2+x)^2}\right)\right) = 0$$

$$\frac{-2+a_0}{x\,(1+x)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(\frac{1}{2x+2} - \frac{1}{x}\right) dx} \\
&= (x+2)\, \mathrm{e}^{\frac{\ln(1+x)}{2} - \ln(x)} \\
&= \frac{(x+2)\sqrt{1+x}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2}{x^2(1+x)}\, dx} \\
&= z_1 e^{-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x+2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2(1+x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\ln(1+x) + \frac{4}{x+2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{x+2}{x}\right) + c_2 \left(\frac{x+2}{x} \left(\ln(1+x) + \frac{4}{x+2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x+2)}{x} + \frac{c_2(\ln(1+x)(x+2)+4)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x+2)}{x} + \frac{c_2(\ln(1+x)(x+2)+4)}{x}$$

Verified OK.

2.667.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y''+x^2y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y'}{1+x}+\frac{2y}{x^2(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{1+x}-\frac{2y}{x^2(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{1+x},P_3(x)=-\frac{2}{x^2(1+x)}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x^2(1+x)\,y''+x^2y'-2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-2u+1)\left(\frac{d}{du}y(u)\right)-2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_ku^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 u^{-1+r} + \left(a_1(1+r)^2 - 2a_0(r^2+1)\right) u^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - 2a_k(k^2+2kr+r^2+1) + \right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term must be 0

  $$a_1(1+r)^2 - 2a_0(r^2+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 - 2a_k(k^2+1) + a_{k-1}(k-1)^2 = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2)^2 - 2a_{k+1}\left((k+1)^2+1\right) + k^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - 4a_{k+1}}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-4k a_{k+1}-4a_{k+1}}{(k+2)^2}, a_1-2a_0=0\right]$$

- Revert the change of variables $u=1+x$

$$\left[y=\sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+2}=-\frac{k^2 a_k-2k^2 a_{k+1}-4k a_{k+1}-4a_{k+1}}{(k+2)^2}, a_1-2a_0=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(x^2*(1+x)*diff(y(x),x$2)+x^2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(x+2)}{x}+\frac{c_2(\ln(x+1)x+2\ln(x+1)+4)}{x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 30

```
DSolve[x^2*(1+x)*y''[x]+x^2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{c_1(x+2)+c_2(x+2)\log(x+1)+4c_2}{x}$$

2.668 problem 683

2.668.1 Maple step by step solution 6291

Internal problem ID [8158]
Internal file name [OUTPUT/7091_Sunday_June_05_2022_05_29_12_PM_29630090/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 683.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

[[_2nd_order, _with_linear_symmetries]]

$$\boxed{x^2 y'' + x(x^2+6)\, y' + 6y = 0}$$

Writing the ode as

$$x^2 y'' + (x^3+6x)\, y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^3+6x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 14}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 14 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{7}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1276: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{7}{2x} - \frac{49}{4x^3} + \frac{343}{4x^5} - \frac{12005}{16x^7} + \frac{117649}{16x^9} - \frac{2470629}{32x^{11}} + \frac{27176919}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+14}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4}+\frac{7}{2}\right) + (0) \\ &= \frac{x^2}{4}+\frac{7}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{7}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{7}{2}\right) - (0) \\ &= \frac{7}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= 3 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{7}{2}}{\frac{1}{2}} - 1\right) &= -4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4}+\frac{7}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	3	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^{+} \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (6x + 2a_2) + 2\left(\frac{x}{2}\right)(3x^2 + 2xa_2 + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{7}{2}\right)\right) = 0 \\ -a_2 x^2 + (-2a_1 + 6)\, x - 3a_0 + 2a_2 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 3, a_2 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 3x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^3 + 3x\right) \mathrm{e}^{\int \frac{x}{2} dx} \\ &= \left(x^3 + 3x\right) \mathrm{e}^{\frac{x^2}{4}} \\ &= x\left(x^2 + 3\right) \mathrm{e}^{\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^3+6x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x^2}{4} - 3\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-\frac{x^2}{4}}}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 + 3}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+6x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2} - 6\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 \left(x^2 + 3\right)^2} dx \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x^2+3}{x^2}\right) + c_2\left(\frac{x^2+3}{x^2}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2\left(x^2+3\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2(x^2+3)^2}dx\right)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2(x^2+3)^2}dx\right)}{x^2}
$$

Verified OK.

2.668.1 Maple step by step solution

Let's solve

$x^2y'' + (x^3+6x)\,y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y}{x^2} - \frac{(x^2+6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2+6)y'}{x} + \frac{6y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^2+6}{x}, P_3(x) = \frac{6}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 6$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 6$$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + x(x^2+6)\, y' + 6y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

  - ○ Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3+r)\,(2+r)\, x^r + a_1(4+r)\,(3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+3)\,(k+r+2) + a_{k-2}(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4+r)\,(3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+3)(k+r+2)+a_{k-2}(k-2+r)=0$$

- Shift index using $k->k+2$

$$a_{k+2}(k+5+r)(k+4+r)+a_k(k+r)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a_k(k+r)}{(k+5+r)(k+4+r)}$$

- Recursion relation for $r=-3$

$$a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}$$

- Solution for $r=-3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}, a_1=0\right]$$

- Recursion relation for $r=-2$; series terminates at $k=2$

$$a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)}$$

- Solution for $r=-2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{a_k(k-2)}{(k+3)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-3}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2}=-\frac{a_k(k-3)}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k(k-2)}{(k+3)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x^2*diff(y(x),x$2)+x*(6+x^2)*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2+3)}{x^2} + \frac{c_2(x^2+3)\left(\int \frac{e^{-\frac{x^2}{2}}}{x^2(x^2+3)^2}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 65

```
DSolve[x^2*y''[x]+x*(6+x^2)*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{2\pi}c_2x(x^2+3)\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)-12c_1x(x^2+3)+2c_2e^{-\frac{x^2}{2}}(x^2+2)}{12x^3}$$

2.669 problem 684

2.669.1 Maple step by step solution 6303

Internal problem ID [8159]
Internal file name [`OUTPUT/7092_Sunday_June_05_2022_05_29_15_PM_11531404/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 684.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(1-x)\,y' - y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2+x\right)y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1278: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 2x + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-2x + 3}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2 - 2x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2 + x}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-(1+x)\,e^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-(1+x)\,e^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x-1)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x-1)}{x}$$

Verified OK.

2.669.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2} + \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(x-1)\, y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k + 1 - m$

$$x^{m} \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{2} \cdot y''$ to series expansion

$$x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_{0}(1+r)\,(-1+r)\, x^{r} + \left(\sum_{k=1}^{\infty} \left(a_{k}(k+r+1)\,(k+r-1) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_{k}(k+r+1) - a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+2+r) - a_{k}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_{k}}{k+2+r}$$

- Recursion relation for $r = -1$

$$a_{k+1} = \frac{a_{k}}{k+1}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k-1}, a_{k+1} = \frac{a_{k}}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = \frac{a_{k}}{k+3}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+1}, a_{k+1} = \frac{a_{k}}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_{k} x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_{k} x^{k+1}\right), a_{k+1} = \frac{a_{k}}{k+1}, b_{k+1} = \frac{b_{k}}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*(1-x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+1)}{x} + \frac{c_2 e^x}{x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]+x*(1-x)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_2 e^x - c_1(x+1)}{x}$$

2.670 problem 685

2.670.1 Maple step by step solution 6313

Internal problem ID [8160]
Internal file name [OUTPUT/7093_Sunday_June_05_2022_05_29_19_PM_59417582/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 685.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x(x+3)\, y' + 4y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2 - 3x\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{2x} - \frac{5}{2x^2} + \frac{15}{2x^3} - \frac{115}{4x^4} + \frac{495}{4x^5} - \frac{2285}{4x^6} + \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+6x-1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{6x-1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{6x-1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 6. Dividing this by leading coefficient in t which is 4 gives $\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{3}{2}\right) - (0) \\ &= \frac{3}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2x} + \frac{1}{2} \\
&= \frac{1+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 6x - 1}{4x^2}\right) \right) = 0
$$

$$
\frac{1 - a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\
&= (1 + x)\, e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (1 + x)\, \sqrt{x}\, e^{\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (1+x)\, x^2 \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{-x} + (-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((1+x)\, x^2 \mathrm{e}^x\right) + c_2\left((1+x)\, x^2 \mathrm{e}^x \left(\frac{\mathrm{e}^{-x} + (-x-1)\,\mathrm{expIntegral}_1(x)}{1+x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x)\, x^2 \mathrm{e}^x - c_2\left(-1 + \mathrm{e}^x (1+x)\,\mathrm{expIntegral}_1(x)\right) x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1(1+x)\, x^2 \mathrm{e}^x - c_2\left(-1 + \mathrm{e}^x (1+x)\,\mathrm{expIntegral}_1(x)\right) x^2
$$

Verified OK.

2.670.1 Maple step by step solution

Let's solve

$x^2 y'' + (-x^2 - 3x) y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x+3)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x+3)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x+3}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -3$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' - x(x+3) y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-2+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 2$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-2)^2 - a_{k-1}(k+r-1) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}(k+r-1)^2 - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r)}{(k+r-1)^2}$$

- Recursion relation for $r = 2$

  $$a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 38

```
dsolve(x^2*diff(y(x),x$2)-x*(x+3)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^2 (x+1) + c_2 x^2 \left(-\operatorname{expIntegral}_1(x)\, x - \operatorname{expIntegral}_1(x) + e^{-x}\right) e^x$$

✓ Solution by Mathematica
Time used: 0.068 (sec). Leaf size: 34

```
DSolve[x^2*y''[x]-x*(x+3)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 (c_2 e^x (x+1) \operatorname{ExpIntegralEi}(-x) + c_1 e^x (x+1) + c_2)$$

2.671 problem 686

2.671.1 Maple step by step solution 6323

Internal problem ID [8161]
Internal file name [OUTPUT/7094_Sunday_June_05_2022_05_29_22_PM_44121196/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 686.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x^2y' - 2y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' - 2y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1282: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{4}{x^4} + \frac{16}{x^6} - \frac{80}{x^8} + \frac{448}{x^{10}} - \frac{2688}{x^{12}} + \frac{16896}{x^{14}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2+8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{1}{4} + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2+8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x} - \frac{1}{2} \\
&= -\frac{x+2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+8}{4x^2}\right)\right) &= 0 \\
\frac{-2 + a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{1}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-\frac{x}{2} - \ln(x)} \\
&= \frac{(x+2)\, \mathrm{e}^{-\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x+2}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(x-2)\,\mathrm{e}^{x}}{x+2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x+2}{x}\right) + c_2\left(\frac{x+2}{x}\left(\frac{(x-2)\,\mathrm{e}^{x}}{x+2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1(x+2)}{x} + \frac{c_2(x-2)\,\mathrm{e}^{x}}{x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1(x+2)}{x} + \frac{c_2(x-2)\,\mathrm{e}^{x}}{x}
$$

Verified OK.

2.671.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x^2y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  - Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) - a_{k-1}(k-1+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) - a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r = -1$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k-1)}{(k+1)(k-2)}$$

- Apply recursion relation for $k = 0$

$$a_1 = \frac{a_0}{2}$$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

$$y = a_0 \cdot \left(\frac{x}{2} + 1\right)$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(\frac{x}{2}+1\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = \frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x+2)}{x} + \frac{c_2 e^x (x-2)}{x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 72

```
DSolve[x^2*y''[x]-x^2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2e^{x/2}\left((c_1x+2ic_2)\cosh\left(\frac{x}{2}\right)-(ic_2x+2c_1)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}\sqrt{x}}$$

2.672 problem 687

2.672.1 Maple step by step solution 6333

Internal problem ID [8162]
Internal file name [OUTPUT/7095_Sunday_June_05_2022_05_29_26_PM_67393217/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 687.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x^2y' - (3x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' + (-3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= -3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1284: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{3}{x} - \frac{7}{x^2} + \frac{42}{x^3} - \frac{301}{x^4} + \frac{2394}{x^5} - \frac{20342}{x^6} + \frac{180852}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 12x + 8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\
&= \frac{1}{4} + \frac{12x + 8}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned}
b &= (3) - (0) \\
&= 3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{1}{2}} - 0\right) &&= 3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{1}{2}} - 0\right) &&= -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 12x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	3	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= 3 - (2) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} + \left(\frac{1}{2}\right) \\
&= \frac{2}{x} + \frac{1}{2} \\
&= \frac{4 + x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(1) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
\frac{4 - a_0}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4 + x) \, e^{\int \left(\frac{2}{x} + \frac{1}{2}\right) dx} \\
&= (4 + x) \, e^{\frac{x}{2} + 2 \ln(x)} \\
&= (4 + x) \, e^{\frac{x}{2}} x^2
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{x^2}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (4+x)\,\mathrm{e}^{x} x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(-x^3-3x^2+2x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\,x^3(4+x)}{24\,(4+x)\,x^3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)\,\mathrm{e}^{x} x^2\right) \\
&\quad + c_2\left((4+x)\,\mathrm{e}^{x} x^2 \left(\frac{(-x^3-3x^2+2x-2)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(x)\,x^3(4+x)}{24\,(4+x)\,x^3} \right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(4+x)\,\mathrm{e}^{x} x^2 + \frac{c_2\left(\mathrm{e}^{x} x^3(4+x)\,\mathrm{expIntegral}_1(x) - x^3 - 3x^2 + 2x - 2\right)}{24x} \tag{1}
$$

Verification of solutions

$$y = c_1(4+x)\,\mathrm{e}^x x^2 + \frac{c_2(\mathrm{e}^x x^3(4+x)\,\mathrm{expIntegral}_1(x) - x^3 - 3x^2 + 2x - 2)}{24x}$$

Verified OK.

2.672.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' + (-3x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(3x+2)y}{x^2} + y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' - \frac{(3x+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -1, P_3(x) = -\frac{3x+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x^2y' + (-3x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$
  - ◦ Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 (1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)\,(k+r-2) - a_{k-1} (k+2+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r+1)\,(k+r-2) - a_{k-1} (k+2+r) = 0$
- Shift index using $k- >k + 1$

  $a_{k+1} (k+2+r)\,(k-1+r) - a_k (k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k (k+r+3)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k (k+2)}{(k+1)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

$a_{k+1} = \frac{a_k(k+2)}{(k+1)(k-2)}$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}$

- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k(k+5)}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 70

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)-(3*x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 \mathrm{e}^x (x+4) - \frac{c_2(-\operatorname{expIntegral}_1(x)\, x^4 + \mathrm{e}^{-x} x^3 - 4\operatorname{expIntegral}_1(x)\, x^3 + 3x^2 \mathrm{e}^{-x} - 2\,\mathrm{e}^{-x} x + 2\,\mathrm{e}^{-x})\,\mathrm{e}^x}{24x}$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]-x^2*y'[x]-(3*x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{24} c_2 e^x (x+4) x^2 \operatorname{ExpIntegralEi}(-x) + c_1 e^x (x+4) x^2 - \frac{c_2(x^3 + 3x^2 - 2x + 2)}{24x}$$

2.673 problem 688

2.673.1 Maple step by step solution 6343

Internal problem ID [8163]
Internal file name [OUTPUT/7096_Sunday_June_05_2022_05_29_29_PM_54683234/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 688.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(5-x)\,y' + 4y = 0}$$

Writing the ode as

$$x^2y'' + \left(-x^2+5x\right)y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2+5x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1286: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{13}{2x^2} - \frac{65}{2x^3} - \frac{819}{4x^4} - \frac{5785}{4x^5} - \frac{43797}{4x^6} - \frac{347425}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-10x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-10x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 10x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{1}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2 - 10x - 1}{4x^2}\right)\right) = 0
$$

$$
\frac{(a_1 + 4)\, x + 2a_0 + a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2, a_1 = -4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 4x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 4x + 2\right) e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= \left(x^2 - 4x + 2\right) e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= \left(x^2 - 4x + 2\right) \sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+5x}{x^2}\, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{5 \ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x}{2}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2 - 4x + 2}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+5x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x - 5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(-x^2 + 4x - 2)\, \text{expIntegral}_1(-x) - e^x(-3 + x)}{4x^2 - 16x + 8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2 - 4x + 2}{x^2}\right) \\
&\quad + c_2 \left(\frac{x^2 - 4x + 2}{x^2} \left(\frac{(-x^2 + 4x - 2)\, \text{expIntegral}_1(-x) - e^x(-3 + x)}{4x^2 - 16x + 8}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1(x^2-4x+2)}{x^2} + \frac{c_2((-x^2+4x-2)\,\text{expIntegral}_1(-x) - e^x(-3+x))}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-4x+2)}{x^2} + \frac{c_2((-x^2+4x-2)\,\text{expIntegral}_1(-x) - e^x(-3+x))}{4x^2}$$

Verified OK.

2.673.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2+5x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(x-5)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-5)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x-5}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 5$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$$x^2 y'' - x(x-5)\, y' + 4y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+2)^2 - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = -2$$

- Each term in the series must be 0, giving the recursion relation

$$a_k (k+r+2)^2 - a_{k-1}(k+r-1) = 0$$

- Shift index using $k->k+1$

$$a_{k+1}(k+3+r)^2 - a_k (k+r) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k (k+r)}{(k+3+r)^2}$$

- Recursion relation for $r = -2$; series terminates at $k = 2$

$$a_{k+1} = \frac{a_k (k-2)}{(k+1)^2}$$

- Apply recursion relation for $k = 0$

$$a_1 = -2a_0$$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{2}$
- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - 2x + \frac{1}{2}x^2\right)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve(x^2*diff(y(x),x$2)+x*(5-x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 4x + 2)}{x^2} + \frac{c_2\left(\frac{x^2\,\mathrm{expIntegral}_1(-x)}{4} + \frac{x\,\mathrm{e}^x}{4} - \mathrm{expIntegral}_1\left(-x\right)x - \frac{3\,\mathrm{e}^x}{4} + \frac{\mathrm{expIntegral}_1(-x)}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 48

```
DSolve[x^2*y''[x]+x*(5-x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2(x^2-4x+2)\,\text{ExpIntegralEi}(x)+4c_1(x^2-4x+2)-c_2e^x(x-3)}{4x^2}$$

2.674 problem 689

2.674.1 Maple step by step solution 6354

Internal problem ID [8164]
Internal file name [OUTPUT/7097_Sunday_June_05_2022_05_29_33_PM_75451789/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 689.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y'' + 4x(1-x)\, y' + (2x-9)\, y = 0}$$

Writing the ode as

$$4x^2y'' + \left(-4x^2+4x\right) y' + (2x-9)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x^2+4x \\ C &= 2x-9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1288: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x + 8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x+8}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-4x+8}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x - 2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{e^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 + 4x}{4x^2} \, dx} \\
&= z_1 e^{\frac{x}{2} - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^x}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+4x}{4x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{x-\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(-e^{-x}\left(x^2+2x+2\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^x}{x^{\frac{3}{2}}}\right) + c_2\left(\frac{e^x}{x^{\frac{3}{2}}}\left(-e^{-x}\left(x^2+2x+2\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{x^{\frac{3}{2}}} - \frac{c_2\left(x^2+2x+2\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{x^{\frac{3}{2}}} - \frac{c_2\left(x^2+2x+2\right)}{x^{\frac{3}{2}}}$$

Verified OK.

2.674.1 Maple step by step solution

Let's solve

$4x^2y'' + (-4x^2 + 4x)\, y' + (2x - 9)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-1)y'}{x} - \frac{(2x-9)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x} + \frac{(2x-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x-1}{x}, P_3(x) = \frac{2x-9}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{9}{4}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4x(x-1)\, y' + (2x - 9)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(3+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+3)\,(2k+2r-3) - 2a_{k-1}(2k+2r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(3+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+r-\frac{3}{2}\right)\left(\left(k+r+\frac{3}{2}\right) a_k - a_{k-1}\right) = 0$$

- Shift index using $k->k+1$

$$4\left(k-\frac{1}{2}+r\right)\left(\left(k+\frac{5}{2}+r\right) a_{k+1} - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{2a_k}{2k+5+2r}$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+2}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{2a_k}{2k+8}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k}{2k+8}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{2a_k}{2k+2}, b_{k+1} = \frac{2b_k}{2k+8}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1-x)*diff(y(x),x)+(2*x-9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 + 2x + 2)}{x^{\frac{3}{2}}} + \frac{c_2 e^x}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 30

```
DSolve[4*x^2*y''[x]+4*x*(1-x)*y'[x]+(2*x-9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x^{3/2}}$$

2.675 problem 690

Internal problem ID [8165]
Internal file name [`OUTPUT/7098_Sunday_June_05_2022_05_29_36_PM_47019247/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 690.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__y__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2x(x+2)\,y' + 2(1+x)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(2x^2+4x\right)y' + (2x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2+4x \\ C &= 2x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+2}{x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x+2 \\ t &= x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+2}{x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1290: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{5}{8x^4} + \frac{7}{8x^5} - \frac{21}{16x^6} + \frac{33}{16x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+2}{x} \\ &= Q + \frac{R}{x} \\ &= (1) + \left(\frac{2}{x}\right) \\ &= 1 + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 2. Dividing this by leading coefficient in t which is 1 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 0\right) &= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+2}{x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + (1) \\ &= 1 + \frac{1}{x} \\ &= 1 + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(1 + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right)^2 - \left(\frac{x+2}{x}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(1+\frac{1}{x}\right) dx} \\ &= x \, \mathrm{e}^{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x^2+4x}{x^2} \, dx} \\ &= z_1 e^{-x-2\ln(x)} \\ &= z_1 \left(\frac{\mathrm{e}^{-x}}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x-4\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{2 \operatorname{expIntegral}_1(2x)\, x - \mathrm{e}^{-2x}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{2 \,\text{expIntegral}_1 (2x)\, x - e^{-2x}}{x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2(2 \,\text{expIntegral}_1 (2x)\, x - e^{-2x})}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2(2 \,\text{expIntegral}_1 (2x)\, x - e^{-2x})}{x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)+2*x*(2+x)*diff(y(x),x)+2*(1+x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + \frac{c_2(2 \,\text{expIntegral}_1 (2x)\, x - e^{-2x})}{x^2}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+2*x*(2+x)*y'[x]+2*(1+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2c_2 x \operatorname{ExpIntegralEi}(-2x) + c_1 x - c_2 e^{-2x}}{x^2}$$

2.676 problem 691

2.676.1 Maple step by step solution 6371

Internal problem ID [8166]
Internal file name [OUTPUT/7099_Sunday_June_05_2022_05_29_39_PM_1573639/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 691.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^{2}y'' - x(1-x)\,y' + (1-x)\,y = 0}$$

Writing the ode as

$$x^{2}y'' + \left(x^{2}-x\right)y' + (1-x)\,y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2} \\ B &= x^{2}-x \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 2x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 2x - 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 2x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{3}{4x^4} + \frac{5}{4x^5} - \frac{9}{4x^6} + \frac{17}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2x - 1}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2x - 1}{4x^2}\right) \\ &= \frac{1}{4} + \frac{2x - 1}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(\frac{1}{2}\right) - (0)\\ &= \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= \frac{1}{2}\\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 2x - 1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^+ - (\alpha_{c_1}^+)\\ &= \frac{1}{2} - \left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2x} + \frac{1}{2} \\ &= \frac{1 + x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x} + \frac{1}{2}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 2x - 1}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{2x} + \frac{1}{2}\right) dx} \\ &= \sqrt{x}\, e^{\frac{x}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x^2 - x}{x^2} \, dx} \\ &= z_1 e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\, e^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(-\operatorname{expIntegral}_1(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(-\operatorname{expIntegral}_1(x)))
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x - \operatorname{expIntegral}_1(x)\, c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 x - \operatorname{expIntegral}_1(x)\, c_2 x$$

Verified OK.

2.676.1 Maple step by step solution

Let's solve

$x^2 y'' + (x^2 - x)\, y' + (1 - x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{(x-1)y}{x^2} - \frac{(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x-1)y'}{x} - \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{x-1}{x}, P_3(x) = -\frac{x-1}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

$x^2y'' + x(x-1)\, y' + (1-x)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)^2 + a_{k-1}(k-2+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+r)^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 1$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r-1)^2 + a_{k-1}(k-2+r) = 0$$

- Shift index using $k- >k+1$

  $$a_{k+1}(k+r)^2 + a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k(k+r-1)}{(k+r)^2}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{a_k k}{(k+1)^2}$$

- Solution for $r = 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k k}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-x*(1-x)*diff(y(x),x)+(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + \text{expIntegral}_1(x)\, x c_2$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[x^2*y''[x]-x*(1-x)*y'[x]+(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2 \,\text{ExpIntegralEi}(-x) + c_1)$$

2.677 problem 692

2.677.1 Maple step by step solution 6378

Internal problem ID [8167]
Internal file name [OUTPUT/7100_Sunday_June_05_2022_05_29_45_PM_98933838/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 692.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4x(2x+1)\,y' + (4x-1)\,y = 0}$$

Writing the ode as

$$4x^2y'' + \left(8x^2 + 4x\right)y' + (4x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x^2 + 4x \\ C &= 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1293: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 0 - 0\\&= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&= z_1 e^{-\int \frac{1}{2}\frac{8x^2+4x}{4x^2}\,dx}\\&= z_1 e^{-x-\frac{\ln(x)}{2}}\\&= z_1\left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-2x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx\\&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-2x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-2x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 e^{-2x}}{\sqrt{x}} + \frac{c_2}{2\sqrt{x}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 e^{-2x}}{\sqrt{x}} + \frac{c_2}{2\sqrt{x}}$$

Verified OK.

2.677.1 Maple step by step solution

Let's solve

$4x^2 y'' + (8x^2 + 4x) y' + (4x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x-1)y}{4x^2} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} + \frac{(4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4x(2x+1)\,y' + (4x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-1}(2k+2r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k-\frac{1}{2}+r\right)\left(\left(k+r+\frac{1}{2}\right)a_k+2a_{k-1}\right)=0$
- Shift index using $k->k+1$

  $4\left(k+r+\frac{1}{2}\right)\left(\left(k+\frac{3}{2}+r\right)a_{k+1}+2a_k\right)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{4a_k}{2k+3+2r}$
- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+1}=-\frac{4a_k}{2k+2}$
- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}, a_{k+1}=-\frac{4a_k}{2k+2}\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{4a_k}{2k+4}$
- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{4a_k}{2k+4}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{4a_k}{2k+2}, b_{k+1}=-\frac{4b_k}{2k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*(1+2*x)*diff(y(x),x)+(4*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2 e^{-2x}}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 26

```
DSolve[4*x^2*y''[x]+4*x*(1+2*x)*y'[x]+(4*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{2c_1 e^{-2x} + c_2}{2\sqrt{x}}$$

2.678 problem 693

Internal problem ID [8168]
Internal file name [OUTPUT/7101_Sunday_June_05_2022_05_29_48_PM_74000155/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 693.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x(4+x)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + \left(x^2+4x\right)y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+4x \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4+x}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4+x \\ t &= 4x \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4+x}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1295: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{5}{x^4} + \frac{14}{x^5} - \frac{42}{x^6} + \frac{132}{x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4+x}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{1}{x}\right) \\ &= \frac{1}{4} + \frac{1}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - 0\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{\frac{1}{2}} - 0\right) &&= -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4+x}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{x} \\ &= \frac{1}{2} + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{4+x}{4x}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\
&= x \, \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2+4x}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - 2\ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+4x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x-4\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{expIntegral}_1(x)\, x - \mathrm{e}^{-x}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{\operatorname{expIntegral}_1(x)\, x - \mathrm{e}^{-x}}{x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2(\operatorname{expIntegral}_1(x)\, x - \mathrm{e}^{-x})}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2(\operatorname{expIntegral}_1(x)\, x - \mathrm{e}^{-x})}{x^2}
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x^2*diff(y(x),x$2)+x*(4+x)*diff(y(x),x)+(2+x)*y(x)=0,y(x), singsol=all)
```

$$
y(x) = \frac{c_1}{x} + \frac{c_2(-\operatorname{expIntegral}_1(x)\, x + \mathrm{e}^{-x})}{x^2}
$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+x*(4+x)*y'[x]+(2+x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2 x \,\text{ExpIntegralEi}(-x) + c_1 x - c_2 e^{-x}}{x^2}$$

2.679 problem 694

2.679.1 Maple step by step solution 6397

Internal problem ID [8169]
Internal file name [`OUTPUT/7102_Sunday_June_05_2022_05_29_51_PM_65868928/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 694.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{9}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1296: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(i) \\
&= -\frac{1}{x} - i \\
&= -\frac{1}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{2ia_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - i
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - i)\, \mathrm{e}^{\int \left(-\frac{1}{x} - i\right) dx} \\
&= (x - i)\, \mathrm{e}^{-ix - \ln(x)} \\
&= \frac{(x - i)\, \mathrm{e}^{-ix}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x-i)\,\mathrm{e}^{-ix}}{x^{\frac{3}{2}}} - \frac{c_2 (ix-1)\,\mathrm{e}^{ix}}{2x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x-i)\,e^{-ix}}{x^{\frac{3}{2}}} - \frac{c_2(ix-1)\,e^{ix}}{2x^{\frac{3}{2}}}$$

Verified OK.

2.679.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-9)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-9)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-9}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{9}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2-9)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(3+2r)\,(-3+2r)\, x^r + a_1(5+2r)\,(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+3)\,(2k+2r-3) + 4a_{\ldots}$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{3}{2}, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(5+2r)\,(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 9) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 9\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+7}$

- Recursion relation for $r = -\frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+4k-8}$
- Solution for $r = -\frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+28k+40}$
- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+28k+40}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+4k-8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+28k+40}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-9/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{ix}(x+i)}{x^{\frac{3}{2}}} + \frac{c_2 e^{-ix}(x-i)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 44

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-9/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((c_1x + c_2)\cos(x) + (c_2x - c_1)\sin(x))}{x^{3/2}}$$

2.680 problem 695

2.680.1 Maple step by step solution 6404

Internal problem ID [8170]
Internal file name [OUTPUT/7103_Sunday_June_05_2022_05_29_55_PM_24268680/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 695.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1298: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}$$

Verified OK.

2.680.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=-\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.681 problem 696

2.681.1 Maple step by step solution 6415

Internal problem ID [8171]
Internal file name [OUTPUT/7104_Sunday_June_05_2022_05_29_58_PM_68935455/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 696.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2xy'' + 5(1-2x)\,y' - 5y = 0}$$

Writing the ode as

$$2xy'' + (-10x+5)\,y' - 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -10x+5 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{100x^2 - 60x + 5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 100x^2 - 60x + 5$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{100x^2 - 60x + 5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1300: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{25}{4} - \frac{15}{4x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{5}{2} - \frac{3}{4x} - \frac{1}{20x^2} - \frac{3}{200x^3} - \frac{1}{200x^4} - \frac{9}{5000x^5} - \frac{137}{200000x^6} - \frac{543}{2000000x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{5}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{5}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{25}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{100x^2 - 60x + 5}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{25}{4}\right) + \left(\frac{-60x + 5}{16x^2}\right) \\ &= \frac{25}{4} + \frac{-60x + 5}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -60. Dividing this by leading coefficient in t which is 16 gives $-\frac{15}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{15}{4}\right) - (0) \\ &= -\frac{15}{4} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{5}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= -\frac{3}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{15}{4}}{\frac{5}{2}} - 0\right) &&= \frac{3}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{100x^2 - 60x + 5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{3}{4} - \left(-\frac{1}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4x} + (-)\left(\frac{5}{2}\right) \\
&= -\frac{1}{4x} - \frac{5}{2} \\
&= -\frac{1}{4x} - \frac{5}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4x} - \frac{5}{2}\right)(1) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x} - \frac{5}{2}\right)^2 - \left(\frac{100x^2 - 60x + 5}{16x^2}\right)\right) = 0
$$

$$
\frac{-1 + 10a_0}{2x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{10}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{10}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{10}\right) e^{\int \left(-\frac{1}{4x} - \frac{5}{2}\right) dx} \\
&= \left(x + \frac{1}{10}\right) e^{-\frac{5x}{2} - \frac{\ln(x)}{4}} \\
&= \frac{(1 + 10x)\, e^{-\frac{5x}{2}}}{10x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10x+5}{2x}\,dx} \\
&= z_1 e^{\frac{5x}{2}-\frac{5\ln(x)}{4}} \\
&= z_1 \left(\frac{e^{\frac{5x}{2}}}{x^{\frac{5}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1+10x}{10x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10x+5}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5x-\frac{5\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{100\sqrt{x}\,e^{5x}}{(1+10x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1+10x}{10x^{\frac{3}{2}}}\right) + c_2\left(\frac{1+10x}{10x^{\frac{3}{2}}}\left(\int \frac{100\sqrt{x}\,e^{5x}}{(1+10x)^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1(1+10x)}{10x^{\frac{3}{2}}} + \frac{c_2(100x+10)\left(\int \frac{\sqrt{x}\,e^{5x}}{(1+10x)^2}dx\right)}{x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+10x)}{10x^{\frac{3}{2}}} + \frac{c_2(100x+10)\left(\int \frac{\sqrt{x}\,e^{5x}}{(1+10x)^2}dx\right)}{x^{\frac{3}{2}}}$$

Verified OK.

2.681.1 Maple step by step solution

Let's solve

$2xy'' + (-10x+5)\,y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y}{2x} + \frac{5(2x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5(2x-1)y'}{2x} - \frac{5y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{5(2x-1)}{2x}, P_3(x) = -\frac{5}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{5}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-10x+5)\,y' - 5y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+5+2r) - 5a_k(2k+2r+1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+\frac{5}{2}+r\right)(k+1+r)\, a_{k+1} - 10\left(k+r+\frac{1}{2}\right) a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{5(2k+2r+1)a_k}{(2k+5+2r)(k+1+r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}\right]$
- Recursion relation for $r = -\frac{3}{2}$; series terminates at $k = 1$

$a_{k+1} = \frac{5(2k-2)a_k}{(2k+2)\left(k-\frac{1}{2}\right)}$

- Apply recursion relation for $k = 0$

  $a_1 = 10a_0$

- Terminating series solution of the ODE for $r = -\frac{3}{2}$. Use reduction of order to find the second

  $y = a_0 \cdot (1 + 10x)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot (1 + 10x), a_{k+1} = \frac{5(2k+1)a_k}{(2k+5)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(2*x*diff(y(x),x$2)+5*(1-2*x)*diff(y(x),x)-5*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(10x+1)}{x^{\frac{3}{2}}} + \frac{c_2(10x+1)\left(\int \frac{\sqrt{x}\,\mathrm{e}^{5x}}{(10x+1)^2}dx\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 40

```
DSolve[2*x*y''[x]+5*(1-2*x)*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 L_{-\frac{1}{2}}^{\frac{3}{2}}(5x) + \frac{c_1(10x+1)}{10\sqrt{5}x^{3/2}}$$

2.682 problem 697

2.682.1 Maple step by step solution 6422

Internal problem ID [8172]
Internal file name [`OUTPUT/7105_Sunday_June_05_2022_05_30_02_PM_94411291/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 697.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1302: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos\left(x\right)}{\sqrt{x}}\right) + c_2\left(\frac{\cos\left(x\right)}{\sqrt{x}}\left(\tan\left(x\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos\left(x\right)}{\sqrt{x}} + \frac{c_2 \sin\left(x\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos\left(x\right)}{\sqrt{x}} + \frac{c_2 \sin\left(x\right)}{\sqrt{x}}
$$

Verified OK.

2.682.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

  - ◦ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.683 problem 698

2.683.1 Maple step by step solution 6433

Internal problem ID [8173]
Internal file name [`OUTPUT/7106_Sunday_June_05_2022_05_30_04_PM_57477091/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 698.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (x+n)\,y' + (n+1)\,y = 0}$$

Writing the ode as

$$xy'' + (x+n)\,y' + (n+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x+n \\ C &= n+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= n^2 - 2xn + x^2 - 2n - 4x \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1304: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{\frac{1}{4}n^2 - \frac{1}{2}n}{x^2} + \frac{-\frac{n}{2} - 1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}n^2 - \frac{1}{2}n$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{n}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{n}{2} + 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} - \frac{2}{x^3} - \frac{132}{x^7} - \frac{14}{x^5} - \frac{5}{x^4} - \frac{1}{x^2} - \frac{42}{x^6} - \frac{3n^6}{2x^7} - \frac{3n^5}{2x^6} - \frac{n}{2x} - \frac{161n}{x^6} - \frac{588n}{x^7} - \frac{3n^4}{2x^5} - \frac{3n^3}{2x^4} - \frac{3n^2}{2x^3} - \frac{3n}{2x^2} - \frac{77n^5}{2x^7} - \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{(-2n-4)\,x + n^2 - 2n}{4x^2}\right) \\
&= \frac{1}{4} + \frac{(-2n-4)\,x + n^2 - 2n}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is $-2n - 4$. Dividing this by leading coefficient in t which is 4 gives $-\frac{n}{2} - 1$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{n}{2} - 1\right) - (0) \\
&= -\frac{n}{2} - 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{n}{2} - 1}{\frac{1}{2}} - 0\right) &= -\frac{n}{2} - 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{n}{2} - 1}{\frac{1}{2}} - 0\right) &= \frac{n}{2} + 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{n}{2}$	$-\frac{n}{2} + 1$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{n}{2} - 1$	$\frac{n}{2} + 1$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{n}{2} + 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{n}{2} + 1 - \left(\frac{n}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{n}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{n}{2x} - \frac{1}{2} \\
&= \frac{n - x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{n}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{n}{2x^2}\right) + \left(\frac{n}{2x} - \frac{1}{2}\right)^2 - \left(\frac{n^2 - 2xn + x^2 - 2n - 4x}{4x^2}\right)\right) = 0$$

$$\frac{n + a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -n\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - n$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x - n)\, \mathrm{e}^{\int \left(\frac{n}{2x} - \frac{1}{2}\right) dx} \\
&= (x - n)\, \mathrm{e}^{-\frac{x}{2} + \frac{n \ln(x)}{2}} \\
&= -(n - x)\, x^{\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+n}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{n\ln(x)}{2}} \\
&= z_1 \left(x^{-\frac{n}{2}} \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-n)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+n}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-n\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-n)\,\mathrm{e}^{-x}\right) + c_2\left((x-n)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1(x-n)\,\mathrm{e}^{-x} - c_2(n-x)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right) \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1(x-n)\,\mathrm{e}^{-x} - c_2(n-x)\,\mathrm{e}^{-x}\left(\int \frac{x^{-n}\mathrm{e}^{x}}{(n-x)^2}dx\right)
$$

Verified OK.

2.683.1 Maple step by step solution

Let's solve

$xy'' + (x+n)\,y' + (n+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(n+1)y}{x} - \frac{(x+n)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x+n)y'}{x} + \frac{(n+1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ∘ Define functions

  $\left[P_2(x) = \frac{x+n}{x}, P_3(x) = \frac{n+1}{x}\right]$

  ∘ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = n$

  ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ∘ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (x+n)\,y' + (n+1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ∘ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ∘ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(n-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(n+k+r) + a_k(n+k+r+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(n-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, -n+1\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(n+k+r) + a_k(n+k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(n+k+r+1)}{(k+1+r)(n+k+r)}$$

- Recursion relation for $r=0$

$$a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}\right]$$

- Recursion relation for $r=-n+1$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}$$

- Solution for $r=-n+1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-n+1}, a_{k+1} = -\frac{a_k(k+2)}{(k+2-n)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-n+1}\right), a_{k+1} = -\frac{a_k(n+k+1)}{(k+1)(n+k)}, b_{k+1} = -\frac{b_k(k+2)}{(k+2-n)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x$2)+(x+n)*diff(y(x),x)+(n+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(-x+n) + c_2 e^{-x}(-x+n)\left(\int \frac{e^{x} x^{-n}}{(-x+n)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.524 (sec). Leaf size: 48

```
DSolve[x*y''[x]+(x+n)*y'[x]+(n+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(n-x)\left(c_2\int_1^x \frac{e^{K[1]}K[1]^{-n}}{(n-K[1])^2}dK[1]+c_1\right)$$

2.684 problem 699

Internal problem ID [8174]
Internal file name [`OUTPUT/7107_Sunday_June_05_2022_05_30_08_PM_15730329/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 699.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^4y'' + xy' + y = 0}$$

Writing the ode as

$$x^4y'' + xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-10x^2 + 1}{4x^6} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -10x^2 + 1 \\ t &= 4x^6 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-10x^2 + 1}{4x^6}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1306: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^6$. There is a pole at $x = 0$ of order 6. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = \frac{1}{4x^6} - \frac{5}{2x^4}$$

There is pole in r at $x = 0$ of order 6, hence $v = 3$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^3} - \frac{5}{2x} - \frac{25x}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 3$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^3} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^3}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^4}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $-\frac{5}{2}$. Therefore

$$
\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{1}{2x^3} \\
\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= -1 \\
\alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} + 3\right) &&= 4
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-10x^2 + 1}{4x^6}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	6	$\frac{1}{2x^3}$	-1	4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x^3} - \frac{1}{x} + (-)\,(0) \\
&= \frac{1}{2x^3} - \frac{1}{x} \\
&= \frac{1}{2x^3} - \frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(\frac{1}{2x^3} - \frac{1}{x}\right)(2x + a_1) + \left(\left(-\frac{3}{2x^4} + \frac{1}{x^2}\right) + \left(\frac{1}{2x^3} - \frac{1}{x}\right)^2 - \left(\frac{-10x^2+1}{4x^6}\right)\right) = 0
$$

$$
\frac{(2a_0+2)\,x + a_1}{x^3} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int \left(\frac{1}{2x^3} - \frac{1}{x}\right) dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2} - \ln(x)} \\
&= \frac{\left(x^2 - 1\right) \mathrm{e}^{-\frac{1}{4x^2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^4} \, dx} \\
&= z_1 e^{\frac{1}{4x^2}} \\
&= z_1 \left(\mathrm{e}^{\frac{1}{4x^2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2 - 1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^4} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{1}{2x^2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{1}{2x^2}} x^2}{\left(x^2 - 1\right)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x}\right) + c_2 \left(\frac{x^2-1}{x} \left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{e^{\frac{1}{2x^2}} x^2}{(x^2-1)^2} dx\right)}{x}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(x^4*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2-1)}{x} + \frac{c_2(x^2-1)\left(\int \frac{x^2 e^{\frac{1}{2x^2}}}{(x+1)^2(x-1)^2}dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 61

```
DSolve[x^4*y''[x]+x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2\pi}c_2(x^2-1)\,\text{erfi}\left(\frac{1}{\sqrt{2}x}\right)-4c_1(x^2-1)+2c_2e^{\frac{1}{2x^2}}x}{4x}$$

2.685 problem 700

2.685.1 Maple step by step solution 6453

Internal problem ID [8175]
Internal file name [OUTPUT/7108_Sunday_June_05_2022_05_30_11_PM_82578155/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 700.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + (2x^2 + x)\, y' - 4y = 0}$$

Writing the ode as

$$x^2y'' + (2x^2 + x)\, y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x^2 + x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1307: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{1}{x} + \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{2x} + \frac{7}{4x^2} - \frac{7}{8x^3} - \frac{35}{32x^4} + \frac{133}{64x^5} + \frac{63}{128x^6} - \frac{1239}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 4x + 15}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{4x + 15}{4x^2}\right) \\ &= 1 + \frac{4x + 15}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 4 gives 1. Now b can be found.

$$\begin{aligned} b &= (1) - (0) \\ &= 1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 + 4x + 15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{3}{2}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)(1) \\ &= -\frac{3}{2x} - 1 \\ &= -\frac{3}{2x} - 1 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x} - 1\right)(1) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x} - 1\right)^2 - \left(\frac{4x^2 + 4x + 15}{4x^2}\right)\right) = 0$$

$$\frac{-3 + 2a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{3}{2}\right) \mathrm{e}^{\int \left(-\frac{3}{2x} - 1\right) dx} \\
&= \left(x + \frac{3}{2}\right) \mathrm{e}^{-x - \frac{3\ln(x)}{2}} \\
&= \frac{(3 + 2x)\,\mathrm{e}^{-x}}{2x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2 + x}{x^2} \, dx} \\
&= z_1 e^{-x - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}(2x^2-4x+3)}{4x+6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}\right) + c_2\left(\frac{(3+2x)\,\mathrm{e}^{-2x}}{2x^2}\left(\frac{\mathrm{e}^{2x}(2x^2-4x+3)}{4x+6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3+2x)\,\mathrm{e}^{-2x}}{2x^2} + \frac{c_2(2x^2-4x+3)}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3+2x)\,\mathrm{e}^{-2x}}{2x^2} + \frac{c_2(2x^2-4x+3)}{4x^2}$$

Verified OK.

2.685.1 Maple step by step solution

Let's solve

$x^2y'' + (2x^2 + x)\, y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{x^2} - \frac{(2x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} - \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = -\frac{4}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -4$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(2x + 1)\, y' - 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-2) + 2a_{k-1}(k+r-1) = 0$

- Shift index using $k- >k+1$

  $a_{k+1}(k+3+r)\,(k+r-1) + 2a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k(k+r)}{(k+3+r)(k+r-1)}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+1} = -\frac{2a_k(k-2)}{(k+1)(k-3)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{4a_0}{3}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

  $a_2 = \frac{2a_0}{3}$

- Terminating series solution of the ODE for $r = -2$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right)$

- Recursion relation for $r = 2$

  $a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{2a_k(k+2)}{(k+5)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{2b_k(k+2)}{(k+5)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 35

```
dsolve(x^2*diff(y(x),x$2)+(x+2*x^2)*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x^2 - 4x + 3)}{x^2} + \frac{c_2 e^{-2x}(2x + 3)}{2x^2}$$

✓ Solution by Mathematica
Time used: 0.07 (sec). Leaf size: 44

```
DSolve[x^2*y''[x]+(x+2*x^2)*y'[x]-4*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\frac{2c_1 e^{-2x}(2x + 3)}{x^2} + \frac{c_2(2x^2 - 4x + 3)}{x^2} - 2\right)$$

2.686 problem 701

2.686.1 Maple step by step solution 6462

Internal problem ID [8176]
Internal file name [OUTPUT/7109_Sunday_June_05_2022_05_30_15_PM_69338995/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 701.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(4x^3 - 14x^2 - 2x)\, y'' - (6x^2 - 7x + 1)\, y' + (6x - 1)\, y = 0}$$

Writing the ode as

$$(4x^3 - 14x^2 - 2x)\, y'' + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^3 - 14x^2 - 2x \\ B &= -6x^2 + 7x - 1 \\ C &= 6x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -12x^4 + 156x^3 + 297x^2 - 78x - 3 \\ t &= 16\left(2x^3 - 7x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\left(2x^3 - 7x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(2x^3 - 7x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ of order 2. There is a pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{9}{4x} - \frac{3}{16x^2} + \frac{3}{4\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{3}{4\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2} + \frac{\frac{9}{8} - \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} - \frac{\sqrt{57}}{4}} + \frac{\frac{9}{8} + \frac{29\sqrt{57}}{152}}{x - \frac{7}{4} + \frac{\sqrt{57}}{4}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = \frac{7}{4} + \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = \frac{7}{4} - \frac{\sqrt{57}}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\,(2x^3 - 7x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-12x^4 + 156x^3 + 297x^2 - 78x - 3}{16\,(2x^3 - 7x^2 - x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$\frac{7}{4} + \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\frac{7}{4} - \frac{\sqrt{57}}{4}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^- + \alpha_{c_3}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^-}{x-c_3} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} + (-)(0) \\
&= \frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} \\
&= \frac{-6x^2 + 7x - 1}{8x^3 - 28x^2 - 4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} - \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x - \frac{7}{4} + \frac{\sqrt{57}}{4}\right)} \right)(1) + \left(\left(-\frac{1}{4x^2} + \frac{1}{2\left(x - \frac{7}{4} - \frac{\sqrt{57}}{4}\right)^2} + \frac{1}{2\left(x - \frac{7}{4} + \right.}\right.\right.$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(\frac{1}{4x} - \frac{1}{2\left(x-\frac{7}{4}-\frac{\sqrt{57}}{4}\right)} - \frac{1}{2\left(x-\frac{7}{4}+\frac{\sqrt{57}}{4}\right)}\right) dx} \\
&= (x-1)\, \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln\left(4x-7-\sqrt{57}\right)}{2} - \frac{\ln\left(4x-7+\sqrt{57}\right)}{2}} \\
&= \frac{(x-1)\, x^{\frac{1}{4}}}{\sqrt{4x-7-\sqrt{57}}\,\sqrt{4x-7+\sqrt{57}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6x^2+7x-1}{4x^3-14x^2-2x} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln\left(2x^2-7x-1\right)}{2}} \\
&= z_1 \left(\frac{\sqrt{2x^2-7x-1}}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x-1)\sqrt{2}}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x^2+7x-1}{4x^3-14x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2} + \ln(2x^2-7x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(32x+16)\sqrt{x}}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-1)\sqrt{2}}{4}\right) + c_2 \left(\frac{(x-1)\sqrt{2}}{4}\left(\frac{(32x+16)\sqrt{x}}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x-1)\sqrt{2}}{4} + c_2(8x+4)\sqrt{2}\,\sqrt{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x-1)\sqrt{2}}{4} + c_2(8x+4)\sqrt{2}\,\sqrt{x}
$$

Verified OK.

2.686.1 Maple step by step solution

Let's solve

$(4x^3 - 14x^2 - 2x)\, y'' + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(6x-1)y}{2x(2x^2-7x-1)} + \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(6x^2-7x+1)y'}{2x(2x^2-7x-1)} + \frac{(6x-1)y}{2x(2x^2-7x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{6x^2-7x+1}{2x(2x^2-7x-1)}, P_3(x) = \frac{6x-1}{2x(2x^2-7x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x(2x^2 - 7x - 1) + (-6x^2 + 7x - 1)\, y' + (6x - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k + 2 - m + r)\,(k + 1 - m + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1 + 2r)\, x^{-1+r} + \left(-a_1(1 + r)\,(1 + 2r) - a_0(14r^2 - 21r + 1)\right) x^r + \left(\sum_{k=1}^{\infty} (-a_{k+1}(k + 1 + r)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term must be 0

$$-a_1(1+r)(1+2r) - a_0(14r^2 - 21r + 1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-14a_k + 4a_{k-1} - 2a_{k+1})k^2 + ((-28a_k + 8a_{k-1} - 4a_{k+1})r + 21a_k - 18a_{k-1} - 3a_{k+1})k + (-14a$$

- Shift index using $k- >k+1$

$$(-14a_{k+1} + 4a_k - 2a_{k+2})(k+1)^2 + ((-28a_{k+1} + 8a_k - 4a_{k+2})r + 21a_{k+1} - 18a_k - 3a_{k+2})(k+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} + 8kra_k - 28kra_{k+1} + 4r^2a_k - 14r^2a_{k+1} - 10ka_k - 7ka_{k+1} - 10ra_k - 7ra_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 4kr + 2r^2 + 7k + 7r + 6}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 6ka_k - 21ka_{k+1} + 2a_k - a_{k+1}}{2k^2 + 9k + 10}, -3a_1 + 6a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4k^2a_k - 14k^2a_{k+1} - 10ka_k - 7ka_{k+1} + 6a_k + 6a_{k+1}}{2k^2 + 7k + 6}, -a_1 - a_0 = 0, b_k\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 20

```
dsolve((4*x^3-14*x^2-2*x)*diff(y(x),x$2)-(6*x^2-7*x+1)*diff(y(x),x)+(6*x-1)*y(x)=0,y(x), sin
```

$$y(x) = c_1(x-1) + c_2\sqrt{x}\,(2x+1)$$

✓ Solution by Mathematica
Time used: 6.075 (sec). Leaf size: 26

```
DSolve[(4*x^3-14*x^2-2*x)*y''[x]-(6*x^2-7*x+1)*y'[x]+(6*x-1)*y[x]==0,y[x],x,IncludeSingularS
```

$$y(x) \to c_1(x-1) - 2c_2\sqrt{x}(2x+1)$$

2.687 problem 702

2.687.1 Maple step by step solution 6473

Internal problem ID [8177]
Internal file name [`OUTPUT/7110_Sunday_June_05_2022_05_30_18_PM_64731730/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 702.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x^2y' + (x-2)\, y = 0}$$

Writing the ode as

$$x^2y'' + x^2y' + (x-2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x-2}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^2 + 2x + 2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(-\mathrm{e}^{-x}\left(x^2 + 2x + 2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} - \frac{c_2 \mathrm{e}^{-x}(x^2 + 2x + 2)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} - \frac{c_2 \mathrm{e}^{-x}(x^2 + 2x + 2)}{x}$$

Verified OK.

2.687.1 Maple step by step solution

Let's solve

$x^2y'' + x^2y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{(x-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{(x-2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 1, P_3(x) = \frac{x-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x^2y' + (x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r+1)\,(k+r-2) + a_{k-1}(k+r) = 0$$

- Shift index using $k- >k+1$

$$a_{k+1}(k+2+r)\,(k-1+r) + a_k(k+r+1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r+1)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k k}{(k+1)(k-2)}$$

- Recursion relation for $r = 2$

$$a_{k+1} = -\frac{a_k(k+3)}{(k+4)(k+1)}$$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1}=-\frac{a_k(k+3)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x}+\frac{c_2 \mathrm{e}^{-x}(x^2+2x+2)}{x}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 29

```
DSolve[x^2*y''[x]+x^2*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\rightarrow \frac{c_1-c_2 e^{-x}(x^2+2x+2)}{x}$$

2.688 problem 703

2.688.1 Maple step by step solution 6483

Internal problem ID [8178]
Internal file name [`OUTPUT/7111_Sunday_June_05_2022_05_30_22_PM_10931010/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 703.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - x^2y' + (x-2)\,y = 0}$$

Writing the ode as

$$x^2y'' - x^2y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^{-1}=\frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-4x+8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x+8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x+8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &&= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x - 2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2}{x^2} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{e^x}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\, dx \\ &= y_1\left(-e^{-x}\left(x^2+2x+2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{e^x}{x}\right) + c_2\left(\frac{e^x}{x}\left(-e^{-x}\left(x^2+2x+2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x^2-2x-2)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{x} + \frac{c_2(-x^2-2x-2)}{x}$$

Verified OK.

2.688.1 Maple step by step solution

Let's solve

$x^2y'' - x^2y' + (x-2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = y' - \frac{(x-2)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{(x-2)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

  $\left[P_2(x) = -1, P_3(x) = \frac{x-2}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$

  ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x^2y' + (x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ○ Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r+1}$$

- Shift index using $k->k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k (k+r+1)\,(k+r-2) - a_{k-1}(k+r-2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\,(a_k (k+r+1) - a_{k-1}) = 0$$

- Shift index using $k->k+1$

$$(k-1+r)\,(a_{k+1}(k+2+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+2+r}$$

- Recursion relation for $r=-1$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r=-1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k}{k+4}$$

- Solution for $r=2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)-x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 + 2x + 2)}{x} + \frac{c_2 e^x}{x}$$

✓ Solution by Mathematica
Time used: 0.03 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]-x^2*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 e^x - c_2(x^2 + 2x + 2)}{x}$$

2.689 problem 704

2.689.1 Maple step by step solution 6492

Internal problem ID [8179]
Internal file name [OUTPUT/7112_Sunday_June_05_2022_05_30_25_PM_31538583/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 704.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-4x)\,y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5yx}{16}=0}$$

Writing the ode as

$$\left(-4x^3+x^2\right)y''+\left(-\frac{1}{4}x-x^2\right)y'-\frac{5yx}{16}=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4x^3+x^2 \\ B &= -\frac{1}{4}x-x^2 \\ C &= -\frac{5x}{16} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -192x^2 - 36x + 9$$
$$t = 64\left(4x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 64(4x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \frac{1}{4}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{16x} + \frac{9}{64x^2} - \frac{3}{16\left(x - \frac{1}{4}\right)^2} - \frac{9}{16\left(x - \frac{1}{4}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{9}{64}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \left\{2, -\frac{1}{2}, \frac{9}{2}\right\} \end{aligned}$$

For the pole at $x = \frac{1}{4}$ let b be the coefficient of $\frac{1}{\left(x - \frac{1}{4}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-192x^2 - 36x + 9}{64\left(4x^2 - x\right)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{1,2,3\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, -\frac{1}{2}, \frac{9}{2}\}$
$\frac{1}{4}$	2	$\{1,2,3\}$

Order of r at ∞	E_∞
2	$\{1,2,3\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\; e_2 = 1,\; e_\infty = 3$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(3-(2+(1)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}+\frac{1}{(x-(\frac{1}{4}))}\right)\\ &= \frac{1}{x}+\frac{1}{2x-\frac{1}{2}}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x} + \frac{1}{2x - \frac{1}{2}}\right) w + \frac{576x^2 - 92x - 9}{64x^2 (4x - 1)^2} = 0$$

Solving for ω gives

$$\omega = \frac{24x - 4 + 5\sqrt{1 - 4x}}{8x (4x - 1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{24x-4+5\sqrt{1-4x}}{8x(4x-1)} dx} \\ &= \frac{(4x - 1)^{\frac{1}{4}} \sqrt{x}\, 2^{\frac{3}{4}} \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{5}{4}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-\frac{1}{4}x - x^2}{-4x^3 + x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(4x-1)}{4} + \frac{\ln(x)}{8}} \\ &= z_1 \left(\frac{x^{\frac{1}{8}}}{(4x - 1)^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{1}{8}} 2^{\frac{3}{4}} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}}}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{4}x - x^2}{-4x^3+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(4x-1)}{2}+\frac{\ln(x)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{4\sqrt{2}}{\sqrt{4x-1}\, \left(\sqrt{1-4x}+1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{1}{8}} 2^{\frac{3}{4}} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}}}{4} \right) \\
&\quad + c_2 \left(\frac{x^{\frac{1}{8}} 2^{\frac{3}{4}} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}}}{4} \left(\int \frac{4\sqrt{2}}{\sqrt{4x-1}\, \left(\sqrt{1-4x}+1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx \right) \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{8}} 2^{\frac{3}{4}} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}}}{4} + 2c_2 x^{\frac{1}{8}} 2^{\frac{1}{4}} \left(\sqrt{1-4x} + 1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}} \left(\int \frac{1}{\sqrt{4x-1}\, \left(\sqrt{1-4x}+1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx \right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{8}} 2^{\frac{3}{4}} \left(\sqrt{1-4x}+1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}}}{4} + 2c_2 x^{\frac{1}{8}} 2^{\frac{1}{4}} \left(\sqrt{1-4x} + 1\right) \left(\frac{\sqrt{1-4x}+1}{\sqrt{x}}\right)^{\frac{1}{4}} \left(\int \frac{1}{\sqrt{4x-1}\,\left(\sqrt{1-4x}+1\right)^2 \sqrt{\frac{\sqrt{1-4x}+1}{\sqrt{x}}}} dx\right)$$

Verified OK.

2.689.1 Maple step by step solution

Let's solve

$\left(-4x^3+x^2\right) y'' + \left(-\frac{1}{4}x - x^2\right) y' - \frac{5yx}{16} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y}{16x(4x-1)} - \frac{(1+4x)y'}{4x(4x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+4x)y'}{4x(4x-1)} + \frac{5y}{16x(4x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+4x}{4x(4x-1)}, P_3(x) = \frac{5}{16x(4x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{1}{4}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$$16xy''(4x-1)+(16x+4)y'+5y=0$$

- Assume series solution for y

$$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m=0..1$

$$x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

  - Convert $x^m \cdot y''$ to series expansion for $m=1..2$

$$x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-4a_0r(-5+4r)\,x^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-4a_{k+1}(k+1+r)\,(4k-1+4r)+a_k(8k+8r-1)\,(8k+8r-5\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-4r(-5+4r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{5}{4}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-16(k+1+r)\left(k-\frac{1}{4}+r\right)a_{k+1}+64\left(k+r-\frac{1}{8}\right)a_k\left(k+r-\frac{5}{8}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{(8k+8r-1)a_k(8k+8r-5)}{4(k+1+r)(4k-1+4r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{(8k-1)a_k(8k-5)}{4(k+1)(4k-1)}$$

- Solution for $r=0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(8k-1) a_k (8k-5)}{4(k+1)(4k-1)}\right]$$

- Recursion relation for $r = \frac{5}{4}$

$$a_{k+1} = \frac{(8k+9) a_k (8k+5)}{4\left(k+\frac{9}{4}\right)(4k+4)}$$

- Solution for $r = \frac{5}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{4}}, a_{k+1} = \frac{(8k+9) a_k (8k+5)}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{4}}\right), a_{k+1} = \frac{(8k-1) a_k (8k-5)}{4(k+1)(4k-1)}, b_{k+1} = \frac{(8k+9) b_k (8k+5)}{4\left(k+\frac{9}{4}\right)(4k+4)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 69

```
dsolve(x^2*(1-4*x)*diff(y(x),x$2)+((1-(5/4))*x-(6-4*(5/4))*x^2)*diff(y(x),x)+(5/4)*(1-(5/4))
```

$$y(x) = c_1 \left(\frac{x\left(1+i\sqrt{-1+4x}\right)}{i\sqrt{-1+4x}-1}\right)^{\frac{5}{8}} + c_2 \left(\frac{x\left(i\sqrt{-1+4x}-1\right)}{1+i\sqrt{-1+4x}}\right)^{\frac{5}{8}}$$

✓ Solution by Mathematica

Time used: 0.355 (sec). Leaf size: 111

```
DSolve[x^2*(1-4*x)*y''[x]+((1-(5/4))*x-(6-4*(5/4))*x^2)*y'[x]+(5/4)*(1-(5/4))*x*y[x]==0,y[x]
```

$$y(x) \to \frac{\sqrt[8]{x}\sqrt[4]{4x-1}\left(5c_1\left(\sqrt{4x-1}-i\right)^{5/4}+ic_2\left(\sqrt{4x-1}+i\right)^{5/4}\right)}{5\sqrt[4]{1-4x}\sqrt[8]{\sqrt{4x-1}-i}\sqrt[8]{\sqrt{4x-1}+i}}$$

2.690 problem 705

2.690.1 Maple step by step solution 6503

Internal problem ID [8180]
Internal file name [`OUTPUT/7113_Sunday_June_05_2022_05_30_29_PM_53334239/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 705.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + (x^2+x)\,y' + (-9+x)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^2+x)\,y' + (-9+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2+x \\ C &= -9+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 35}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 2x + 35 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 35}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{35}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{35}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{17}{2x^2} + \frac{17}{2x^3} - \frac{255}{4x^4} - \frac{833}{4x^5} + \frac{3213}{4x^6} + \frac{21709}{4x^7} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 2x + 35}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-2x + 35}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-2x + 35}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 2x + 35}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{2}$	$-\frac{5}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{5}{2}\right) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{5}{2x} + \left(\frac{1}{2}\right) \\
&= -\frac{5}{2x} + \frac{1}{2} \\
&= \frac{x - 5}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{5}{2x} + \frac{1}{2}\right)(2x + a_1) + \left(\left(\frac{5}{2x^2}\right) + \left(-\frac{5}{2x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 2x + 35}{4x^2}\right)\right) = 0
$$

$$
\frac{(-a_1 - 8)\,x - 2a_0 - 5a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 20, a_1 = -8\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 8x + 20
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 8x + 20\right) e^{\int \left(-\frac{5}{2x} + \frac{1}{2}\right) dx} \\
&= \left(x^2 - 8x + 20\right) e^{\frac{x}{2} - \frac{5\ln(x)}{2}} \\
&= \frac{\left(x^2 - 8x + 20\right) e^{\frac{x}{2}}}{x^{\frac{5}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^2-8x+20}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^2-8x+20}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-8x+20}{x^3}\right) + c_2 \left(\frac{x^2-8x+20}{x^3}\left(-\frac{(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^2-8x+20}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(x^2-8x+20)}{x^3} - \frac{c_2(x^3+9x^2+36x+60)\,\mathrm{e}^{-x}}{x^3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2 - 8x + 20)}{x^3} - \frac{c_2(x^3 + 9x^2 + 36x + 60)\,\mathrm{e}^{-x}}{x^3}$$

Verified OK.

2.690.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2 + x)\,y' + (-9 + x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-9+x)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{(-9+x)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{-9+x}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -9$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1 + x)\,y' + (-9 + x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ∘ Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ∘ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - ∘ Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - ∘ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(3+r)\,(-3+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3+r)\,(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-3, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)\,(k+r-3) + a_{k-1}(k+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+4+r)\,(k-2+r) + a_k(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+1)}{(k+4+r)(k-2+r)}$
- Recursion relation for $r = -3$; series terminates at $k = 2$

  $a_{k+1} = -\frac{a_k(k-2)}{(k+1)(k-5)}$
- Apply recursion relation for $k = 0$

$a_1 = -\frac{2a_0}{5}$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{8}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{20}$

- Terminating series solution of the ODE for $r = -3$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right)$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k(k+4)}{(k+7)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{2}{5}x + \frac{1}{20}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+7)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(x^2*diff(y(x),x$2)+(x+x^2)*diff(y(x),x)+(x-9)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2 - 8x + 20)}{x^3} + \frac{c_2 e^{-x}(x^3 + 9x^2 + 36x + 60)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]+(x+x^2)*y'[x]+(x-9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1((x-8)x + 20) - c_2 e^{-x}(x^3 + 9x^2 + 36x + 60)}{x^3}$$

2.691 problem 706

2.691.1 Maple step by step solution 6514

Internal problem ID [8181]
Internal file name [OUTPUT/7114_Sunday_June_05_2022_05_30_32_PM_23478193/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 706.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + x(1+x)\,y' + (3x-1)\,y = 0}$$

Writing the ode as

$$x^2y'' + (x^2+x)\,y' + (3x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x^2 + x \\ C &= 3x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10x + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 10x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{5}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{5}{2x} - \frac{11}{2x^2} - \frac{55}{2x^3} - \frac{671}{4x^4} - \frac{4565}{4x^5} - \frac{33231}{4x^6} - \frac{253275}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2-10x+3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-10x+3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-10x+3}{4x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -10. Dividing this by leading coefficient in t which is 4 gives $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}b &= \left(-\frac{5}{2}\right) - (0) \\ &= -\frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 0\right) &&= \frac{5}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 10x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{3}{2x} - \frac{1}{2} \\
&= -\frac{-3+x}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{3}{2x^2}\right) + \left(\frac{3}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-10x+3}{4x^2}\right)\right) = 0
$$

$$
\frac{3+a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = -3 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (-3+x)\,\mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{2}\right) dx} \\
&= (-3+x)\,\mathrm{e}^{-\frac{x}{2} + \frac{3\ln(x)}{2}} \\
&= (-3+x)\,x^{\frac{3}{2}}\mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^2+x}{x^2}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (-3+x)\,x\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(-x^3+3x^2)\,\mathrm{expIntegral}_1(-x)-\mathrm{e}^{x}(x^2-2x-1)}{6x^2\,(-3+x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left((-3+x)\,x\,\mathrm{e}^{-x}\right) \\
&\quad + c_2\left((-3+x)\,x\,\mathrm{e}^{-x}\left(\frac{(-x^3+3x^2)\,\mathrm{expIntegral}_1(-x)-\mathrm{e}^{x}(x^2-2x-1)}{6x^2\,(-3+x)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(-3+x)\,x\,\mathrm{e}^{-x} + \frac{c_2(-x^2\mathrm{e}^{-x}(-3+x)\,\mathrm{expIntegral}_1(-x)-x^2+2x+1)}{6x} \tag{1}
$$

Verification of solutions

$$y = c_1(-3+x)\,x\,\mathrm{e}^{-x} + \frac{c_2(-x^2\mathrm{e}^{-x}(-3+x)\,\mathrm{expIntegral}_1\,(-x) - x^2 + 2x + 1)}{6x}$$

Verified OK.

2.691.1 Maple step by step solution

Let's solve

$x^2y'' + (x^2+x)\,y' + (3x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-1)y}{x^2} - \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{(3x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{3x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(1+x)\,y' + (3x-1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r)\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+2+r) = 0$
- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)\,(k+r) + a_k(k+r+3) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+r+3)}{(k+2+r)(k+r)}$
- Recursion relation for $r = -1$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$a_{k+1} = -\frac{a_k(k+2)}{(k+1)(k-1)}$

- Recursion relation for $r = 1$

$a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k(k+4)}{(k+3)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(x^2*diff(y(x),x$2)+x*(x+1)*diff(y(x),x)+(3*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\, \mathrm{e}^{-x}(x-3) + \frac{c_2(\mathrm{expIntegral}_1(-x)\, x^3 + \mathrm{e}^x x^2 - 3x^2\, \mathrm{expIntegral}_1(-x) - 2x\, \mathrm{e}^x - \mathrm{e}^x)\, \mathrm{e}^{-x}}{6x}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 66

```
DSolve[x^2*y''[x]+x*(x+1)*y'[x]+(3*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_2(x-3)x^2\, \mathrm{ExpIntegralEi}(x) + 6c_1x^3 - x^2(c_2e^x + 18c_1) + 2c_2e^x x + c_2e^x)}{6x}$$

2.692 problem 707

2.692.1 Maple step by step solution 6524

Internal problem ID [8182]
Internal file name [`OUTPUT/7115_Sunday_June_05_2022_05_30_36_PM_5880073/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 707.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - (x^2 + 4x)\, y' + 4y = 0}$$

Writing the ode as

$$x^2y'' + (-x^2 - 4x)\, y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x^2 - 4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 8x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 8x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 8x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1321: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{2}{x^2} + \frac{8}{x^3} - \frac{36}{x^4} + \frac{176}{x^5} - \frac{912}{x^6} + \frac{4928}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+8x+8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{8x+8}{4x^2}\right) \\
&= \frac{1}{4} + \frac{8x+8}{4x^2}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$
\begin{aligned}
b &= (2) - (0) \\
&= 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &= 2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &= -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 8x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty}\\ &= \frac{2}{x} + \left(\frac{1}{2}\right)\\ &= \frac{2}{x} + \frac{1}{2}\\ &= \frac{4+x}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{2}{x} + \frac{1}{2}\right)(0) + \left(\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 + 8x + 8}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\,dx}\\ &= \mathrm{e}^{\int \left(\frac{2}{x} + \frac{1}{2}\right)dx}\\ &= \mathrm{e}^{\frac{x}{2}} x^2\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-4x}{x^2}\,dx}\\ &= z_1 e^{\frac{x}{2} + 2\ln(x)}\\ &= z_1\left(\mathrm{e}^{\frac{x}{2}} x^2\right)\end{aligned}$$

Which simplifies to

$$y_1 = x^4 e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(-x^2+x-2)\, e^{-x} + \text{expIntegral}_1(x)\, x^3}{6x^3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^4 e^x) + c_2 \left(x^4 e^x \left(\frac{(-x^2+x-2)\, e^{-x} + \text{expIntegral}_1(x)\, x^3}{6x^3}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x x^4 + \frac{c_2 x (\text{expIntegral}_1(x)\, x^3 e^x - x^2 + x - 2)}{6} \tag{1}$$

Verification of solutions

$$y = c_1 e^x x^4 + \frac{c_2 x (\text{expIntegral}_1(x)\, x^3 e^x - x^2 + x - 2)}{6}$$

Verified OK.

2.692.1 Maple step by step solution

Let's solve

$x^2y'' + (-x^2 - 4x)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2} + \frac{(4+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(4+x)y'}{x} + \frac{4y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{4+x}{x}, P_3(x) = \frac{4}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 4$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - x(4 + x)\, y' + 4y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)\,(-4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r-1)\,(k+r-4) - a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{1, 4\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\,(a_k(k+r-4) - a_{k-1}) = 0$

- Shift index using $k- >k+1$

$(k+r)\,(a_{k+1}(k-3+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k-3+r}$

- Recursion relation for $r = 1$

$a_{k+1} = \frac{a_k}{k-2}$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 2$

$a_{k+1} = \frac{a_k}{k-2}$

- Recursion relation for $r = 4$

$a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 4$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+1} = \frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(x^2*diff(y(x),x$2)-(x^2+4*x)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x x^4 - \frac{c_2 x\, e^x \left(-\operatorname{expIntegral}_1(x)\, x^3 + x^2 e^{-x} - e^{-x} x + 2\, e^{-x}\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]-(x^2+4*x)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^x x^4 - \frac{1}{6} c_1 x \left(e^x x^3 \operatorname{ExpIntegralEi}(-x) + x^2 - x + 2\right)$$

2.693 problem 708

Internal problem ID [8183]
Internal file name [OUTPUT/7116_Sunday_June_05_2022_05_30_40_PM_82224006/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 708.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - (3x+2)\,y' + \frac{(2x-1)\,y}{x} = 0}$$

Writing the ode as

$$2x^2y'' + (-3x-2)\,y' + \left(2 - \frac{1}{x}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= -3x-2 \\ C &= 2 - \frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5x^2 + 36x + 4}{16x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5x^2 + 36x + 4$$
$$t = 16x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5x^2 + 36x + 4}{16x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v{\geq}4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{9}{4x^3} + \frac{1}{4x^4} + \frac{5}{16x^2}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2x^2} + \frac{9}{4x} - \frac{19}{4} + \frac{171x}{8} - \frac{475x^2}{4} + \frac{11799x^3}{16} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{9}{4}$. Therefore

$$\begin{aligned} b &= \left(\frac{9}{4}\right) - (0) \\ &= \frac{9}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= \frac{13}{4} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{\frac{9}{4}}{\frac{1}{2}} + 2\right) &= -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5x^2 + 36x + 4}{16x^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5x^2 + 36x + 4}{16x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2x^2}$	$\frac{13}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{4} - \left(-\frac{5}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x^2} - \frac{5}{4x} + (-)(0) \\
&= -\frac{1}{2x^2} - \frac{5}{4x} \\
&= \frac{-5x-2}{4x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x^2} - \frac{5}{4x}\right)(1) + \left(\left(\frac{1}{x^3} + \frac{5}{4x^2}\right) + \left(-\frac{1}{2x^2} - \frac{5}{4x}\right)^2 - \left(\frac{5x^2 + 36x + 4}{16x^4}\right)\right) = 0$$

$$\frac{-2 + 5a_0}{2x^2} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{2}{5}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + \frac{2}{5}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{5}\right) e^{\int \left(-\frac{1}{2x^2} - \frac{5}{4x}\right) dx} \\
&= \left(x + \frac{2}{5}\right) e^{\frac{1}{2x} - \frac{5\ln(x)}{4}} \\
&= \frac{(5x + 2)\, e^{\frac{1}{2x}}}{5x^{\frac{5}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-2}{2x^2} \, dx} \\
&= z_1 e^{-\frac{1}{2x} + \frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{3}{4}} e^{-\frac{1}{2x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{5x + 2}{5\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-2}{2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{1}{x}+\frac{3\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{25x^{\frac{5}{2}} \mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{5x+2}{5\sqrt{x}}\right) + c_2 \left(\frac{5x+2}{5\sqrt{x}} \left(\int \frac{25x^{\frac{5}{2}} \mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(5x+2)}{5\sqrt{x}} + \frac{c_2(25x+10)\left(\int \frac{x^{\frac{5}{2}} \mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(5x+2)}{5\sqrt{x}} + \frac{c_2(25x+10)\left(\int \frac{x^{\frac{5}{2}} \mathrm{e}^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(2*x^2*diff(y(x),x$2)-(3*x+2)*diff(y(x),x)+(2*x-1)/x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(5x+2)}{\sqrt{x}} + \frac{c_2(5x+2)\left(\int \frac{x^{\frac{5}{2}} e^{-\frac{1}{x}}}{(5x+2)^2} dx\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 70

```
DSolve[2*x^2*y''[x]-(3*x+2)*y'[x]+(2*x-1)/x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2\sqrt{\pi}c_2(5x+2)\text{erf}\left(\frac{1}{\sqrt{x}}\right)}{3\sqrt{x}} + \frac{2}{3}c_2e^{-1/x}\left(x^2-4x-2\right) + \frac{c_1(5x+2)}{5\sqrt{x}}$$

2.694 problem 709

2.694.1 Maple step by step solution 6542

Internal problem ID [8184]
Internal file name [`OUTPUT/7117_Sunday_June_05_2022_05_30_43_PM_98918614/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 709.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\, y''+\left(-2x+\frac{3}{2}\right) y'-\frac{y}{4}=0}$$

Writing the ode as

$$\left(-x^{2}+x\right) y''+\left(-2x+\frac{3}{2}\right) y'-\frac{y}{4}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^{2}+x \\ B &= -2x+\frac{3}{2} \\ C &= -\frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^2 + 4x - 3}{16\,(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4x^2 + 4x - 3$$
$$t = 16(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^2 + 4x - 3}{16\,(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1324: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} - \frac{3}{16(x-1)^2} - \frac{3}{16x^2} + \frac{1}{8x-8}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4x^2 + 4x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^2+4x-3}{16\left(x^2-x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4x} + \frac{1}{4x-4} + (-)(0) \\
&= \frac{1}{4x} + \frac{1}{4x-4} \\
&= \frac{2x-1}{4x(x-1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{4x} + \frac{1}{4x-4}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{1}{4(x-1)^2}\right) + \left(\frac{1}{4x} + \frac{1}{4x-4}\right)^2 - \left(\frac{-4x^2+4x-3}{16(x^2-x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{1}{4x-4}\right) dx} \\
&= x^{\frac{1}{4}} (x-1)^{\frac{1}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x+\frac{3}{2}}{-x^2+x} \, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4} - \frac{\ln(x-1)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}} (x-1)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+\frac{3}{2}}{-x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}-\frac{\ln(x-1)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}}\right) + c_2\left(\frac{(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}} + \frac{c_2(x(x-1))^{\frac{1}{4}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)\right)}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x(x-1))^{\frac{1}{4}}}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}} + \frac{c_2(x(x-1))^{\frac{1}{4}}\left(-\ln(2)+\ln\left(2x-1+2\sqrt{x\,(x-1)}\right)\right)}{x^{\frac{3}{4}}\,(x-1)^{\frac{1}{4}}}$$

Verified OK.

2.694.1 Maple step by step solution

Let's solve

$\left(-x^2+x\right) y''+\left(-2x+\frac{3}{2}\right) y'-\frac{y}{4}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(4x-3)y'}{2x(x-1)}-\frac{y}{4x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(4x-3)y'}{2x(x-1)}+\frac{y}{4x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4x-3}{2x(x-1)}, P_3(x)=\frac{1}{4x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0}=\frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4y''x(x-1)+(8x-6)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m=0..1$

  $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y' =\sum_{k=-1+m}^{\infty} a_{k+1-m} \left(k+1-m+r\right) x^{k+r}$

- Convert $x^{m}\cdot y''$ to series expansion for $m =1..2$

  $x^{m}\cdot y'' =\sum_{k=0}^{\infty} a_{k} \left(k+r\right) \left(k+r-1\right) x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y'' =\sum_{k=-2+m}^{\infty} a_{k+2-m} \left(k+2-m+r\right) \left(k+1-m+r\right) x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0} r \left(1+2r\right) x^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-2a_{k+1} \left(k+1+r\right) \left(2k+3+2r\right)+a_{k} \left(2k+2r+1\right)^{2}\right) x^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r \left(1+2r\right) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, -\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k} \left(2k+2r+1\right)^{2}-4\left(k+1+r\right) a_{k+1} \left(k+\frac{3}{2}+r\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_{k} \left(2k+2r+1\right)^{2}}{2\left(k+1+r\right) \left(2k+3+2r\right)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_{k} \left(2k+1\right)^{2}}{2\left(k+1\right) \left(2k+3\right)}$

- Solution for $r = 0$

  $\left[y =\sum_{k=0}^{\infty} a_{k} x^{k}, a_{k+1} = \frac{a_{k} \left(2k+1\right)^{2}}{2\left(k+1\right) \left(2k+3\right)}\right]$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = \frac{2a_{k} k^{2}}{\left(k+\frac{1}{2}\right) \left(2k+2\right)}$

- Solution for $r = -\frac{1}{2}$

  $\left[y =\sum_{k=0}^{\infty} a_{k} x^{k-\frac{1}{2}}, a_{k+1} = \frac{2a_{k} k^{2}}{\left(k+\frac{1}{2}\right) \left(2k+2\right)}\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k(2k+1)^2}{2(k+1)(2k+3)}, b_{k+1}=\frac{2b_k k^2}{(k+\frac{1}{2})(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*(1-x)*diff(y(x),x$2)+(3/2-2*x)*diff(y(x),x)-1/4*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1}{\sqrt{x}}+\frac{c_2\ln\left(x-\frac{1}{2}+\sqrt{x\,(x-1)}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 51

```
DSolve[x*(1-x)*y''[x]+(3/2-2*x)*y'[x]-1/4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{c_1}{\sqrt{x}}-\frac{2c_2\sqrt{x-1}\log\left(\sqrt{x-1}-\sqrt{x}\right)}{\sqrt{-((x-1)x)}}$$

2.695 problem 710

Internal problem ID [8185]
Internal file name [`OUTPUT/7118_Sunday_June_05_2022_05_30_47_PM_53483008/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 710.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x(1-x)\,y''+xy'-y=0}$$

Writing the ode as

$$\left(-2x^{2}+2x\right)y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= -2x^{2}+2x\\ B &= x\\ C &= -1\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x + 8}{16x(x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x + 8$$
$$t = 16x(x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x + 8}{16x(x - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1326: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{2x} + \frac{5}{16(x-1)^2} - \frac{1}{2(x-1)}
$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{-3x+8}{16x(x-1)^2}
$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-3x+8}{16x\left(x-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{3}{4}-\left(\frac{3}{4}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{x} - \frac{1}{4(x-1)} + (0) \\
&= \frac{1}{x} - \frac{1}{4(x-1)} \\
&= \frac{1}{x} - \frac{1}{4x-4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{x} - \frac{1}{4(x-1)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{4(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{4(x-1)}\right)^2 - \left(\frac{-3x+8}{16x(x-1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x} - \frac{1}{4(x-1)}\right) dx} \\
&= \frac{x}{(x-1)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{-2x^2+2x} \, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{4}} \\
&= z_1 \left((x-1)^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x-1)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}}{x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}}{x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2\left(\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2\left(\arctan\left(\sqrt{x-1}\right) x - \sqrt{x-1}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(2*x*(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2\left(\arctan\left(\sqrt{x-1}\right)x - \sqrt{x-1}\right)$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 43

```
DSolve[2*x*(1-x)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[4]{2}\left(c_2x\text{arctanh}\left(\sqrt{1-x}\right) + c_1x - c_2\sqrt{1-x}\right)$$

2.696 problem 711

2.696.1 Maple step by step solution 6558

Internal problem ID [8186]
Internal file name [`OUTPUT/7119_Sunday_June_05_2022_05_30_50_PM_68658250/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 711.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{2x(1-x)\,y''+(1-11x)\,y'-10y=0}$$

Writing the ode as

$$\left(-2x^2+2x\right)y''+(1-11x)\,y'-10y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^2+2x \\ B &= 1-11x \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3x^2 + 66x - 3 \\ t &= 16\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x} + \frac{15}{4(x-1)^2} - \frac{3}{16x^2} - \frac{15}{4(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 66x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 66x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{4x} - \frac{3}{2\left(x - 1\right)} + (-)\left(0\right) \\
&= \frac{3}{4x} - \frac{3}{2\left(x - 1\right)} \\
&= -\frac{3(1 + x)}{4x\left(x - 1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{4x} - \frac{3}{2(x-1)}\right)(1) + \left(\left(-\frac{3}{4x^2} + \frac{3}{2(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{3}{2(x-1)}\right)^2 - \left(\frac{-3x^2 + 66x - 3}{16(x^2-x)^2}\right)\right) = \frac{-3 + 3a_0}{2x(x-1)} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (1+x)\, \mathrm{e}^{\int \left(\frac{3}{4x} - \frac{3}{2(x-1)}\right) dx} \\
&= (1+x)\, \mathrm{e}^{\frac{3\ln(x)}{4} - \frac{3\ln(x-1)}{2}} \\
&= \frac{(1+x)\, x^{\frac{3}{4}}}{(x-1)^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-11x}{-2x^2+2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} - \frac{5\ln(x-1)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{4}} (x-1)^{\frac{5}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(1+x)\sqrt{x}}{(x-1)^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-11x}{-2x^2+2x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}-5\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{2x^2+12x+2}{(1+x)\sqrt{x}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(1+x)\sqrt{x}}{(x-1)^4}\right) + c_2 \left(\frac{(1+x)\sqrt{x}}{(x-1)^4} \left(\frac{2x^2+12x+2}{(1+x)\sqrt{x}}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1(1+x)\sqrt{x}}{(x-1)^4} + \frac{c_2(2x^2+12x+2)}{(x-1)^4} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1(1+x)\sqrt{x}}{(x-1)^4} + \frac{c_2(2x^2+12x+2)}{(x-1)^4}$$

Verified OK.

2.696.1 Maple step by step solution

Let's solve

$(-2x^2+2x)\,y''+(1-11x)\,y'-10y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(-1+11x)y'}{2x(x-1)}-\frac{5y}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(-1+11x)y'}{2x(x-1)}+\frac{5y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{-1+11x}{2x(x-1)},P_3(x)=\frac{5}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-1+11x)\,y'+10y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

  $$x^{m}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_{0}r(-1+2r)\, x^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_{k}(2k+2r+5)\,(k+r+2)\right) x^{k-}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1}+2\left(k+r+\frac{5}{2}\right) a_{k}(k+r+2)=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=\frac{(2k+2r+5) a_{k}(k+r+2)}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$

  $$a_{k+1}=\frac{(2k+5) a_{k}(k+2)}{(k+1)(2k+1)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k}, a_{k+1}=\frac{(2k+5) a_{k}(k+2)}{(k+1)(2k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+1}=\frac{(2k+6) a_{k}\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_{k}x^{k+\frac{1}{2}}, a_{k+1}=\frac{(2k+6) a_{k}\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{(2k+5)a_k(k+2)}{(k+1)(2k+1)}, b_{k+1} = \frac{(2k+6)b_k\left(k+\frac{5}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(2*x*(1-x)*diff(y(x),x$2)+(1-11*x)*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2+6x+1)}{(x-1)^4} + \frac{c_2\sqrt{x}\,(x+1)}{(x-1)^4}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 35

```
DSolve[2*x*(1-x)*y''[x]+(1-11*x)*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1\sqrt{x}(x+1) - 2c_2(x^2+6x+1)}{(x-1)^4}$$

2.697 problem 712

2.697.1 Maple step by step solution 6567

Internal problem ID [8187]
Internal file name [`OUTPUT/7120_Sunday_June_05_2022_05_30_53_PM_21746331/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 712.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\,y''+\frac{(1-2x)\,y'}{3}+\frac{20y}{9}=0}$$

Writing the ode as

$$\left(-x^2+x\right)y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+x \\ B &= -\frac{2x}{3}+\frac{1}{3} \\ C &= \frac{20}{9} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 72x^2 - 72x - 5$$
$$t = 36\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{72x^2 - 72x - 5}{36\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{41}{18x} - \frac{5}{36(x-1)^2} - \frac{5}{36x^2} + \frac{41}{18(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{72x^2 - 72x - 5}{36(x^2 - x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{72x^2 - 72x - 5}{36\,(x^2-x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{6}$	$\frac{1}{6}$
1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned}d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= 2 - (1)\\ &= 1\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{6x} + \frac{5}{6\,(x-1)} + (0) \\
&= \frac{1}{6x} + \frac{5}{6\,(x-1)} \\
&= \frac{6x-1}{6x\,(x-1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{6x} + \frac{5}{6\,(x-1)}\right)(1) + \left(\left(-\frac{1}{6x^2} - \frac{5}{6\,(x-1)^2}\right) + \left(\frac{1}{6x} + \frac{5}{6\,(x-1)}\right)^2 - \left(\frac{72x^2-72x-5}{36\,(x^2-x)^2}\right)\right) = \frac{-1-6a_0}{3x\,(x-1)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{1}{6}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{1}{6}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{1}{6}\right) e^{\int \left(\frac{1}{6x} + \frac{5}{6(x-1)}\right) dx} \\
&= \left(x - \frac{1}{6}\right) e^{\frac{\ln(x)}{6} + \frac{5\ln(x-1)}{6}} \\
&= \left(x - \frac{1}{6}\right) x^{\frac{1}{6}} (x-1)^{\frac{5}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\, dx} \\
&= z_1 e^{-\frac{\ln(x(x-1))}{6}} \\
&= z_1 \left(\frac{1}{\left(x\left(x-1\right)\right)^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(6x-1\right) x^{\frac{1}{6}} \left(x-1\right)^{\frac{5}{6}}}{6 \left(x\left(x-1\right)\right)^{\frac{1}{6}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{2x}{3}+\frac{1}{3}}{-x^2+x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x-1))}{3}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\left(-324x+270\right) x^{\frac{2}{3}}}{\left(x-1\right)^{\frac{2}{3}} \left(30x-5\right)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(6x-1\right) x^{\frac{1}{6}} \left(x-1\right)^{\frac{5}{6}}}{6 \left(x\left(x-1\right)\right)^{\frac{1}{6}}}\right) + c_2 \left(\frac{\left(6x-1\right) x^{\frac{1}{6}} \left(x-1\right)^{\frac{5}{6}}}{6 \left(x\left(x-1\right)\right)^{\frac{1}{6}}} \left(\frac{\left(-324x+270\right) x^{\frac{2}{3}}}{\left(x-1\right)^{\frac{2}{3}} \left(30x-5\right)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (6x-1) x^{\frac{1}{6}} \left(x-1\right)^{\frac{5}{6}}}{6 \left(x\left(x-1\right)\right)^{\frac{1}{6}}} - \frac{9 c_2 (6x-5) \left(x-1\right)^{\frac{1}{6}} x^{\frac{5}{6}}}{5 \left(x\left(x-1\right)\right)^{\frac{1}{6}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(6x-1)\,x^{\frac{1}{6}}(x-1)^{\frac{5}{6}}}{6\,(x\,(x-1))^{\frac{1}{6}}} - \frac{9c_2(6x-5)\,(x-1)^{\frac{1}{6}}\,x^{\frac{5}{6}}}{5\,(x\,(x-1))^{\frac{1}{6}}}$$

Verified OK.

2.697.1 Maple step by step solution

Let's solve

$(-x^2+x)\,y''+\left(-\frac{2x}{3}+\frac{1}{3}\right)y'+\frac{20y}{9}=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(2x-1)y'}{3x(x-1)}+\frac{20y}{9x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(2x-1)y'}{3x(x-1)}-\frac{20y}{9x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x-1}{3x(x-1)}, P_3(x)=-\frac{20}{9x(x-1)}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=\frac{1}{3}$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $9y''x(x-1)+(6x-3)\,y'-20y=0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-3a_0 r(-2+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-3a_{k+1}(k+1+r)\,(3k+1+3r) + a_k(3k+3r+4)\,(3k+3r-5\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-3r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-9\left(k+\frac{1}{3}+r\right)(k+1+r)\, a_{k+1} + 9\left(k+r-\frac{5}{3}\right)\left(k+r+\frac{4}{3}\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(3k+3r-5)(3k+3r+4)a_k}{3(3k+1+3r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{2}{3}$; series terminates at $k = 1$

$a_{k+1} = \frac{(3k-3)(3k+6)a_k}{3(3k+3)\left(k+\frac{5}{3}\right)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{6a_0}{5}$

- Terminating series solution of the ODE for $r = \frac{2}{3}$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(-\frac{6x}{5} + 1\right)$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + b_0 \cdot \left(-\frac{6x}{5} + 1\right), a_{k+1} = \frac{(3k-5)(3k+4)a_k}{3(3k+1)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*(1-x)*diff(y(x),x$2)+1/3*(1-2*x)*diff(y(x),x)+20/9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(6x-5)\,x^{\frac{2}{3}} + c_2(6x-1)\,(x-1)^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 51

```
DSolve[x*(1-x)*y''[x]+1/3*(1-2*x)*y'[x]+20/9*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_2 \sqrt[3]{-((x-1)x)} Q_1^{\frac{2}{3}}(2x-1) + \frac{c_1 x^{2/3}(6x-5)}{3\,\text{Gamma}\left(\frac{4}{3}\right)}$$

2.698 problem 713

2.698.1 Maple step by step solution 6577

Internal problem ID [8188]
Internal file name [OUTPUT/7121_Sunday_June_05_2022_05_30_57_PM_7262723/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 713.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y'' + \frac{3(-x^2+2)\, y}{(1-x^2)^2} = 0}$$

Writing the ode as

$$4y'' + \frac{(-3x^2+6)\, y}{(x^2-1)^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= \frac{-3x^2+6}{(x^2-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \\ = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 - 6}{4\left(x^2 - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3x^2 - 6$$
$$t = 4\left(x^2 - 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 - 6}{4\left(x^2 - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(1+x)^2} - \frac{3}{16(x-1)^2} + \frac{9}{16(x-1)} - \frac{9}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2-6}{4(x^2-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2-6}{4\left(x^2-1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$
-1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\,(x-1)} + \frac{3}{4\,(1+x)} + (0) \\
&= \frac{3}{4\,(x-1)} + \frac{3}{4\,(1+x)} \\
&= \frac{3x}{2x^2 - 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\,(x-1)} + \frac{3}{4\,(1+x)}\right)(0) + \left(\left(-\frac{3}{4\,(x-1)^2} - \frac{3}{4\,(1+x)^2}\right) + \left(\frac{3}{4\,(x-1)} + \frac{3}{4\,(1+x)}\right)^2 - \left(\frac{3}{4(
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4(x-1)} + \frac{3}{4(1+x)}\right) dx} \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}
\end{aligned}
$$

Which simplifies to

$$y_1 = (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \int \frac{1}{(x-1)^{\frac{3}{2}} (1+x)^{\frac{3}{2}}}\, dx \\ &= (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left((x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}}\right) + c_2\left((x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} \left(-\frac{x}{\sqrt{x-1}\,\sqrt{1+x}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} - c_2 (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}} x \tag{1}$$

Verification of solutions

$$y = c_1 (x-1)^{\frac{3}{4}} (1+x)^{\frac{3}{4}} - c_2 (x-1)^{\frac{1}{4}} (1+x)^{\frac{1}{4}} x$$

Verified OK.

2.698.1 Maple step by step solution

Let's solve

$4y'' + \frac{(-3x^2+6)y}{(x^2-1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3(x^2-2)y}{4(x^2-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3(x^2-2)y}{4(x^2-1)^2} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{3(x^2-2)}{4(x^2-1)^2}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 0$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = \frac{3}{16}$

  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $4y''(x^2-1)^2 + (-3x^2+6)\,y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(4u^4 - 16u^3 + 16u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-3u^2 + 6u + 3)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$
- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$
- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$
- Shift index using $k- >k + 2 - m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)(-3+4r)\, u^r + \left(a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3)\right) u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(4k+4r -\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)(-3+4r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$
- Each term must be 0

  $a_1(3+4r)(1+4r) - 2a_0(8r^2-8r-3) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{2a_0(8r^2-8r-3)}{16r^2+16r+3}$
- Each term in the series must be 0, giving the recursion relation

  $4(4a_k + a_{k-2} - 4a_{k-1})\, k^2 + 4(2(4a_k + a_{k-2} - 4a_{k-1})\, r - 4a_k - 5a_{k-2} + 12a_{k-1})\, k + 4(4a_k + a_{k-2}$
- Shift index using $k- >k + 2$

  $4(4a_{k+2} + a_k - 4a_{k+1})\, (k+2)^2 + 4(2(4a_{k+2} + a_k - 4a_{k+1})\, r - 4a_{k+2} - 5a_k + 12a_{k+1})\, (k+2) + 4$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} + 8kra_k - 32kra_{k+1} + 4r^2a_k - 16r^2a_{k+1} - 4ka_k - 16ka_{k+1} - 4ra_k - 16ra_{k+1} - 3a_k + 6a_{k+1}}{16k^2+32kr+16r^2+48k+48r+35}$
- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{4k^2a_k - 16k^2a_{k+1} - 2ka_k - 24ka_{k+1} - \frac{15}{4}a_k + a_{k+1}}{16k^2+56k+48}$

- Solution for $r = \frac{1}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48}, a_1 = -\frac{9a_0}{8}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$$a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15a_{k+1}}{16k^2 + 72k + 80}$$

- Solution for $r = \frac{3}{4}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{3}{4}}, a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} + 2k a_k - 40k a_{k+1} - \frac{15}{4} a_k - 15a_{k+1}}{16k^2 + 72k + 80}, a_1 = -\frac{3a_0}{8}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{4k^2 a_k - 16k^2 a_{k+1} - 2k a_k - 24k a_{k+1} - \frac{15}{4} a_k + a_{k+1}}{16k^2 + 56k + 48},\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(4*diff(y(x),x$2)+3*(2-x^2)/(1-x^2)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2-1)^{\frac{1}{4}} x + c_2(x^2-1)^{\frac{3}{4}}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 51

```
DSolve[4*y''[x]+3*(2-x^2)/(1-x^2)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2-1}\left(c_2 Q_{\frac{1}{2}}^{\frac{1}{2}}(x) + \frac{\sqrt{\frac{2}{\pi}} c_1 x}{\sqrt[4]{1-x^2}}\right)$$

2.699 problem 714

2.699.1 Maple step by step solution 6588

Internal problem ID [8189]
Internal file name [OUTPUT/7122_Sunday_June_05_2022_05_31_00_PM_77242800/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 714.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{u'' - \frac{2u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' - \frac{2u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{a^2x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= a^2x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$
u = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-xa-1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-xa - \ln(x)} \\
&= \frac{(xa+1)\, e^{-xa}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{2}{x}}{1}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa+1)\,\mathrm{e}^{-xa}}{a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{-\frac{2}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{2\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a}\right) + c_2 \left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a}\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2} \tag{1}
$$

Verification of solutions

$$
u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2}
$$

Verified OK.

2.699.1 Maple step by step solution

Let's solve

$u'' - \frac{2u'}{x} - a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  u''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $-a^2 ux + u''x - 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  ◦ Convert u' to series expansion

$$u' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$u' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

  $$x \cdot u'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- Each term must be 0

  $$a_1(1+r)\,(-2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k-2+r) - a^2 a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+2+r)\,(k+r-1) - a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}$$

- Solution for $r = 0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k-1)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}$$

- Solution for $r = 3$

$$\left[u = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a^2 a_k}{(k+5)(k+2)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[u = \left(\sum_{k=0}^{\infty} b_k x^k\right) + \left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2} = \frac{a^2 b_k}{(k+2)(k-1)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+5)(k+2)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(u(x),x$2)-2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = c_1 e^{ax}(ax - 1) + \frac{c_2 e^{-ax}(ax + 1)}{a}$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 68

```
DSolve[u''[x]-2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{a\sqrt{-iax}}$$

2.700 problem 715

2.700.1 Maple step by step solution 6595

Internal problem ID [8190]
Internal file name [`OUTPUT/7123_Sunday_June_05_2022_05_31_03_PM_86431013/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 715.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{2u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (a^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{ax\,\mathrm{csgn}(a)}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(u_1)^2}\, dx \\ &= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2}\, dx \\ &= u_1\left(-\frac{\mathrm{csgn}\,(a)\,\mathrm{e}^{-2ax\,\mathrm{csgn}(a)}}{2a}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1u_1 + c_2u_2 \\
&= c_1\left(\frac{e^{ax\ \mathrm{csgn}(a)}}{x}\right) + c_2\left(\frac{e^{ax\ \mathrm{csgn}(a)}}{x}\left(-\frac{\mathrm{csgn}\left(a\right)e^{-2ax\ \mathrm{csgn}(a)}}{2a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are t

Simplifying the solution $u = \frac{c_1 e^{ax\ \mathrm{csgn}(a)}}{x} - \frac{c_2\ \mathrm{csgn}(a)e^{-ax\ \mathrm{csgn}(a)}}{2ax}$ to $u = \frac{c_1 e^{xa}}{x} - \frac{c_2 e^{-xa}}{2ax}$

Verification of solutions

$$
u = \frac{c_1 e^{xa}}{x} - \frac{c_2 e^{-xa}}{2ax}
$$

Verified OK.

2.700.1 Maple step by step solution

Let's solve

$u'' + \frac{2u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2 ux + u'' x + 2u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

  - Shift index using $k->k-1$

    $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

  - Convert u' to series expansion

    $u' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$

  - Shift index using $k->k+1$

    $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$

  - Convert $x \cdot u''$ to series expansion

    $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)(k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(1+r) x^{-1+r} + a_1 (1+r)(2+r) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k+2+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1 (1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)-a^2 a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)-a^2 a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a^2 a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=\frac{a^2 a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[u=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a^2 a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{a^2 a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[u=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a^2 a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[u=\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2}=\frac{a^2 b_k}{(k+1)(k+2)}, 0=0, c_{k+2}=\frac{a^2 c_k}{(k+2)(k+3)}, 2c_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sinh(ax)}{x} + \frac{c_2 \cosh(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 35

```
DSolve[u''[x]+2/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{2ac_1 e^{-ax} + c_2 e^{ax}}{2ax}$$

2.701 problem 716

2.701.1 Maple step by step solution 6602

Internal problem ID [8191]
Internal file name [`OUTPUT/7124_Sunday_June_05_2022_05_31_06_PM_16076600/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 716.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{2u'}{x} + a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{2u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-a^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2}{x}}{1}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1} dx}}{(u_1)^2} \, dx \\
&= u_1 \int \frac{e^{-2\ln(x)}}{(u_1)^2} \, dx \\
&= u_1 \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x} \right) + c_2 \left(\frac{\mathrm{e}^{\sqrt{-a^2}\, x}}{x} \left(\frac{\sqrt{-a^2}\, \mathrm{e}^{-2\sqrt{-a^2}\, x}}{2a^2} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\, x}}{x} + \frac{c_2 \sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2 x} \tag{1}
$$

Verification of solutions

$$
u = \frac{c_1 \mathrm{e}^{\sqrt{-a^2}\, x}}{x} + \frac{c_2 \sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2 x}
$$

Verified OK.

2.701.1 Maple step by step solution

Let's solve

$u'' + \frac{2u'}{x} + a^2 u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- □ Check to see if $x_0 = 0$ is a regular singular point

  - ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = a^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$a^2 u x + u'' x + 2u' = 0$$

- Assume series solution for u

  $$u = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot u$ to series expansion

  $$x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

  $$u' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$u' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

  $$x \cdot u'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k + r + 1)\,(k + r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(1+r)\,x^{-1+r}+a_1(1+r)\,(2+r)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+2+r)+a^2a_{k-1}\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-1,0\}$$

- Each term must be 0

$$a_1(1+r)\,(2+r)=0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)\,(k+2+r)+a^2a_{k-1}=0$$

- Shift index using $k->k+1$

$$a_{k+2}(k+2+r)\,(k+3+r)+a^2a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{a^2a_k}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{a^2a_k}{(k+1)(k+2)}$$

- Solution for $r=-1$

$$\left[u=\sum_{k=0}^{\infty}a_kx^{k-1},a_{k+2}=-\frac{a^2a_k}{(k+1)(k+2)},0=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{a^2a_k}{(k+2)(k+3)}$$

- Solution for $r=0$

$$\left[u=\sum_{k=0}^{\infty}a_kx^{k},a_{k+2}=-\frac{a^2a_k}{(k+2)(k+3)},2a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[u=\left(\sum_{k=0}^{\infty}b_kx^{k-1}\right)+\left(\sum_{k=0}^{\infty}c_kx^{k}\right),b_{k+2}=-\frac{a^2b_k}{(k+1)(k+2)},0=0,c_{k+2}=-\frac{a^2c_k}{(k+2)(k+3)},2c_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)+2/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 \sin(ax)}{x} + \frac{c_2 \cos(ax)}{x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 42

```
DSolve[u''[x]+2/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{e^{-iax}\left(2c_1 - \frac{ic_2 e^{2iax}}{a}\right)}{2x}$$

2.702 problem 717

2.702.1 Maple step by step solution 6613

Internal problem ID [8192]
Internal file name [`OUTPUT/7125_Sunday_June_05_2022_05_31_09_PM_91297312/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 717.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{u'' + \frac{4u'}{x} - a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{4u'}{x} - a^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= -a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{1}{a\,x^2} - \frac{1}{2a^3x^4} + \frac{1}{2a^5x^6} - \frac{5}{8a^7x^8} + \frac{7}{8a^9x^{10}} - \frac{21}{16a^{11}x^{12}} + \frac{33}{16a^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{2}{x^2}\right) \\ &= a^2 + \frac{2}{x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{a^2x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(a) \\
&= -\frac{1}{x} - a \\
&= \frac{-xa-1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - a\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - a\right)^2 - \left(\frac{a^2x^2+2}{x^2}\right)\right) &= 0 \\
\frac{2aa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{1}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{1}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{1}{a}\right) e^{\int \left(-\frac{1}{x} - a\right) dx} \\
&= \left(x + \frac{1}{a}\right) e^{-xa - \ln(x)} \\
&= \frac{(xa+1)\, e^{-xa}}{ax}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1u_1 + c_2u_2 \\
&= c_1\left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}\right) + c_2\left(\frac{(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3}\left(\frac{(xa-1)\,\mathrm{e}^{2xa}}{2\,(xa+1)\,a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(xa+1)\,\mathrm{e}^{-xa}}{a\,x^3} + \frac{c_2(xa-1)\,\mathrm{e}^{xa}}{2a^2x^3}$$

Verified OK.

2.702.1 Maple step by step solution

Let's solve

$u'' + \frac{4u'}{x} - a^2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4}{x}, P_3(x) = -a^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $-a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot u$ to series expansion

  $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k-1$

  $x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert u' to series expansion

  $u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$

- Convert $x \cdot u''$ to series expansion

  $x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(3+r)\, x^{-1+r} + a_1(1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, 0\}$

- Each term must be 0

  $a_1(1+r)\,(4+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)\,(k+4+r) - a^2 a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+5+r) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+2+r)(k+5+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}$

- Solution for $r = -3$

  $\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = \frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}$

- Solution for $r = 0$

  $\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^k\right), b_{k+2} = \frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = \frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)-a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1 e^{ax}(ax - 1)}{x^3} + \frac{c_2 e^{-ax}(ax + 1)}{x^3 a}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 68

```
DSolve[u''[x]+4/x*u'[x]-a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to \frac{\sqrt{\frac{2}{\pi}}((iac_2x + c_1)\sinh(ax) - (ac_1x + ic_2)\cosh(ax))}{ax^{5/2}\sqrt{-iax}}$$

2.703 problem 718

2.703.1 Maple step by step solution 6624

Internal problem ID [8193]
Internal file name [OUTPUT/7126_Sunday_June_05_2022_05_31_12_PM_75849863/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 718.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' + \frac{4u'}{x} + a^2 u = 0}$$

Writing the ode as

$$u'' + \frac{4u'}{x} + a^2 u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{4}{x} \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = u e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2x^2 + 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-a^2x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - a^2$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ia - \frac{i}{a\,x^2} - \frac{i}{2a^3x^4} - \frac{i}{2a^5x^6} - \frac{5i}{8a^7x^8} - \frac{7i}{8a^9x^{10}} - \frac{21i}{16a^{11}x^{12}} - \frac{33i}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = ia$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= ia \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{-a^2x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-a^2) + \left(\frac{2}{x^2}\right) \\ &= \frac{2}{x^2} - a^2\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned}b &= (0) - (0) \\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= ia \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{ia} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{ia} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-a^2x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	ia	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(ia) \\
&= -\frac{1}{x} - ia \\
&= -\frac{1}{x} - ia
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - ia\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - ia\right)^2 - \left(\frac{-a^2x^2+2}{x^2}\right)\right) &= 0 \\
\frac{2iaa_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{i}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{i}{a}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{i}{a}\right) e^{\int \left(-\frac{1}{x} - ia\right) dx} \\
&= \left(x - \frac{i}{a}\right) e^{-ixa - \ln(x)} \\
&= \frac{(xa - i)\, e^{-ixa}}{xa}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{4}{x}}{1}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
u_1 = \frac{(xa - i)\,\mathrm{e}^{-ixa}}{x^3 a}
$$

The second solution u_2 to the original ode is found using reduction of order

$$
u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{4}{x}}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-4\ln(x)}}{(u_1)^2}\,dx \\
&= u_1 \left(\frac{(ixa - 1)\,\mathrm{e}^{2ixa}}{2a\,(-xa + i)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{(xa - i)\,\mathrm{e}^{-ixa}}{x^3 a}\right) + c_2 \left(\frac{(xa - i)\,\mathrm{e}^{-ixa}}{x^3 a}\left(\frac{(ixa - 1)\,\mathrm{e}^{2ixa}}{2a\,(-xa + i)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
u = \frac{c_1 (xa - i)\,\mathrm{e}^{-ixa}}{x^3 a} - \frac{c_2 (ixa - 1)\,\mathrm{e}^{ixa}}{2a^2 x^3} \tag{1}
$$

Verification of solutions

$$u = \frac{c_1(xa-i)\,e^{-ixa}}{x^3a} - \frac{c_2(ixa-1)\,e^{ixa}}{2a^2x^3}$$

Verified OK.

2.703.1 Maple step by step solution

Let's solve

$u'' + \frac{4u'}{x} + a^2u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = a^2\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 4$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $a^2ux + u''x + 4u' = 0$

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot u$ to series expansion

    $x \cdot u = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k->k-1$

  $$x \cdot u = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert u' to series expansion

  $$u' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$u' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

- Convert $x \cdot u''$ to series expansion

  $$x \cdot u'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot u'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(3+r)\, x^{-1+r} + a_1 (1+r)\,(4+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-3, 0\}$$

- Each term must be 0

  $$a_1 (1+r)\,(4+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+r+1)\,(k+4+r) + a^2 a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)\,(k+5+r) + a^2 a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2+r)(k+5+r)}$$

- Recursion relation for $r=-3$

  $$a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}$$

- Solution for $r = -3$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{a^2 a_k}{(k-1)(k+2)}, -2a_1 = 0\right]$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}$$

- Solution for $r = 0$

  $$\left[u = \sum_{k=0}^{\infty} a_k x^{k}, a_{k+2} = -\frac{a^2 a_k}{(k+2)(k+5)}, 4a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[u = \left(\sum_{k=0}^{\infty} b_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} c_k x^{k}\right), b_{k+2} = -\frac{a^2 b_k}{(k-1)(k+2)}, -2b_1 = 0, c_{k+2} = -\frac{a^2 c_k}{(k+2)(k+5)}, 4c_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(u(x),x$2)+4/x*diff(u(x),x)+a^2*u(x)=0,u(x), singsol=all)
```

$$u(x) = \frac{c_1(\cos(ax)\,ax - \sin(ax))}{x^3} + \frac{c_2(\cos(ax) + \sin(ax)\,ax)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 57

```
DSolve[u''[x]+4/x*u'[x]+a^2*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to -\frac{\sqrt{\frac{2}{\pi}}((ac_1x + c_2)\cos(ax) + (ac_2x - c_1)\sin(ax))}{x^{3/2}(ax)^{3/2}}$$

2.704 problem 719

2.704.1 Maple step by step solution 6635

Internal problem ID [8194]
Internal file name [`OUTPUT/7127_Sunday_June_05_2022_05_31_16_PM_26299811/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 719.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - a^2 y - \frac{6y}{x^2} = 0}$$

Writing the ode as

$$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -a^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = a^2x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{a^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = a^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx a + \frac{3}{a\,x^2} - \frac{9}{2a^3x^4} + \frac{27}{2a^5x^6} - \frac{405}{8a^7x^8} + \frac{1701}{8a^9x^{10}} - \frac{15309}{16a^{11}x^{12}} + \frac{72171}{16a^{13}x^{14}} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = a$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= a \end{aligned} \quad (10)$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = a^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{a^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (a^2) + \left(\frac{6}{x^2}\right) \\ &= a^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= a \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{a} - 0\right) &&= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{a} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{a^2x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	a	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(a) \\
&= -\frac{2}{x} - a \\
&= \frac{-xa - 2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{x} - a\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - a\right)^2 - \left(\frac{a^2x^2 + 6}{x^2}\right)\right) = 0
$$

$$
\frac{2axa_1 + 4aa_0 - 6x - 4a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{3}{a^2}, a_1 = \frac{3}{a}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + \frac{3x}{a} + \frac{3}{a^2}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{\int \left(-\frac{2}{x} - a\right) dx} \\
&= \left(x^2 + \frac{3x}{a} + \frac{3}{a^2}\right) e^{-xa - 2\ln(x)} \\
&= \frac{(a^2x^2 + 3xa + 3)\, e^{-xa}}{a^2x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2}\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= \frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2} \int \frac{1}{\frac{(a^2x^2+3xa+3)^2\mathrm{e}^{-2xa}}{a^4x^4}}\,dx\\ &= \frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2}\left(\frac{(a^2x^2-3xa+3)\,\mathrm{e}^{2xa}}{2a\,(a^2x^2+3xa+3)}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(\frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2}\right) + c_2\left(\frac{(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2}\left(\frac{(a^2x^2-3xa+3)\,\mathrm{e}^{2xa}}{2a\,(a^2x^2+3xa+3)}\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2} + \frac{c_2\mathrm{e}^{xa}(a^2x^2-3xa+3)}{2a^3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(a^2x^2+3xa+3)\,\mathrm{e}^{-xa}}{a^2x^2} + \frac{c_2\mathrm{e}^{xa}(a^2x^2-3xa+3)}{2a^3x^2}$$

Verified OK.

2.704.1 Maple step by step solution

Let's solve

$y'' + \left(-a^2 - \frac{6}{x^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(a^2x^2+6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(a^2x^2+6)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{a^2x^2+6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + (-a^2x^2 - 6)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-3+r)x^r + a_1(3+r)(-2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-3) - a^2 a_{k-2})x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-3) - a^2 a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+4+r)(k+r-1) - a^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a^2 a_k}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

  $a_{k+2} = \frac{a^2 a_k}{(k+2)(k-3)}$

- Solution for $r = -2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=\frac{a^2 a_k}{(k+2)(k-3)}, a_1=0\right]$$

- Recursion relation for $r=3$

$$a_{k+2}=\frac{a^2 a_k}{(k+7)(k+2)}$$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{a^2 a_k}{(k+7)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} b_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} c_k x^{k+3}\right), b_{k+2}=\frac{a^2 b_k}{(k+2)(k-3)}, b_1=0, c_{k+2}=\frac{a^2 c_k}{(k+7)(k+2)}, c_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-a^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \mathrm{e}^{-ax}(a^2x^2+3ax+3)}{x^2a^2}+\frac{c_2 \mathrm{e}^{ax}(a^2x^2-3ax+3)}{3x^2}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 90

```
DSolve[y''[x]-a^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}\left((a^2c_2x^2 - 3iac_1x + 3c_2)\cosh(ax) + i(c_1(a^2x^2+3) + 3iac_2x)\sinh(ax)\right)}{a^2x^{3/2}\sqrt{-iax}}$$

2.705 problem 720

2.705.1 Maple step by step solution 6646

Internal problem ID [8195]
Internal file name [OUTPUT/7128_Sunday_June_05_2022_05_31_20_PM_56880608/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 720.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + n^2 y - \frac{6y}{x^2} = 0}$$

Writing the ode as

$$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= n^2 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-n^2x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -n^2x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-n^2x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -n^2 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx in - \frac{3i}{n\,x^2} - \frac{9i}{2n^3x^4} - \frac{27i}{2n^5x^6} - \frac{405i}{8n^7x^8} - \frac{1701i}{8n^9x^{10}} - \frac{15309i}{16n^{11}x^{12}} - \frac{72171i}{16n^{13}x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = in$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= in \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -n^2$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-n^2x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-n^2) + \left(\frac{6}{x^2}\right) \\ &= -n^2 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= in \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{in} - 0\right) &= 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{in} - 0\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-n^2x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	in	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)\,(in) \\ &= -\frac{2}{x} - in \\ &= -\frac{2}{x} - in\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - in\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - in\right)^2 - \left(\frac{-n^2x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{(2ina_1 - 6)\,x + 4ina_0 - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{n^2}, a_1 = -\frac{3i}{n}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - \frac{3ix}{n} - \frac{3}{n^2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{\int \left(-\frac{2}{x} - in\right) dx} \\ &= \left(x^2 - \frac{3ix}{n} - \frac{3}{n^2}\right) e^{-inx - 2\ln(x)} \\ &= \frac{(n^2x^2 - 3inx - 3)\, e^{-inx}}{x^2 n^2}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \int \frac{1}{\frac{(n^2x^2-3inx-3)^2\mathrm{e}^{-2inx}}{x^4n^4}}\,dx \\ &= \frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} \left(\frac{(in^2x^2 - 3xn - 3i)\,\mathrm{e}^{2inx}}{6\left(-\frac{1}{3}n^2x^2 + inx + 1\right)n} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\right) \\ &\quad + c_2\left(\frac{(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2}\left(\frac{(in^2x^2 - 3xn - 3i)\,\mathrm{e}^{2inx}}{6\left(-\frac{1}{3}n^2x^2 + inx + 1\right)n}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3xn - 3i)}{2n^3x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(n^2x^2 - 3inx - 3)\,\mathrm{e}^{-inx}}{x^2n^2} - \frac{c_2\mathrm{e}^{inx}(in^2x^2 - 3xn - 3i)}{2n^3x^2}$$

Verified OK.

2.705.1 Maple step by step solution

Let's solve

$y'' + \left(n^2 - \frac{6}{x^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(n^2x^2-6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(n^2x^2-6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{n^2x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

$x^2y'' + (n^2x^2 - 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-3+r)\, x^r + a_1(3+r)\,(-2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r-3) + n^2 a_{k-2}\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 3\}$

- Each term must be 0

  $a_1(3+r)\,(-2+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-3) + n^2 a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}(k+4+r)\,(k+r-1) + n^2 a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{n^2 a_k}{(k+4+r)(k+r-1)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0\right]$

- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{n^2 a_k}{(k+7)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = -\frac{n^2 a_k}{(k+2)(k-3)}, a_1 = 0, b_{k+2} = -\frac{n^2 b_k}{(k+7)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 67

```
dsolve(diff(y(x),x$2)+n^2*y(x)=6*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(\cos(nx)\, x^2n^2 - 3\sin(nx)\, nx - 3\cos(nx))}{x^2} + \frac{c_2(\sin(nx)\, x^2n^2 + 3\cos(nx)\, nx - 3\sin(nx))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 79

```
DSolve[y''[x]+n^2*y[x]==6*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\sqrt{x}((c_2(-n^2)\, x^2 + 3c_1nx + 3c_2)\cos(nx) + (c_1(n^2x^2 - 3) + 3c_2nx)\sin(nx))}{(nx)^{5/2}}$$

2.706 problem 721

2.706.1 Maple step by step solution 6653

Internal problem ID [8196]
Internal file name [OUTPUT/7129_Sunday_June_05_2022_05_31_24_PM_5540904/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 721.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(-x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{-x}}{\sqrt{x}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{-x}}{\sqrt{x}} + \frac{c_2 e^{x}}{2\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^{-x}}{\sqrt{x}} + \frac{c_2 e^{x}}{2\sqrt{x}}
$$

Verified OK.

2.706.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(-x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+1) y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(4x^2+1) y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{4x^2+1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (-4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) - 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) - 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) - 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = \frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-(x^2+1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x)}{\sqrt{x}} + \frac{c_2 \cosh(x)}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.031 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+x*y'[x]-(x^2+1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(c_2 e^{2x} + 2c_1)}{2\sqrt{x}}$$

2.707 problem 722

2.707.1 Maple step by step solution 6664

Internal problem ID [8197]
Internal file name [OUTPUT/7130_Sunday_June_05_2022_05_31_29_PM_83245013/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 722.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \frac{(-9a^2 + 4x^2)\, y}{4a^2} = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -\frac{9}{4} + \frac{x^2}{a^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2a^2 - x^2}{a^2 x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2a^2 - x^2$$
$$t = a^2 x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2a^2 - x^2}{a^2 x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = a^2x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{a^2} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx -\frac{33ia^{13}}{16x^{14}} - \frac{21ia^{11}}{16x^{12}} - \frac{7ia^9}{8x^{10}} - \frac{5ia^7}{8x^8} - \frac{ia^5}{2x^6} - \frac{ia^3}{2x^4} - \frac{ia}{x^2} + \frac{i}{a} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{a}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{i}{a} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -\frac{1}{a^2}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2a^2 - x^2}{a^2 x^2} \\ &= Q + \frac{R}{a^2 x^2} \\ &= \left(-\frac{1}{a^2}\right) + \left(\frac{2}{x^2}\right) \\ &= -\frac{1}{a^2} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{i}{a} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{i}{a}} - 0\right) &= 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{i}{a}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2a^2 - x^2}{a^2 x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{i}{a}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\left(\frac{i}{a}\right) \\ &= -\frac{1}{x} - \frac{i}{a} \\ &= -\frac{ix + a}{xa}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - \frac{i}{a}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{i}{a}\right)^2 - \left(\frac{2a^2 - x^2}{a^2 x^2}\right)\right) = 0$$

$$\frac{2ia_0 - 2a}{xa} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -ia\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = -ia + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= (-ia + x)\, e^{\int \left(-\frac{1}{x} - \frac{i}{a}\right) dx} \\ &= (-ia + x)\, e^{-\frac{ix}{a} - \ln(x)} \\ &= \frac{(-ia + x)\, e^{-\frac{ix}{a}}}{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia + x)}{2ix + 2a}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} \left(\frac{a\,\mathrm{e}^{\frac{2ix}{a}}(ia + x)}{2ix + 2a}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(-ia + x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix - a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(-ia+x)\,\mathrm{e}^{-\frac{ix}{a}}}{x^{\frac{3}{2}}} - \frac{c_2(ix-a)\,a\,\mathrm{e}^{\frac{ix}{a}}}{2x^{\frac{3}{2}}}$$

Verified OK.

2.707.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(-\frac{9}{4} + \frac{x^2}{a^2}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(9a^2-4x^2)y}{4a^2x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{(9a^2-4x^2)y}{4a^2x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y''a^2 + 4xy'a^2 - (9a^2-4x^2)\,y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)a^2+4\left(\frac{d}{dt}y(t)\right)a^2-\left(9a^2-4x^2\right)y(t)=0$$

- Simplify

$$-9y(t)\,a^2+4y(t)\,x^2+4a^2\left(\frac{d^2}{dt^2}y(t)\right)=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=\frac{(9a^2-4x^2)\,y(t)}{4a^2}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)-\frac{(9a^2-4x^2)\,y(t)}{4a^2}=0$$

- Characteristic polynomial of ODE

$$r^2-\frac{9a^2-4x^2}{4a^2}=0$$

- Factor the characteristic polynomial

$$\frac{4r^2a^2-9a^2+4x^2}{4a^2}=0$$

- Roots of the characteristic polynomial

$$r=\left(\frac{\sqrt{9a^2-4x^2}}{2a},-\frac{\sqrt{9a^2-4x^2}}{2a}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,t}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,t}{2a}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\mathrm{e}^{\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}+c_2\mathrm{e}^{-\frac{\sqrt{9a^2-4x^2}\,\ln(x)}{2a}}$$

- Simplify

$$y=c_1x^{\frac{\sqrt{9a^2-4x^2}}{2a}}+c_2x^{-\frac{\sqrt{9a^2-4x^2}}{2a}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(4*x^2-9*a^2)/(4*a^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{\frac{ix}{a}}(-ix+a)}{x^{\frac{3}{2}}} + \frac{c_2 e^{-\frac{ix}{a}}(ix+a)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 62

```
DSolve[x^2*y''[x]+x*y'[x]+(4*x^2-9*a^2)/(4*a^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}\left((ac_2 + c_1 x)\cos\left(\frac{x}{a}\right) + (c_2 x - ac_1)\sin\left(\frac{x}{a}\right)\right)}{x\sqrt{\frac{x}{a}}}$$

2.708 problem 723

2.708.1 Maple step by step solution 6674

Internal problem ID [8198]
Internal file name [OUTPUT/7131_Sunday_June_05_2022_05_31_33_PM_80146755/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 723.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{25}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 + 6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0
$$

$$
\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -3, a_1 = -3i\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 - 3ix - 3
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) e^{-ix - 2\ln(x)} \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix} (ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2 \mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}$$

Verified OK.

2.708.1 Maple step by step solution

Let's solve

$x^2 y'' + xy' + \left(x^2 - \frac{25}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{25}{4}$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(5+2r)\,(-5+2r)\, x^r+a_1(7+2r)\,(-3+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5)+4a_{\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(5+2r)\,(-5+2r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{5}{2}, \frac{5}{2}\right\}$

- Each term must be 0

  $a_1(7+2r)\,(-3+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r+5)\,(2k+2r-5)+4a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(2k+9+2r)\,(2k-1+2r)+4a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}$
- Solution for $r = -\frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0\right]$$
- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-25/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{ix}(x^2 + 3ix - 3)}{x^{\frac{5}{2}}} + \frac{c_2 e^{-ix}(x^2 - 3ix - 3)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 59

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-25/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{\sqrt{\frac{2}{\pi}}\left(\left(-c_2x^2+3c_1x+3c_2\right)\cos(x)+\left(c_1\left(x^2-3\right)+3c_2x\right)\sin(x)\right)}{x^{5/2}}$$

2.709 problem 724

2.709.1 Maple step by step solution 6685

Internal problem ID [8199]
Internal file name [OUTPUT/7132_Sunday_June_05_2022_05_31_37_PM_18646242/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 724.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + qy' - \frac{2y}{x^2} = 0}$$

Writing the ode as

$$y'' + qy' - \frac{2y}{x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= q \\ C &= -\frac{2}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2q^2 + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2q^2 + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2q^2 + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{q^2}{4} + \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{q}{2} + \frac{2}{q\,x^2} - \frac{4}{q^3 x^4} + \frac{16}{q^5 x^6} - \frac{80}{q^7 x^8} + \frac{448}{q^9 x^{10}} - \frac{2688}{q^{11} x^{12}} + \frac{16896}{q^{13} x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{q}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{q}{2}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{q^2}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 q^2 + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{q^2}{4}\right) + \left(\frac{2}{x^2}\right) \\ &= \frac{q^2}{4} + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{q}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{q}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{q}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2q^2 + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{q}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{q}{2}\right) \\
&= -\frac{1}{x} - \frac{q}{2} \\
&= -\frac{qx + 2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{q}{2}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{q}{2}\right)^2 - \left(\frac{x^2 q^2 + 8}{4x^2}\right)\right) &= 0 \\
\frac{q a_0 - 2}{x} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = \frac{2}{q}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + \frac{2}{q}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x + \frac{2}{q}\right) e^{\int \left(-\frac{1}{x} - \frac{q}{2}\right) dx} \\
&= \left(x + \frac{2}{q}\right) e^{-\frac{qx}{2} - \ln(x)} \\
&= \frac{(qx + 2)\, e^{-\frac{qx}{2}}}{qx}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{q}{1}\,dx} \\
&= z_1 e^{-\frac{qx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{qx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{q}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-qx}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\right) + c_2 \left(\frac{(qx+2)\,\mathrm{e}^{-qx}}{qx}\left(\frac{(qx-2)\,\mathrm{e}^{qx}}{q\,(qx+2)}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2 x} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(qx+2)\,\mathrm{e}^{-qx}}{qx} + \frac{c_2(qx-2)}{q^2x}$$

Verified OK.

2.709.1 Maple step by step solution

Let's solve

$y'' + qy' - \frac{2y}{x^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = q, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $qy'x^2 + x^2y'' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

- Shift index using $k->k-1$

  $$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+r)\,(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(k+r+1)\,(k+r-2) + q a_{k-1}(k-1+r) = 0$$

- Shift index using $k->k+1$

  $$a_{k+1}(k+2+r)\,(k-1+r) + q a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{q a_k(k+r)}{(k+2+r)(k-1+r)}$$

- Recursion relation for $r=-1$; series terminates at $k=1$

  $$a_{k+1} = -\frac{q a_k(k-1)}{(k+1)(k-2)}$$

- Apply recursion relation for $k=0$

  $$a_1 = -\frac{q a_0}{2}$$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $$y = a_0 \cdot \left(-\frac{qx}{2} + 1\right)$$

- Recursion relation for $r=2$

  $$a_{k+1} = -\frac{q a_k(k+2)}{(k+4)(k+1)}$$

- Solution for $r=2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{q a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot \left(-\frac{qx}{2}+1\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), b_{k+1} = -\frac{q b_k (k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+q*diff(y(x),x)=2*y(x)/x^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(qx-2)}{x} + \frac{c_2 e^{-qx}(qx+2)}{qx}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 80

```
DSolve[y''[x]+q*y'[x]==2*y[x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{q x^{3/2} e^{-\frac{qx}{2}}\left(2(ic_2qx+2c_1)\sinh\left(\frac{qx}{2}\right)-2(c_1qx+2ic_2)\cosh\left(\frac{qx}{2}\right)\right)}{\sqrt{\pi}(-iqx)^{5/2}}$$

2.710 problem 725

2.710.1 Maple step by step solution 6694

Internal problem ID [8200]
Internal file name [`OUTPUT/7133_Sunday_June_05_2022_05_31_41_PM_74713393/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 725.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + 3y' + 4yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 2ix \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{x}\, dx} \\ &= z_1 e^{-\frac{3 \ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i \mathrm{e}^{2i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-i x^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-i x^2}}{x^2} \left(-\frac{i \mathrm{e}^{2i x^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2}
$$

Verified OK.

2.710.1 Maple step by step solution

Let's solve

$x y'' + 3y' + 4y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + 4x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 3y' + 4yx^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k - 3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)(3+r)\, x^r + a_2(2+r)(4+r)\, x^{1+r} + a_3(3+r)(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- > k+3$

  $a_{k+4}(k+4+r)(k+6+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$

- Recursion relation for $r = -2$

  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin(x^2)}{x^2}+\frac{c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 41

```
DSolve[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.711 problem 726

2.711.1 Maple step by step solution 6703

Internal problem ID [8201]
Internal file name [`OUTPUT/7134_Sunday_June_05_2022_05_31_45_PM_725963/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 726.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x+2)\,y''+2xy'-2y=0}$$

Writing the ode as

$$\left(-x^3+2x^2\right)y''+2xy'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3+2x^2 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (x^2 - 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1357: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x} + \frac{3}{4(x-2)^2} + \frac{3}{4x^2} - \frac{3}{4(x-2)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2-2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \frac{3}{2\left(x-2\right)} + (-)\left(0\right) \\
&= -\frac{1}{2x} + \frac{3}{2\left(x-2\right)} \\
&= \frac{1+x}{x\left(x-2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} + \frac{3}{2(x-2)}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{3}{2(x-2)^2}\right) + \left(-\frac{1}{2x} + \frac{3}{2(x-2)}\right)^2 - \left(\frac{3}{(x^2-2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2x} + \frac{3}{2(x-2)}\right) dx} \\ &= \frac{(x-2)^{\frac{3}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{-x^3+2x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} + \frac{\ln(x-2)}{2}} \\ &= z_1 \left(\frac{\sqrt{x-2}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-2)^2}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{-x^3+2x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)+\ln(x-2)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{1-x}{(x-2)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x-2)^2}{x}\right) + c_2 \left(\frac{(x-2)^2}{x} \left(\frac{1-x}{(x-2)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x-2)^2}{x} + \frac{c_2 (1-x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x-2)^2}{x} + \frac{c_2 (1-x)}{x}
$$

Verified OK.

2.711.1 Maple step by step solution

Let's solve

$(-x^3 + 2x^2)\, y'' + 2xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{(x-2)x^2} + \frac{2y'}{x(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2y'}{x(x-2)} + \frac{2y}{(x-2)x^2} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2}{x(x-2)}, P_3(x) = \frac{2}{(x-2)x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

• Multiply by denominators

  $y''(x-2)\,x^2 - 2xy' + 2y = 0$

• Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0(1+r)\,(-1+r)\,x^r + \left(\sum_{k=1}^{\infty} \left(-2a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+r-1)\,(k-2+r)\right) x^{k+\right.$

• $a_0$cannot be 0 by assumption, giving the indicial equation

$-2(1+r)(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+r-1)\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k(k+r+1)\right)=0$

- Shift index using $k->k+1$

  $-2(k+r)\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}(k+2+r)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{(k+r-1)a_k}{2(k+2+r)}$

- Recursion relation for $r=-1$; series terminates at $k=2$

  $a_{k+1}=\frac{(k-2)a_k}{2(k+1)}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Apply recursion relation for $k=1$

  $a_2=-\frac{a_1}{4}$

- Express in terms of a_0

  $a_2=\frac{a_0}{4}$

- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y=a_0\cdot\left(\frac{1}{4}x^2-x+1\right)$

- Recursion relation for $r=1$

  $a_{k+1}=\frac{ka_k}{2(k+3)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+1}, a_{k+1}=\frac{ka_k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(\frac{1}{4}x^2-x+1\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+1}\right), b_{k+1}=\frac{kb_k}{2(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x^2*(2-x)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x-1)}{x} + xc_2$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 24

```
DSolve[x^2*(2-x)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1(x-2)^2 + c_2(x-1)}{x}$$

2.712 problem 727

Internal problem ID [8202]
Internal file name [`OUTPUT/7135_Sunday_June_05_2022_05_31_48_PM_25935636/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 727.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1359: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

2.713 problem 728

2.713.1 Maple step by step solution 6719

Internal problem ID [8203]
Internal file name [`OUTPUT/7136_Sunday_June_05_2022_05_31_51_PM_85863296/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 728.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - 2(1+x)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1360: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\mathrm{e}^x + \frac{c_2x^3\mathrm{e}^x}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^x + \frac{c_2x^3\mathrm{e}^x}{3}
$$

Verified OK.

2.713.1 Maple step by step solution

Let's solve

$xy'' + (-2x-2)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x - 2)\,y' + (x + 2)\,y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

  - Shift index using $k- >k + 1 - m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k + 1 - m + r)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k + 1 + r)\,(k + r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-3 + r)\, x^{-1+r} + (a_1(1 + r)\,(-2 + r) - 2a_0(-1 + r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k + 1 + r)\,(k - 2 + r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term must be 0

  $a_1(1+r)(-2+r)-2a_0(-1+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r)-2a_k k-2a_k r+2a_k+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Series not valid for $r=0$, division by 0 in the recursion relation at $k=1$

  $a_{k+2}=\frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$

- Recursion relation for $r=3$

  $a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=\frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1-4a_0=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+(x+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^3$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 23

```
DSolve[x*y''[x]-2*(x+1)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} e^x \left(c_2 x^3 + 3 c_1\right)$$

2.714 problem 729

2.714.1 Maple step by step solution 6728

Internal problem ID [8204]
Internal file name [`OUTPUT/7137_Sunday_June_05_2022_05_31_54_PM_99386052/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 729.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3xy'' - 2(3x-1)\,y' + (3x-2)\,y = 0}$$

Writing the ode as

$$3xy'' + (-6x+2)\,y' + (3x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3x \\ B &= -6x+2 \\ C &= 3x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -2$$
$$t = 9x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1362: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{9x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{3} - \left(\frac{1}{3}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{3x} + (-)\,(0) \\ &= \frac{1}{3x} \\ &= \frac{1}{3x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x}\right)(0) + \left(\left(-\frac{1}{3x^2}\right) + \left(\frac{1}{3x}\right)^2 - \left(-\frac{2}{9x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{3x} dx} \\ &= x^{\frac{1}{3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6x+2}{3x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{3}} \\ &= z_1 \left(\frac{\mathrm{e}^x}{x^{\frac{1}{3}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x+2}{3x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x - \frac{2\ln(x)}{3}}}{(y_1)^2} \, dx \\ &= y_1 \left(3x^{\frac{1}{3}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(3x^{\frac{1}{3}}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^x + 3c_2 \mathrm{e}^x x^{\frac{1}{3}} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x + 3c_2 \mathrm{e}^x x^{\frac{1}{3}}$$

Verified OK.

2.714.1 Maple step by step solution

Let's solve

$3xy'' + (-6x + 2)\, y' + (3x - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{3x} + \frac{2(3x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(3x-1)y'}{3x} + \frac{(3x-2)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(3x-1)}{3x}, P_3(x) = \frac{3x-2}{3x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{2}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $3xy''+(-6x+2)\,y'+(3x-2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..1$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+3r)\,x^{-1+r}+(a_1(1+r)\,(2+3r)-2a_0(1+3r))\,x^r+\left(\sum_{k=1}^{\infty}(a_{k+1}(k+1+r)\,(3k+2+3r$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r)=0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(1+r)(2+3r) - 2a_0(1+3r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $3(k+1+r)\left(k+\frac{2}{3}+r\right)a_{k+1} - 6a_k k - 6a_k r - 2a_k + 3a_{k-1} = 0$

- Shift index using $k- >k+1$

  $3(k+2+r)\left(k+\frac{5}{3}+r\right)a_{k+2} - 6a_{k+1}(k+1) - 6ra_{k+1} - 2a_{k+1} + 3a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{6ka_{k+1}+6ra_{k+1}-3a_k+8a_{k+1}}{(k+2+r)(3k+5+3r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(k+2)(3k+5)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(k+2)(3k+5)}, 2a_1 - 2a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{\left(k+\frac{7}{3}\right)(3k+6)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{\left(k+\frac{7}{3}\right)(3k+6)}, 4a_1 - 4a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(k+2)(3k+5)}, 2a_1 - 2a_0 = 0, b_{k+2} = \frac{6kb_{k+1}-3b_k+10b_k}{\left(k+\frac{7}{3}\right)(3k+6)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(3*x*diff(y(x),x$2)-2*(3*x-1)*diff(y(x),x)+(3*x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{x} + c_2 x^{\frac{1}{3}} \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 21

```
DSolve[3*x*y''[x]-2*(3*x-1)*y'[x]+(3*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} \left(3c_2 \sqrt[3]{x} + c_1\right)$$

2.715 problem 730

2.715.1 Maple step by step solution 6738

Internal problem ID [8205]
Internal file name [OUTPUT/7138_Sunday_June_05_2022_05_31_57_PM_17417457/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 730.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)\,y''-(x-1)\,y'+y=0}$$

Writing the ode as

$$(x^2+x)\,y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+x \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 10x - 1$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1364: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{x} + \frac{2}{(1+x)^2} - \frac{1}{4x^2} + \frac{2}{1+x}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 10x - 1}{4\left(x^2 + x\right)^2}$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x^2-10x-1}{4\left(x^2+x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=\frac{1}{2}$ then

$$\begin{aligned}d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right)\\ &= \frac{1}{2}-\left(-\frac{1}{2}\right)\\ &= 1\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{1+x} + \frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{1+x} + \frac{1}{2x} \\
&= -\frac{x-1}{2x\,(1+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{1+x} + \frac{1}{2x}\right)(1) + \left(\left(\frac{1}{(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{1+x} + \frac{1}{2x}\right)^2 - \left(\frac{-x^2-10x-1}{4\,(x^2+x)^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x\,(1+x)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, \mathrm{e}^{\int \left(-\frac{1}{1+x} + \frac{1}{2x}\right) dx} \\
&= (x-1)\, \mathrm{e}^{\frac{\ln(x)}{2} - \ln(1+x)} \\
&= \frac{(x-1)\sqrt{x}}{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{x^2+x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}+\ln(1+x)} \\
&= z_1\left(\frac{1+x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{x^2+x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) - \frac{4}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(\ln(x) - \frac{4}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-1) + c_2(\ln(x)(x-1) - 4)
$$

Verified OK.

2.715.1 Maple step by step solution

Let's solve

$(x^2+x)\,y''+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(1+x)}+\frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-1)y'}{x(1+x)}+\frac{y}{x(1+x)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x-1}{x(1+x)},P_3(x)=\frac{1}{x(1+x)}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=-2$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)\,y''+(1-x)\,y'+y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-u)\left(\frac{d^2}{du^2}y(u)\right)+(2-u)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- ○ Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2\right) u^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)(k-2+r) + a_k(k+r-1)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-1)^2}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k-1)^2}{(k+1)(k-2)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - \frac{u}{2}\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right)\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0\left(-\frac{x}{2} + \frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x*(x+1)*diff(y(x),x$2)-(x-1)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-1) + c_2(x\ln(x) - \ln(x) - 4)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 23

```
DSolve[x*(x+1)*y''[x]-(x-1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

2.716 problem 731

2.716.1 Maple step by step solution 6747

Internal problem ID [8206]
Internal file name [`OUTPUT/7139_Sunday_June_05_2022_05_32_00_PM_61950034/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 731.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2x)\,y''-2(1+x)\,y'+2y=0}$$

Writing the ode as

$$\left(x^2+2x\right)y''+(-2x-2)\,y'+2y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= -2x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1366: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2} + \frac{3}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} + (-)\,(0) \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} \\
&= \frac{x+3}{x\,(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-x-1}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}}\right) + c_2 \left(\frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}}\left(\frac{-x-1}{x^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}
$$

Verified OK.

2.716.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (-2x-2)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x+2)} + \frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{2(1+x)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$$

- □ Check to see if x_0 is a regular singular point
  - ○ Define functions

    $$\left[P_2(x) = -\frac{2(1+x)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$$
  - ○ $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $$\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = -1$$
  - ○ $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$
  - ○ $x = -2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$
- • Multiply by denominators

  $$y''x(x+2) + (-2x-2)\,y' + 2y = 0$$
- • Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $$(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$$
- • Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$
- □ Rewrite ODE with series expansions
  - ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$
  - ○ Shift index using $k- >k+1-m$

    $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$
  - ○ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$
  - ○ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2)\,a_{k+1} + a_k(k+r-2)\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2 x^2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 19

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

2.717 problem 732

2.717.1 Maple step by step solution 6756

Internal problem ID [8207]
Internal file name [`OUTPUT/7140_Sunday_June_05_2022_05_32_03_PM_12675842/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 732.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+2x)\,y''-2(1+x)\,y'+2y=0}$$

Writing the ode as

$$\begin{align*}(x^2+2x)\,y''+(-2x-2)\,y'+2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=x^2+2x\\ B&=-2x-2 \tag{3}\\ C&=2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1368: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2} + \frac{3}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{\left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} + (-)\,(0) \\
&= -\frac{1}{2\,(x+2)} + \frac{3}{2x} \\
&= \frac{x+3}{x\,(x+2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1 \left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{3}{2}} \sqrt{x(x+2)}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-x-1}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}}\right) + c_2 \left(\frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}} \left(\frac{-x-1}{x^2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2 \sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2 \sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}
$$

Verified OK.

2.717.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (-2x-2)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x+2)} + \frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2(1+x)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2(1+x)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$

- $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = -1$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

- $x = -2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (-2x-2)\,y' + 2y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_{0} r(-2+r) u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1} (k+1+r)(k+r-1) + a_{k} (k+r-1)(k+r-2)\right) u^{k+r}\right)$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left((-2k-2r-2) a_{k+1} + a_{k} (k+r-2)\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_{k} (k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_{k} (k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_{1} = -a_{0}$

- Apply recursion relation for $k = 1$

  $a_{2} = -\frac{a_{1}}{4}$

- Express in terms of a_{0}

  $a_{2} = \frac{a_{0}}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_{0} \cdot \left(1-u+\frac{1}{4} u^{2}\right)$

- Revert the change of variables $u = x+2$

  $\left[y = \frac{a_{0} x^{2}}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_{k} k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_{k} u^{k+2}, a_{k+1} = \frac{a_{k} k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2 x^2$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 19

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2 - c_2(x+1)$$

2.718 problem 733

Internal problem ID [8208]
Internal file name [`OUTPUT/7141_Sunday_June_05_2022_05_32_06_PM_65515785/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 733.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y'' - 2xy' + 2y = 0}$$

Writing the ode as

$$(x^2+1)\, y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = \left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2 x}{(x-i)^2 (x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x - c_1(x - i)^2$$

2.719 problem 734

Internal problem ID [8209]
Internal file name [`OUTPUT/7142_Sunday_June_05_2022_05_32_09_PM_16641482/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 734.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__y__method__2", "second__order__ode__non__constant__coeff__transformation__on__B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = (x^2+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1371: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 0 \\
&= 4
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}
$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\
&= \frac{x-2i}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\frac{}{(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1\left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2\, x}{(x-i)^2\,(x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2\, x}{(x-i)^2\,(x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

2.720 problem 735

2.720.1 Maple step by step solution 6777

Internal problem ID [8210]
Internal file name [OUTPUT/7143_Sunday_June_05_2022_05_32_12_PM_25436375/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 735.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2}
$$

Verified OK.

2.720.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

  Rewrite ODE with series expansions

  $$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x \, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.721 problem 736

2.721.1 Maple step by step solution 6783

Internal problem ID [8211]
Internal file name [`OUTPUT/7144_Sunday_June_05_2022_05_32_15_PM_28072288/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 736.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1 \left(\mathrm{e}^{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{x^2}\right) + c_2 \left(e^{x^2}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{x^2} + c_2 x\, e^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{x^2} + c_2 x\, e^{x^2}
$$

Verified OK.

2.721.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x \, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.722 problem 737

2.722.1 Maple step by step solution 6793

Internal problem ID [8212]
Internal file name [`OUTPUT/7145_Sunday_June_05_2022_05_32_17_PM_94292626/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 737.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x-3)\,y''-xy'+y=0}$$

Writing the ode as

$$(2x-3)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x-3 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 8x + 18}{4(2x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 8x + 18$$
$$t = 4(2x-3)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 8x + 18}{4(2x-3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{33}{64\left(x - \frac{3}{2}\right)^2} - \frac{5}{16\left(x - \frac{3}{2}\right)}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} - \frac{5}{8x} - \frac{11}{16x^2} - \frac{1}{32x^3} + \frac{245}{64x^4} + \frac{2591}{128x^5} + \frac{21117}{256x^6} + \frac{154743}{512x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 8x + 18}{16x^2 - 48x + 36} \\ &= Q + \frac{R}{16x^2 - 48x + 36} \\ &= \left(\frac{1}{16}\right) + \left(\frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36}\right) \\ &= \frac{1}{16} + \frac{-5x + \frac{63}{4}}{16x^2 - 48x + 36} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -5. Dividing this by leading coefficient in t which is 16 gives $-\frac{5}{16}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{5}{16}\right) - (0) \\ &= -\frac{5}{16} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &= -\frac{5}{8} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{5}{16}}{\frac{1}{4}} - 0\right) &= \frac{5}{8} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$\frac{3}{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{5}{8}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{5}{8}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{5}{8} - \left(-\frac{3}{8}\right) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4} \\
&= -\frac{x}{4x - 6}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4}\right)(1) + \left(\left(\frac{3}{8\left(x - \frac{3}{2}\right)^2}\right) + \left(-\frac{3}{8\left(x - \frac{3}{2}\right)} - \frac{1}{4}\right)^2 - \left(\frac{x^2 - 8x + 18}{4\left(2x - 3\right)^2}\right)\right) = 0$$

$$\frac{a_0}{2x - 3} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, \mathrm{e}^{\int \left(-\frac{3}{8\left(x-\frac{3}{2}\right)}-\frac{1}{4}\right)dx} \\
&= (x)\, \mathrm{e}^{-\frac{x}{4}-\frac{3\ln(2x-3)}{8}} \\
&= \frac{x\,\mathrm{e}^{-\frac{x}{4}}}{(2x-3)^{\frac{3}{8}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{2x-3}\, dx} \\
&= z_1 e^{\frac{x}{4}+\frac{3\ln(2x-3)}{8}} \\
&= z_1 \left((2x-3)^{\frac{3}{8}}\, \mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{2x-3}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{2}+\frac{3\ln(2x-3)}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\int \frac{\mathrm{e}^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2 x\left(\int \frac{\mathrm{e}^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x + c_2 x\left(\int \frac{\mathrm{e}^{\frac{x}{2}}(2x-3)^{\frac{3}{4}}}{x^2}dx\right)
$$

Verified OK.

2.722.1 Maple step by step solution

Let's solve

$(2x-3)\,y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x-3} + \frac{xy'}{2x-3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{2x-3} + \frac{y}{2x-3} = 0$

□ Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{x}{2x-3}, P_3(x) = \frac{1}{2x-3}\right]$

  ◦ $\left(x-\frac{3}{2}\right)\cdot P_2(x)$ is analytic at $x = \frac{3}{2}$

    $\left(\left(x-\frac{3}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{3}{2}} = -\frac{3}{4}$

  ◦ $\left(x-\frac{3}{2}\right)^2\cdot P_3(x)$ is analytic at $x = \frac{3}{2}$

$$\left(\left(x-\frac{3}{2}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=\frac{3}{2}} = 0$$

- ○ $x = \frac{3}{2}$is a regular singular point

  Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  $x_0 = \frac{3}{2}$

- Multiply by denominators

  $(2x-3)\,y''-xy'+y=0$

- Change variables using $x = u+\frac{3}{2}$ so that the regular singular point is at $u=0$

  $2u\left(\frac{d^2}{du^2}y(u)\right)+\left(-u-\frac{3}{2}\right)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ○ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-1}$

- ○ Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $\frac{a_0 r(-7+4r)u^{-1+r}}{2}+\left(\sum_{k=0}^{\infty}\left(\frac{a_{k+1}(k+1+r)(4k-3+4r)}{2}-a_k(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $\frac{r(-7+4r)}{2}=0$

- Values of r that satisfy the indicial equation

$r \in \left\{0, \frac{7}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{3}{4}+r\right)a_{k+1}-a_k(k+r-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{2a_k(k+r-1)}{(k+1+r)(4k-3+4r)}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{2a_k(k-1)}{(k+1)(4k-3)}$

- Apply recursion relation for $k=0$

  $a_1=\frac{2a_0}{3}$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot\left(1+\frac{2u}{3}\right)$

- Revert the change of variables $u=x-\frac{3}{2}$

  $\left[y=\frac{2a_0x}{3}\right]$

- Recursion relation for $r=\frac{7}{4}$

  $a_{k+1}=\frac{2a_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}$

- Solution for $r=\frac{7}{4}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{7}{4}}, a_{k+1}=\frac{2a_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$

- Revert the change of variables $u=x-\frac{3}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k\left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}, a_{k+1}=\frac{2a_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$

- Combine solutions and rename parameters

  $\left[y=\frac{2a_0x}{3}+\left(\sum_{k=0}^{\infty}b_k\left(x-\frac{3}{2}\right)^{k+\frac{7}{4}}\right), b_{k+1}=\frac{2b_k\left(k+\frac{3}{4}\right)}{\left(k+\frac{11}{4}\right)(4k+4)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve((2*x-3)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 x \left(\int \frac{(-3+2x)^{\frac{3}{4}} e^{\frac{x}{2}}}{x^2} dx \right)$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 63

```
DSolve[(2*x-3)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\ 2^{3/4}(2x-3)\left(c_2(2x-3)^{3/4}L_{-\frac{3}{4}}^{\frac{7}{4}}\left(\frac{x}{2}-\frac{3}{4}\right)+\frac{4\sqrt{2}c_1x}{2x-3}\right)$$

2.723 problem 738

2.723.1 Maple step by step solution 6804

Internal problem ID [8213]
Internal file name [OUTPUT/7146_Sunday_June_05_2022_05_32_21_PM_86127357/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 738.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' - 3y = 0}$$

Writing the ode as

$$y'' - xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} + \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1378: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + \frac{5}{2x} - \frac{25}{4x^3} + \frac{125}{4x^5} - \frac{3125}{16x^7} + \frac{21875}{16x^9} - \frac{328125}{32x^{11}} + \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+10}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} + \frac{5}{2}\right) + (0) \\ &= \frac{x^2}{4} + \frac{5}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $\frac{5}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{5}{2}\right) - (0) \\ &= \frac{5}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} + \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	2	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + \left(\frac{x}{2}\right) \\ &= \frac{x}{2} \\ &= \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(\frac{x}{2}\right)(2x + a_1) + \left(\left(\frac{1}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + \frac{5}{2}\right)\right) &= 0 \\ -a_1 x - 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\int \frac{x}{2} dx} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}} \\
&= \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(\left(x^2+1\right) \mathrm{e}^{\frac{x^2}{2}} \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\left(x^2+1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 (x^2+1)\, e^{\frac{x^2}{2}} + c_2 (x^2+1)\, e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx\right) \tag{1}$$

Verification of solutions

$$y = c_1 (x^2+1)\, e^{\frac{x^2}{2}} + c_2 (x^2+1)\, e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{(x^2+1)^2} dx\right)$$

Verified OK.

2.723.1 Maple step by step solution

Let's solve

$y'' - xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k+3))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$(k^2+3k+2)\, a_{k+2} - a_k(k+3) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k(k+3)}{k^2+3k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x^2}{2}} (x^2+1) + c_2 \mathrm{e}^{\frac{x^2}{2}} (x^2+1) \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{(x^2+1)^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 35

```
DSolve[y''[x]-x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{HermiteH}\left(-3, \frac{x}{\sqrt{2}}\right) + c_2 e^{\frac{x^2}{2}} (x^2 + 1)$$

2.724 problem 739

Internal problem ID [8214]
Internal file name [`OUTPUT/7147_Sunday_June_05_2022_05_32_24_PM_9152215/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 739.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x^2+1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 - 6}{4\left(x^2 + 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 - 6$$
$$t = 4\left(x^2 + 1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 - 6}{4\left(x^2 + 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1380: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(x-i)^2} + \frac{5}{16(x+i)^2} + \frac{7i}{16(x-i)} - \frac{7i}{16(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x^2 - 6}{4(x^2+1)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-x^2 - 6}{4\left(x^2+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
i	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$-i$	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)} \\
&= -\frac{x}{2x^2 + 2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)(1) + \left(\left(\frac{1}{4\left(x - i\right)^2} + \frac{1}{4\left(x + i\right)^2}\right) + \left(-\frac{1}{4\left(x - i\right)} - \frac{1}{4\left(x + i\right)}\right)^2 - \left(\frac{-\ \ldots}{4(\ldots}\right.\right. \ldots \frac{(x^2 + 1 \ldots}{(-x + i)^2 (\ldots}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int \left(-\frac{1}{4(x - i)} - \frac{1}{4(x + i)}\right) dx} \\
&= (x) \, \frac{1}{\left(x^2 + 1\right)^{\frac{1}{4}}} \\
&= \frac{x}{\left(x^2 + 1\right)^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2+1}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{4}} \\
&= z_1\left(\left(x^2+1\right)^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{\ln\left(x^2+1\right)}{2}}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{\operatorname{arcsinh}\left(x\right)x-\sqrt{x^2+1}}{x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{\operatorname{arcsinh}\left(x\right)x-\sqrt{x^2+1}}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2\left(\operatorname{arcsinh}\left(x\right)x-\sqrt{x^2+1}\right) \tag{1}
$$

Verification of solutions

$$y = c_1 x + c_2 \left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2+1}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((1+x^2)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(\operatorname{arcsinh}(x)\, x - \sqrt{x^2+1}\right)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 42

```
DSolve[(1+x^2)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -c_2\sqrt{x^2+1} - c_2 x \log\left(\sqrt{x^2+1} - x\right) + c_1 x$$

2.725 problem 740

2.725.1 Maple step by step solution 6820

Internal problem ID [8215]
Internal file name [`OUTPUT/7148_Sunday_June_05_2022_05_32_27_PM_46655771/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 740.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' + 2y = 0}$$

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2-1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2-1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2-1\right) + c_2 \left(x^2-1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2-1\right)^2} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.725.1 Maple step by step solution

Let's solve

$y'' - xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k->k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1) - a_k(k-2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right)a_{k+2}-a_k(k-2)=0$$

- Recursion relation; series terminates at $k=2$

$$a_{k+2}=\frac{a_k(k-2)}{k^2+3k+2}$$

- Apply recursion relation for $k=0$

$$a_2=-a_0$$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

$$y=A_2x^2+A_1x-a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x-1)^2(x+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 54

```
DSolve[y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

2.726 problem 741

2.726.1 Maple step by step solution 6829

Internal problem ID [8216]
Internal file name [`OUTPUT/7149_Sunday_June_05_2022_05_32_31_PM_47175674/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 741.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-x^2)\,y''-y'+y=0}$$

Writing the ode as

$$(1-x^2)\,y''-y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x - 3}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x - 3$$
$$t = 4(x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x - 3}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(1+x)^2} - \frac{3}{16(x-1)^2} + \frac{7}{16(x-1)} - \frac{7}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2 - 4x - 3}{4(x^2 - 1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{1, 2, 3\}$
-1	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\; e_2 = -1,\; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (1 + (-1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(1))} + \frac{-1}{(x-(-1))}\right) \\ &= \frac{1}{2x-2} - \frac{1}{2\,(1+x)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 - 6}{(1+x)^2\,(x-1)} = 0$$

And solving for p gives

$$p = x + \frac{3}{2}$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2\left(1+x\right)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{x+\frac{3}{2}} + \frac{1}{2x-2} - \frac{1}{2\left(1+x\right)}\right) w + \frac{-8x^3 - 4x^2 + 10x + 7}{4\left(x^2-1\right)^2\left(3+2x\right)} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)}\,x + 2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)} + 2x^2 + 2x + 1}{2\left(3+2x\right)\left(x-1\right)\left(1+x\right)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(1+x)}\,x+2\sqrt{5}\,\sqrt{(x-1)(1+x)}+2x^2+2x+1}{2(3+2x)(x-1)(1+x)}dx} \\ &= \frac{\left(x-1\right)^{\frac{1}{4}}\sqrt{3+2x}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{\frac{1}{4}}}{\left(1+x\right)^{\frac{1}{4}}\sqrt{\frac{5\sqrt{x^2-1}+(3x+2)\sqrt{5}}{\sqrt{x^2-1}\,\sqrt{-\frac{(3+2x)^2}{x^2-1}}}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1-x^2}\,dx} \\ &= z_1 e^{\frac{\operatorname{arctanh}(x)}{2}} \\ &= z_1\left(\sqrt{\frac{1+x}{\sqrt{1-x^2}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1-x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\int \frac{i\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{(3+2x)^2\sqrt{5x-5}}dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}}\right) \\ &\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\,(5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\,(1+x)^{\frac{1}{4}}} \left(\int \frac{i\sqrt{1+x}\,\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x+2\sqrt{5}+\right.}{(3+2x)^2\sqrt{5x-5}}\right.\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\, (5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\, (1+x)^{\frac{1}{4}}} + \frac{i c_2 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\, (5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}} \left(\int \frac{\sqrt{1+x}\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}}{(3+2x)^2\sqrt{5x-5}}dx\right)}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\, (1+x)^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\, (5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\, (1+x)^{\frac{1}{4}}} + \frac{i c_2 \left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{3+2x}\, (5x-5)^{\frac{1}{4}} \sqrt{\frac{1+x}{\sqrt{1-x^2}}} \left(\int \frac{\sqrt{1+x}\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}}}{(3+2x)^2\sqrt{5x-5}}dx\right)}{\sqrt{\frac{i\left(3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}\right)}{3+2x}}\, (1+x)^{\frac{1}{4}}}
$$

Verified OK.

2.726.1 Maple step by step solution

Let's solve

$(1-x^2)\, y'' - y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x^2-1} + \frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x^2-1} - \frac{y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x)=\frac{1}{x^2-1}, P_3(x)=-\frac{1}{x^2-1}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left.\left((1+x)\cdot P_2(x)\right)\right|_{x=-1}=-\frac{1}{2}$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left.\left((1+x)^2\cdot P_3(x)\right)\right|_{x=-1}=0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''+y'-y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+\frac{d}{du}y(u)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u)=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,u^{k+r}$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + a_k(k^2+2kr+r^2-k-r-1)\right)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k-\frac{1}{2}+r\right) a_{k+1} + \left(k^2+(2r-1)\,k+r^2-r-1\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k^2+2kr+r^2-k-r-1)\,a_k}{(k+1+r)\,(2k-1+2r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{(k^2-k-1)\,a_k}{(k+1)\,(2k-1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{(k^2-k-1)\,a_k}{(k+1)\,(2k-1)}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+1} = \frac{(k^2-k-1)\,a_k}{(k+1)\,(2k-1)}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{(k^2+2k-\frac{1}{4})\,a_k}{(k+\frac{5}{2})\,(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})\,a_k}{(k+\frac{5}{2})\,(2k+2)}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{3}{2}}, a_{k+1} = \frac{(k^2+2k-\frac{1}{4})\,a_k}{(k+\frac{5}{2})\,(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{3}{2}}\right), a_{k+1} = \frac{(k^2-k-1)\,a_k}{(k+1)\,(2k-1)}, b_{k+1} = \frac{(k^2+2k-\frac{1}{4})\,b_k}{(k+\frac{5}{2})\,(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 177

```
dsolve((1-x^2)*diff(y(x),x$2)-diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{2x+3}\left(\frac{3\sqrt{5}\,x+2\sqrt{5}-5\sqrt{x^2-1}}{3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{5}} + c_2\sqrt{2x+3}\left(\frac{3\sqrt{5}\,x+2\sqrt{5}+5\sqrt{x^2-1}}{3\sqrt{5}\,x+2\sqrt{5}-5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{-\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{5}}$$

✓ Solution by Mathematica

Time used: 2.8 (sec). Leaf size: 198

```
DSolve[(1-x^2)*y''[x]-y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[4]{x+1}\left(\sqrt{x+1}-\sqrt{x-1}\right)^{-1-\sqrt{5}}\left(-2x+2\sqrt{x-1}\sqrt{x+1}+\sqrt{5}-3\right)e^{-\operatorname{arctanh}\left(x-\sqrt{x-1}\sqrt{x+1}\right)}\left(c_2\int_1^x \frac{e^{2\operatorname{arct}}}{\cdots}\right.}{\sqrt[4]{1-x}}$$

2.727 problem 742

2.727.1 Maple step by step solution 6838

Internal problem ID [8217]
Internal file name [OUTPUT/7150_Sunday_June_05_2022_05_32_35_PM_46037152/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 742.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(1+x)^2 y'' + (1-x^2) y' + (x-1) y = 0}$$

Writing the ode as

$$x(1+x)^2 y'' + (1-x^2) y' + (x-1) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(1+x)^2 \\ B &= 1-x^2 \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-x^2}{x(1+x)^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2} + \ln(1+x)} \\ &= z_1 \left(\frac{1+x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1 + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1-x^2}{x(1+x)^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-\ln(x)+2\ln(1+x)}}{(y_1)^2}\, dx$$
$$= y_1(\ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(1+x) + c_2(1 + x(\ln(x)))$$

Summary

The solution(s) found are the following

$$y = c_1(1+x) + c_2(1+x)\ln(x) \tag{1}$$

Verification of solutions

$$y = c_1(1+x) + c_2(1+x)\ln(x)$$

Verified OK.

2.727.1 Maple step by step solution

Let's solve

$x(1+x)^2 y'' + (1-x^2)\, y' + (x-1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x(1+x)^2} + \frac{(x-1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x(1+x)} + \frac{(x-1)y}{x(1+x)^2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x-1}{x(1+x)}, P_3(x) = \frac{x-1}{x(1+x)^2}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = -2$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 2$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(1+x)^2 y'' - (x-1)(1+x) y' + (x-1) y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - u^2)\left(\frac{d^2}{du^2}y(u)\right) + (-u^2 + 2u)\left(\frac{d}{du}y(u)\right) + (u-2)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k- >k - m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+r)(-2+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+r)(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{1, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_k(k+r-1)(k+r-2) + a_{k-1}(k+r-2)^2 = 0$$

- Shift index using $k->k+1$

$$-a_{k+1}(k+r)(k+r-1) + a_k(k+r-1)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+r}$$

- Recursion relation for $r=1$

$$a_{k+1} = \frac{a_k k}{k+1}$$

- Solution for $r=1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+1}, a_{k+1} = \frac{a_k k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1} = \frac{a_k(k+1)}{k+2}$$

- Solution for $r=2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = \frac{a_k(k+1)}{k+2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = \frac{a_k k}{k+1}, b_{k+1} = \frac{b_k(k+1)}{k+2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*(x+1)^2*diff(y(x),x$2)+(1-x^2)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2(x+1)\ln(x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 17

```
DSolve[x*(x+1)^2*y''[x]+(1-x^2)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x+1)(c_2 \log(x) + c_1)$$

2.728 problem 743

2.728.1 Maple step by step solution 6847

Internal problem ID [8218]
Internal file name [OUTPUT/7151_Sunday_June_05_2022_05_32_38_PM_66396767/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 743.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' - y' + 2y = 0}$$

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1387: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}$$
$$= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)$$
$$= -\frac{1}{2x}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\phi = \theta + \frac{p'}{p}$$
$$= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2 - 12x + 1}{64x^3 + 16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x} + 4x - 1}{4(1+4x)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{(2\sqrt{-x}-1)(1+4x)}}{\sqrt{2\sqrt{-x}+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}(2\sqrt{-x}+1)}{(2\sqrt{-x}-1)(1+4x)} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\right) \\
&\quad + c_2 \left(\frac{e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}} \left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} + \frac{c_2 e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} \left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}} \left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx\right)}{\sqrt{2\sqrt{-x}+1}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} + \frac{c_2 e^{2\sqrt{-x}} \sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} \left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}} \left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx\right)}{\sqrt{2\sqrt{-x}+1}}
$$

Verified OK.

2.728.1 Maple step by step solution

Let's solve

$2xy'' - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$
- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$
- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' - y' + 2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$
- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $2(k+1+r)\left(k-\frac{1}{2}+r\right) a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_k}{(k+1+r)(2k-1+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}\right]$
- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}$
- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(k+1)(2k-1)}, b_{k+1} = -\frac{2b_k}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 75

```
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2i\sqrt{x}} \sqrt{\frac{(1+4x)\,(2i\sqrt{x}-1)}{1+2i\sqrt{x}}} + c_2 e^{-2i\sqrt{x}} \sqrt{\frac{(1+4x)\,(1+2i\sqrt{x})}{2i\sqrt{x}-1}}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 59

```
DSolve[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}(2\sqrt{x}+i) + \frac{1}{8} c_2 e^{-2i\sqrt{x}}(1+2i\sqrt{x})$$

2.729 problem 744

Internal problem ID [8219]
Internal file name [`OUTPUT/7152_Sunday_June_05_2022_05_32_41_PM_87399837/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 744.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + xy' - 2y = 0}$$

Writing the ode as

$$xy'' + xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x+8}{4x} \tag{6}$$

Comparing the above to (5) shows that

$$s = x + 8$$
$$t = 4x$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x+8}{4x}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1389: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x$. There is a pole at $x = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x} - \frac{4}{x^2} + \frac{16}{x^3} - \frac{80}{x^4} + \frac{448}{x^5} - \frac{2688}{x^6} + \frac{16896}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x+8}{4x} \\ &= Q + \frac{R}{4x} \\ &= \left(\frac{1}{4}\right) + \left(\frac{2}{x}\right) \\ &= \frac{1}{4} + \frac{2}{x} \end{aligned}$$

Since the degree of t is 1, then we see that the coefficient of the term 1 in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 0\right) &&= 2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 0\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x+8}{4x}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	2	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 2$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 2 - (1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{x} \\ &= \frac{1}{2} + \frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2} + \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{x^2}\right) + \left(\frac{1}{2} + \frac{1}{x}\right)^2 - \left(\frac{x+8}{4x}\right)\right) = 0$$

$$\frac{2 - a_0}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x+2) \, e^{\int \left(\frac{1}{2} + \frac{1}{x}\right) dx} \\ &= (x+2) \, e^{\frac{x}{2} + \ln(x)} \\ &= (x+2) \, x \, e^{\frac{x}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x} \, dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(e^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x(x+2)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{(-x-1)\,\mathrm{e}^{-x} + (x+2)\,x \ \mathrm{expIntegral}_1\,(x)}{2x\,(x+2)} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x(x+2)) + c_2 \left(x(x+2) \left(\frac{(-x-1)\,\mathrm{e}^{-x} + (x+2)\,x \ \mathrm{expIntegral}_1\,(x)}{2x\,(x+2)} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x(x+2) + c_2 \left(\frac{(-x-1)\,\mathrm{e}^{-x}}{2} + \frac{(x+2)\,x \ \mathrm{expIntegral}_1\,(x)}{2} \right) \tag{1}$$

Verification of solutions

$$y = c_1 x(x+2) + c_2 \left(\frac{(-x-1)\,\mathrm{e}^{-x}}{2} + \frac{(x+2)\,x \ \mathrm{expIntegral}_1\,(x)}{2} \right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2+2x) + c_2\left(\frac{x^2 \operatorname{expIntegral}_1(x)}{2} - \frac{e^{-x}x}{2} + \operatorname{expIntegral}_1(x)\,x - \frac{e^{-x}}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 39

```
DSolve[x*y''[x]+x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x(x+2) - \frac{1}{2}c_2 e^{-x}(e^x(x+2)x\,\text{ExpIntegralEi}(-x)+x+1)$$

2.730 problem 745

2.730.1 Maple step by step solution 6864

Internal problem ID [8220]
Internal file name [`OUTPUT/7153_Sunday_June_05_2022_05_32_45_PM_96040416/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 745.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x(x-1)^2 y''-2y=0}$$

Writing the ode as

$$x(x-1)^2 y''-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x(x-1)^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x\left(x-1\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x(x-1)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x\left(x-1\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1390: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= 1\\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x} + \frac{2}{(x-1)^2} - \frac{2}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= 0\\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	1	0	0	1
1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x} - \frac{1}{x-1} + (0) \\ &= \frac{1}{x} - \frac{1}{x-1} \\ &= -\frac{1}{x\,(x-1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{x-1}\right)^2 - \left(\frac{2}{x(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right)dx} \\ &= \frac{x}{x-1} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{x}{x-1} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x}{x-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{x}{x-1} \int \frac{1}{\frac{x^2}{(x-1)^2}}\, dx \\ &= \frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x}{x-1}\right) + c_2\left(\frac{x}{x-1}\left(x - \frac{1}{x} - 2\ln(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x}{x-1} + \frac{c_2(-2x\ln(x) + x^2 - 1)}{x-1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{x-1} + \frac{c_2(-2x\ln(x) + x^2 - 1)}{x-1}
$$

Verified OK.

2.730.1 Maple step by step solution

Let's solve

$x(x-1)^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{2}{x(x-1)^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^{2} \cdot P_{3}(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)^{2} y'' - 2y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

    $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$$

  - Shift index using $k- >k+2-m$

    $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+r) x^{-1+r} + \left(a_1(1+r) r - 2a_0(r^2 - r + 1)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)(k+r) - 2a_k(k^2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r) r - 2a_0(r^2 - r + 1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1}) k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1}) r + 2a_k - 3a_{k-1} + a_{k+1}) k + (-2a_k + a_{k-1} -$

- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2}) r + 2a_{k+1} - 3a_k + a_{k+2})(k+1) + (-$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kra_k - 4kra_{k+1} + r^2 a_k - 2r^2 a_{k+1} - ka_k - 2ka_{k+1} - ra_k - 2ra_{k+1} - 2a_{k+1}}{k^2 + 2kr + r^2 + 3k + 3r + 2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 6ka_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + ka_k - 6ka_{k+1} - 6a_{k+1}}{k^2 + 5k + 6}, 2a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - ka_k - 2ka_{k+1} - 2a_{k+1}}{k^2 + 3k + 2}, -2a_0 = 0, b_{k+2} = -\frac{k^2 b\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(x*(x-1)^2*diff(y(x),x$2)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{x-1} + \frac{c_2(2x\ln(x) - x^2 + 1)}{x-1}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 33

```
DSolve[x*(x-1)^2*y''[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-c_2 x^2 - c_1 x + 2c_2 x \log(x) + c_2}{x-1}$$

2.731 problem 746

2.731.1 Maple step by step solution 6871

Internal problem ID [8221]
Internal file name [OUTPUT/7154_Sunday_June_05_2022_05_32_48_PM_45032550/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 746.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2xy' + x^2y = 0}$$

Writing the ode as

$$y'' - 2xy' + x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1392: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\, \mathrm{e}^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x^2}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos (x)\, \mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(\cos (x)\, \mathrm{e}^{\frac{x^2}{2}} (\tan (x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos (x)\, \mathrm{e}^{\frac{x^2}{2}} + c_2 \sin (x)\, \mathrm{e}^{\frac{x^2}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \cos (x)\, \mathrm{e}^{\frac{x^2}{2}} + c_2 \sin (x)\, \mathrm{e}^{\frac{x^2}{2}}
$$

Verified OK.

2.731.1 Maple step by step solution

Let's solve

$y'' - 2xy' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - ○ Shift index using $k- >k-2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  - ○ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + (6a_3 - 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k k + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 - 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = 0, a_3 = \frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} - 2a_k k + a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 2a_{k+2}(k+2) + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} - a_k + 4a_{k+2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{\frac{x^2}{2}} \cos(x) + c_2 \mathrm{e}^{\frac{x^2}{2}} \sin(x)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 39

```
DSolve[y''[x]-2*x*y'[x]+x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{\frac{1}{2}x(x-2i)} \left(2c_1 - ic_2 e^{2ix}\right)$$

2.732 problem 747

2.732.1 Maple step by step solution 6881

Internal problem ID [8222]
Internal file name [OUTPUT/7155_Sunday_June_05_2022_05_32_51_PM_9976214/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 747.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x\left(-x^2+2\right)y''-\left(x^2+4x+2\right)\left(\left(1-x\right)y'+y\right)=0}$$

Writing the ode as

$$\left(-x^3+2x\right)y''+\left(x^3+3x^2-2x-2\right)y'+\left(-x^2-4x-2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^3+2x\\B&=x^3+3x^2-2x-2\\C&=-x^2-4x-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\,(x^3 - 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12$$
$$t = 4(x^3 - 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\,(x^3 - 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1394: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{2x} + \frac{3}{4x^2} + \frac{3}{4\left(x - \sqrt{2}\right)^2} + \frac{3}{4\left(x + \sqrt{2}\right)^2} + \frac{-\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x - \sqrt{2}} + \frac{\frac{5\sqrt{2}}{8} - \frac{1}{2}}{x + \sqrt{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + \sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{2x} - \frac{1}{2x^2} - \frac{3}{2x^3} + \frac{21}{4x^4} - \frac{43}{4x^5} + \frac{135}{4x^6} - \frac{147}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading

coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2} \\
&= Q + \frac{R}{4x^6 - 16x^4 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{2x^5 - x^4 - 16x^3 + 20x^2 + 24x + 12}{4x^6 - 16x^4 + 16x^2}
\end{aligned}$$

Since the degree of t is 6, then we see that the coefficient of the term x^5 in the remainder R is 2. Dividing this by leading coefficient in t which is 4 gives $\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{1}{2}\right) - (0) \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) &= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^6 + 2x^5 - 5x^4 - 16x^3 + 24x^2 + 24x + 12}{4\left(x^3 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-\sqrt{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \left(\frac{1}{2}\right) \\
&= \frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2} \\
&= \frac{x^3+x^2-2x-6}{2x^3-4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{2\left(x-\sqrt{2}\right)^2} + \frac{1}{2\left(x+\sqrt{2}\right)^2}\right) + \left(\frac{3}{2x}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{2\left(x-\sqrt{2}\right)} - \frac{1}{2\left(x+\sqrt{2}\right)} + \frac{1}{2}\right) dx} \\
&= \frac{x^{\frac{3}{2}} \mathrm{e}^{\frac{x}{2}}}{\sqrt{x-\sqrt{2}}\,\sqrt{x+\sqrt{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2} + \frac{\ln\left(x^2-2\right)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x^2-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3+3x^2-2x-2}{-x^3+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x)+\ln(x^2-2)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{(x-1)\,\mathrm{e}^{-x}}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \mathrm{e}^x\right) + c_2 \left(x^2 \mathrm{e}^x \left(-\frac{(x-1)\,\mathrm{e}^{-x}}{x^2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x^2 \mathrm{e}^x c_1 + c_2(1-x) \tag{1}
$$

Verification of solutions

$$
y = x^2 \mathrm{e}^x c_1 + c_2(1-x)
$$

Verified OK.

2.732.1 Maple step by step solution

Let's solve

$(-x^3+2x)\, y'' + (x^3+3x^2-2x-2)\, y' + (-x^2-4x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+4x+2)y}{x(x^2-2)} + \frac{(x^3+3x^2-2x-2)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^3+3x^2-2x-2)y'}{x(x^2-2)} + \frac{(x^2+4x+2)y}{x(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x^3+3x^2-2x-2}{x(x^2-2)}, P_3(x) = \frac{x^2+4x+2}{x(x^2-2)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $y''x(x^2-2)+(-x^3-3x^2+2x+2)\,y'+(x^2+4x+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m=0..3$

    $x^m \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m \cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m=1..3$

    $x^m \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r(-2+r)\,x^{-1+r}+(-2a_1(1+r)\,(-1+r)+2a_0(1+r))\,x^r+\big(-2a_2(2+r)\,r+2a_1(2+r)+$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,2\}$

- The coefficients of each power of x must be 0

  $$\left[-2a_1(1+r)(-1+r)+2a_0(1+r)=0, -2a_2(2+r)r+2a_1(2+r)+a_0(-2+r)^2=0\right]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1=\frac{a_0}{-1+r}, a_2=\frac{a_0(r^2-5r+10)}{2(r^2+r-2)}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k-1}(k-3+r)^2-2a_{k+1}(k+r+1)(k+r-1)+(2a_k-a_{k-2})k+(2a_k-a_{k-2})r+2a_k+3a_{k-}$$

- Shift index using $k->k+2$

  $$a_{k+1}(k+r-1)^2-2a_{k+3}(k+3+r)(k+r+1)+(2a_{k+2}-a_k)(k+2)+(2a_{k+2}-a_k)r+2a_{k+2}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3}=\frac{k^2a_{k+1}+2kra_{k+1}+r^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}-a_kr-2ra_{k+1}+2ra_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3+r)(k+r+1)}$$

- Recursion relation for $r=0$

  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+3}=\frac{k^2a_{k+1}-a_kk-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=-\frac{5a_0}{2}\right]$$

- Recursion relation for $r=2$

  $$a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}$$

- Solution for $r=2$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+3}=\frac{k^2a_{k+1}-a_kk+2ka_{k+1}+2ka_{k+2}-a_k+a_{k+1}+10a_{k+2}}{2(k+5)(k+3)}, a_1=a_0, a_2=\frac{a_0}{2}\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), a_{k+3}=\frac{k^2a_{k+1}-ka_k-2ka_{k+1}+2ka_{k+2}+a_k+a_{k+1}+6a_{k+2}}{2(k+3)(k+1)}, a_1=-a_0, a_2=\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*(2-x^2)*diff(y(x),x$2)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x))=0,y(x), singsol=all)
```

$$y(x) = c_1(x - 1) + c_2 e^x x^2$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 21

```
DSolve[x*(2-x^2)*y''[x]-(x^2+4*x+2)*((1-x)*y'[x]+y[x])==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 e^x x^2 + c_2(x - 1)$$

2.733 problem 748

2.733.1 Maple step by step solution 6890

Internal problem ID [8223]
Internal file name [`OUTPUT/7156_Sunday_June_05_2022_05_32_54_PM_45707710/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 748.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(1+x)\, y''-(2x+1)\,(xy'-y)=0}$$

Writing the ode as

$$x^2(1+x)\, y''+\left(-2x^2-x\right) y'+(2x+1)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(1+x) \\ B &= -2x^2-x \\ C &= 2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1 - 4x}{4\left(x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1 - 4x$$
$$t = 4\left(x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-1 - 4x}{4\left(x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1396: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{2x} + \frac{3}{4(1+x)^2} - \frac{1}{4x^2} + \frac{1}{2x+2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-1-4x}{4\left(x^{2}+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_{\infty}^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=0$ then

$$\begin{aligned}d&=\alpha_{\infty}^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right)\\&=0-(0)\\&=0\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+\left((+)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{+}}{x-c_2}\right)+(+)[\sqrt{r}]_\infty\\&=-\frac{1}{2\left(1+x\right)}+\frac{1}{2x}+(0)\\&=-\frac{1}{2\left(1+x\right)}+\frac{1}{2x}\\&=\frac{1}{2x\left(1+x\right)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(1+x)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2\,(1+x)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{2\,(1+x)} + \frac{1}{2x}\right)^2 - \left(\frac{-1-4x}{4\,(x^2+x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(1+x)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{\sqrt{1+x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-x}{x^2(1+x)}\, dx} \\ &= z_1 e^{\frac{\ln(x(1+x))}{2}} \\ &= z_1\left(\sqrt{x\,(1+x)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}\,\sqrt{x\,(1+x)}}{\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x^2-x}{x^2(1+x)}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{\ln(x(1+x))}}{(y_1)^2}\,dx$$
$$= y_1(x + \ln(x))$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\frac{\sqrt{x}\,\sqrt{x\,(1+x)}}{\sqrt{1+x}}\right) + c_2\left(\frac{\sqrt{x}\,\sqrt{x\,(1+x)}}{\sqrt{1+x}}(x + \ln(x))\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x}\,\sqrt{x\,(1+x)}}{\sqrt{1+x}} + \frac{c_2\sqrt{x}\,\sqrt{x\,(1+x)}\,(x+\ln(x))}{\sqrt{1+x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{x}\,\sqrt{x\,(1+x)}}{\sqrt{1+x}} + \frac{c_2\sqrt{x}\,\sqrt{x\,(1+x)}\,(x+\ln(x))}{\sqrt{1+x}}$$

Verified OK.

2.733.1 Maple step by step solution

Let's solve

$x^2(1+x)\,y'' + (-2x^2 - x)\,y' + (2x+1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y}{x^2(1+x)} + \frac{(2x+1)y'}{x(1+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x+1)y'}{x(1+x)} + \frac{(2x+1)y}{x^2(1+x)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x+1}{x(1+x)}, P_3(x) = \frac{2x+1}{x^2(1+x)}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1} = -1$

- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x^2(1+x)\,y'' - x(2x+1)\,y' + (2x+1)\,y = 0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u^2+3u-1)\left(\frac{d}{du}y(u)\right) + (2u-1)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m\cdot\left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, u^{-1+r} + (a_1(1+r)(-1+r) - a_0(2r^2-5r+1))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r-\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term must be 0

  $$a_1(1+r)(-1+r) - a_0(2r^2-5r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(-2a_k + a_{k-1} + a_{k+1})\, k^2 + ((-4a_k + 2a_{k-1} + 2a_{k+1})\, r + 5a_k - 5a_{k-1})\, k + (-2a_k + a_{k-1} + a_{k+1})$$

- Shift index using $k- >k+1$

  $$(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + ((-4a_{k+1} + 2a_k + 2a_{k+2})\, r + 5a_{k+1} - 5a_k)\,(k+1) + (-2a_{k+1} +$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2kr a_k - 4kr a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 3k a_k + k a_{k+1} - 3r a_k + r a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2kr + r^2 + 2k + 2r}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_k + k a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_k + k a_{k+1} + 2a_k + 2a_{k+1}}{k^2 + 2k}$$

- Recursion relation for $r = 2$

  $$a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}$$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4a_{k+1}}{k^2 + 6k + 8}, 3a_1 + a_0 = 0\right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + k a_k - 7k a_{k+1} - 4 a_{k+1}}{k^2+6k+8}, 3a_1 + a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*(1+x)*diff(y(x),x$2)-(1+2*x)*(x*diff(y(x),x)-y(x))=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 x(x + \ln(x))$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 132

```
DSolve[x^2*(1+x)*y''[x]-(1+2*x)*(x*y'[x]+y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^{1+\sqrt{2}}\,\text{Hypergeometric2F1}\left(-\frac{1}{2}+\sqrt{2}-\frac{\sqrt{17}}{2}, -\frac{1}{2}+\sqrt{2}+\frac{\sqrt{17}}{2}, 1+2\sqrt{2}, -x\right) + c_1 x^{1-\sqrt{2}}\,\text{Hypergeometric2F1}\left(\frac{1}{2}\left(-1-2\sqrt{2}-\sqrt{17}\right), \frac{1}{2}\left(-1-2\sqrt{2}+\sqrt{17}\right), 1 - 2\sqrt{2}, -x\right)$$

2.734 problem 749

2.734.1 Maple step by step solution 6899

Internal problem ID [8224]
Internal file name [`OUTPUT/7157_Sunday_June_05_2022_05_32_58_PM_52389439/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 749.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(-x+2)\, x^2 y'' - (-x+4)\, x y' + (3-x)\, y = 0}$$

Writing the ode as

$$\left(-2x^3+4x^2\right) y'' + \left(x^2-4x\right) y' + (3-x)\, y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -2x^3+4x^2 \\ B &= x^2-4x \\ C &= 3-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{16 (x-2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3$$
$$t = 16(x-2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{16 (x-2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1398: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{3}{16(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{16\left(x-2\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{4x-8} + (-)\left(0\right) \\ &= \frac{1}{4x-8} \\ &= \frac{1}{4x-8} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{4x-8}\right)(0) + \left(\left(-\frac{1}{4\left(x-2\right)^2}\right) + \left(\frac{1}{4x-8}\right)^2 - \left(-\frac{3}{16\left(x-2\right)^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int \frac{1}{4x-8}dx} \\ &= (x-2)^{\frac{1}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2-4x}{-2x^3+4x^2}\,dx} \\ &= z_1 e^{\frac{\ln(x)}{2} - \frac{\ln(x-2)}{4}} \\ &= z_1 \left(\frac{\sqrt{x}}{(x-2)^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2-4x}{-2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)-\frac{\ln(x-2)}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(2\sqrt{x-2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(2\sqrt{x-2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x} + 2c_2\sqrt{x}\,\sqrt{x-2} \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x} + 2c_2\sqrt{x}\,\sqrt{x-2}
$$

Verified OK.

2.734.1 Maple step by step solution

Let's solve

$\left(-2x^3+4x^2\right)y'' + \left(x^2-4x\right)y' + (3-x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x-4)y'}{2x(x-2)} - \frac{(-3+x)y}{2x^2(x-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-4)y'}{2x(x-2)} + \frac{(-3+x)y}{2x^2(x-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{x-4}{2x(x-2)}, P_3(x) = \frac{-3+x}{2(x-2)x^2}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{4}$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$2y''(x-2)\, x^2 - x(x-4)\, y' + (-3+x)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

$$x^{m} \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0(-1+2r)\,(-3+2r)\,x^{r} + \left(\sum_{k=1}^{\infty} \left(-a_k(2k+2r-1)\,(2k+2r-3) + a_{k-1}(2k+2r-3)\,(k-2\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-(-1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-4\left(k+r-\frac{3}{2}\right)\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1} + a_k\left(k-\frac{1}{2}+r\right)\right) = 0$$

- Shift index using $k- >k+1$

$$-4\left(k-\frac{1}{2}+r\right)\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k + a_{k+1}\left(k+\frac{1}{2}+r\right)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r-1)a_k}{2k+1+2r}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}\right]$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = \frac{\left(k+\frac{1}{2}\right)a_k}{2k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = \frac{\left(k-\frac{1}{2}\right)a_k}{2k+2}, b_{k+1} = \frac{\left(k+\frac{1}{2}\right)b_k}{2k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(2*(2-x)*x^2*diff(y(x),x$2)-(4-x)*x*diff(y(x),x)+(3-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x^2 - 2x}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 41

```
DSolve[2*(2-x)*x^2*y''[x]-(4-x)*x*y'[x]+(3-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{\sqrt[4]{x-2}\sqrt{x}\left(2c_2\sqrt{x-2} + c_1\right)}{\sqrt[4]{2-x}}$$

2.735 problem 750

Internal problem ID [8225]
Internal file name [`OUTPUT/7158_Sunday_June_05_2022_05_33_01_PM_23889271/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 750.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(1-x)\,x^2y''+(5x-4)\,xy'+(6-9x)\,y=0}$$

Writing the ode as

$$\left(-x^3+x^2\right)y''+\left(5x^2-4x\right)y'+(6-9x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=-x^3+x^2\\ B&=5x^2-4x \\ C&=6-9x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r&=\frac{s}{t} \\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 4}{4x\,(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x + 4 \\ t &= 4x(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 4}{4x\,(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1400: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(x-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{-x+4}{4x\left(x-1\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$\begin{aligned}
d&=\alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&=\frac{1}{2}-\left(\frac{1}{2}\right)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^{-}}{x-c_2}\right)+(-)[\sqrt{r}]_\infty\\
&=\frac{1}{x}-\frac{1}{2\left(x-1\right)}+\left(-\right)\left(0\right)\\
&=\frac{1}{x}-\frac{1}{2\left(x-1\right)}\\
&=\frac{x-2}{2x\left(x-1\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{x}-\frac{1}{2\,(x-1)}\right)(0)+\left(\left(-\frac{1}{x^2}+\frac{1}{2\,(x-1)^2}\right)+\left(\frac{1}{x}-\frac{1}{2\,(x-1)}\right)^2-\left(\frac{-x+4}{4x\,(x-1)^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}z_1(x)&=pe^{\int \omega\,dx}\\&=\mathrm{e}^{\int\left(\frac{1}{x}-\frac{1}{2(x-1)}\right)dx}\\&=\frac{x}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\&=z_1e^{-\int \frac{1}{2}\frac{5x^2-4x}{-x^3+x^2}\,dx}\\&=z_1e^{2\ln(x)+\frac{\ln(x-1)}{2}}\\&=z_1\left(x^2\sqrt{x-1}\right)\end{aligned}$$

Which simplifies to

$$y_1=x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int\frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2-4x}{-x^3+x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)+\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{1}{x}+\ln(x)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x^3) + c_2\left(x^3\left(\frac{1}{x}+\ln(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 + c_2 x^2 (x\ln(x)+1) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 + c_2 x^2 (x\ln(x)+1)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((1-x)*x^2*diff(y(x),x$2)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^3 + c_2x^2(x\ln(x) + 1)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 24

```
DSolve[(1-x)*x^2*y''[x]+(5*x-4)*x*y'[x]+(6-9*x)*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x^2(c_1x - c_2(x\log(x) + 1))$$

2.736 problem 751

2.736.1 Maple step by step solution 6915

Internal problem ID [8226]
Internal file name [`OUTPUT/7159_Sunday_June_05_2022_05_33_04_PM_97436028/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 751.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (4x^2 + 1)\, y' + 4x(x^2 + 1)\, y = 0}$$

Writing the ode as

$$xy'' + (4x^2 + 1)\, y' + (4x^3 + 4x)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 4x^2 + 1 \\ C &= 4x^3 + 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= pe^{\int \omega\, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^2+1}{x}\, dx}\\ &= z_1 e^{-x^2 - \frac{\ln(x)}{2}}\\ &= z_1\left(\frac{e^{-x^2}}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+1}{x}\, dx}}{(y_1)^2}\, dx\\ &= y_1 \int \frac{e^{-2x^2 - \ln(x)}}{(y_1)^2}\, dx\\ &= y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} (\ln (x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \ln (x) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 \mathrm{e}^{-x^2} \ln (x)
$$

Verified OK.

2.736.1 Maple step by step solution

Let's solve

$x y'' + (4x^2 + 1) y' + (4x^3 + 4x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-4x^2 - 4) y - \frac{(4x^2+1) y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2+1) y'}{x} + (4x^2 + 4) y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4x^2+1}{x}, P_3(x) = 4x^2 + 4\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (4x^2 + 1)\, y' + 4x(x^2 + 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 1..3$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\,(k + r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1 + r)^2 x^r + \left(a_2 (2 + r)^2 + 4a_0 (1 + r)\right) x^{1+r} + \left(a_3 (3 + r)^2 + 4a_1 (2 + r)\right) x^{2+r} + ($

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- The coefficients of each power of x must be 0

  $\left[a_1(1+r)^2 = 0, a_2(2+r)^2 + 4a_0(1+r) = 0, a_3(3+r)^2 + 4a_1(2+r) = 0\right]$

- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{4a_0(1+r)}{r^2+4r+4}, a_3 = 0\right\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 4a_{k-1}k + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4)^2 + 4a_{k+2}(k+3) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4(ka_{k+2}+a_k+3a_{k+2})}{(k+4)^2}, a_1 = 0, a_2 = -a_0, a_3 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 e^{-x^2} \ln(x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 21

```
DSolve[x*y''[x]+(4*x^2+1)*y'[x]+4*x*(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 \log(x) + c_1)$$

2.737 problem 752

Internal problem ID [8227]
Internal file name [`OUTPUT/7160_Sunday_June_05_2022_05_33_08_PM_16508863/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 752.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2-9) + (0) \\ &= x^2-9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &&= -5 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(x) \\ &= -x \\ &= -x\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(12x^2 + 6xa_3 + 2a_2) + 2(-x)\left(4x^3 + 3x^2a_3 + 2xa_2 + a_1\right) + \left((-1) + (-x)^2 - (x^2 - 9)\right) = 0 \\ 2a_3x^3 + 4(3 + a_2)\, x^2 + 6(a_1 + a_3)\, x + 8a_0 + 2a_2 = 0\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\ &= \frac{(4x^4 - 12x^2 + 3)\, e^{-\frac{x^2}{2}}}{4}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1 \left(e^{\frac{x^2}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^4 - 3x^2 + \frac{3}{4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2 \left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2 (4x^4 - 12x^2 + 3) \left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 63

```
DSolve[y''[x]-2*x*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) - \frac{1}{12}c_2\left(\sqrt{\pi}\left(-4x^4 + 12x^2 - 3\right)\operatorname{erfi}(x) + 2e^{x^2}x\left(2x^2 - 5\right)\right)$$

2.738 problem 753

Internal problem ID [8228]
Internal file name [OUTPUT/7161_Sunday_June_05_2022_05_33_12_PM_99167358/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 753.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 - 9\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1404: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2-9) + (0) \\ &= x^2 - 9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &&= -5 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &&= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)(x) \\ &= -x \\ &= -x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (12x^2 + 6xa_3 + 2a_2) + 2(-x)(4x^3 + 3x^2 a_3 + 2xa_2 + a_1) + \left((-1) + (-x)^2 - (x^2 - 9)\right) &= 0 \\ 2a_3 x^3 + 4(3 + a_2) x^2 + 6(a_1 + a_3) x + 8a_0 + 2a_2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{ a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0 \right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4} \right) e^{\int -x dx} \\ &= \left(x^4 - 3x^2 + \frac{3}{4} \right) e^{-\frac{x^2}{2}} \\ &= \frac{(4x^4 - 12x^2 + 3) e^{-\frac{x^2}{2}}}{4} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2\left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2\left(4x^4 - 12x^2 + 3\right)\left(\int \frac{\mathrm{e}^{x^2}}{(4x^4-12x^2+3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 63

```
DSolve[y''[x]-2*x*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) - \frac{1}{12}c_2\left(\sqrt{\pi}\left(-4x^4 + 12x^2 - 3\right)\operatorname{erfi}(x) + 2e^{x^2}x\left(2x^2 - 5\right)\right)$$

2.739 problem 754

2.739.1 Maple step by step solution 6941

Internal problem ID [8229]
Internal file name [OUTPUT/7162_Sunday_June_05_2022_05_33_15_PM_27478625/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 754.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-2xy'+12y=0}$$

Writing the ode as

$$(1-x^2)\,y''-2xy'+12y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -2x \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12x^2 - 13 \\ t &= (x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12x^2 - 13}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{25}{4(x-1)} - \frac{25}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12x^2 - 13}{(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12x^2 - 13}{(x^2 - 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 4$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 4 - (1) \\ &= 3 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} + (0) \\ &= \frac{1}{2x - 2} + \frac{1}{2x + 2} \\ &= \frac{x}{x^2 - 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)\left(3x^2 + 2xa_2 + a_1\right) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x \cdots}\right.\right.$$
$$\frac{-6a_2x^2 + (-10a_1 \cdots}{x^{\cdots}}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = -\frac{3}{5}, a_2 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 - \frac{3}{5}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^3 - \frac{3}{5}x\right) e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= \frac{(5x^3 - 3x)\sqrt{x-1}\,\sqrt{1+x}}{5}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{625x}{180x^2 - 108} + \frac{25}{9x} - \frac{25\ln(1+x)}{8} + \frac{25\ln(x-1)}{8}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}}\right) \\ &\quad + c_2 \left(\frac{(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}} \left(\frac{625x}{180x^2 - 108} + \frac{25}{9x} - \frac{25\ln(1+x)}{8} + \frac{25\ln(x-1)}{8}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{c_1(5x^3 - 3x)\sqrt{x^2 - 1}}{5\sqrt{x - 1}\,\sqrt{1 + x}} \\ & - \frac{5c_2(15\ln(1+x)\,x^3 - 15\ln(x-1)\,x^3 - 9\ln(1+x)\,x + 9\ln(x-1)\,x - 30x^2 + 8)\sqrt{x^2 - 1}}{24\sqrt{1 + x}\,\sqrt{x - 1}} \end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(5x^3-3x)\sqrt{x^2-1}}{5\sqrt{x-1}\,\sqrt{1+x}} - \frac{5c_2(15\ln(1+x)\,x^3-15\ln(x-1)\,x^3-9\ln(1+x)\,x+9\ln(x-1)\,x-30x^2+8)\sqrt{x^2-1}}{24\sqrt{1+x}\,\sqrt{x-1}}$$

Verified OK.

2.739.1 Maple step by step solution

Let's solve

$(1-x^2)\,y''-2xy'+12y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2xy'}{x^2-1}+\frac{12y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{12y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{12}{x^2-1}\right]$

  - $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((1+x)\cdot P_2(x))\right|_{x=-1}=1$

  - $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

    $\left.((1+x)^2\cdot P_3(x))\right|_{x=-1}=0$

  - $x=-1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''+2xy'-12y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$$\left(u^2-2u\right)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-12y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- ○ Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- ○ Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- ○ Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+4)\,(k+r-3)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r^2=0$$

- Values of r that satisfy the indicial equation

$$r=0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2+a_k(k+4)\,(k-3)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Recursion relation for $r=0$; series terminates at $k=3$

$$a_{k+1}=\frac{a_k(k+4)(k-3)}{2(k+1)^2}$$

- Apply recursion relation for $k=0$

$a_1 = -6a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{5a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{15a_0}{2}$

- Apply recursion relation for $k = 2$

  $a_3 = -\frac{a_2}{3}$

- Express in terms of a_0

  $a_3 = -\frac{5a_0}{2}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 6u + \frac{15}{2}u^2 - \frac{5}{2}u^3\right)$

- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(\frac{3}{2}x - \frac{5}{2}x^3\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(-\frac{5}{3}x^3 + x\right) + c_2\left(-\frac{5\ln(x+1)\,x^3}{24} + \frac{5\ln(x-1)\,x^3}{24} + \frac{\ln(x+1)\,x}{8} - \frac{\ln(x-1)\,x}{8} + \frac{5x^2}{12} - \frac{1}{9}\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 59

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_1 x(5x^2 - 3) + c_2\left(-\frac{5x^2}{2} - \frac{1}{4}(5x^2 - 3)\,x(\log(1-x) - \log(x+1)) + \frac{2}{3}\right)$$

2.740 problem 755

2.740.1 Maple step by step solution 6951

Internal problem ID [8230]
Internal file name [`OUTPUT/7163_Sunday_June_05_2022_05_33_19_PM_29531278/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 755.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+2)\,y''+2(1+x)\,y'-2y=0}$$

Writing the ode as

$$\left(x^2+2x\right)y''+(2x+2)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2x \\ B &= 2x+2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 + 4x - 1$$
$$t = (x^2 + 2x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{4x} - \frac{5}{4(x+2)} - \frac{1}{4(x+2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 + 4x - 1}{(x^2 + 2x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2+4x-1}{(x^2+2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{1}{2}$	$\frac{1}{2}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x+4} + \frac{1}{2x} + (0) \\
&= \frac{1}{2x+4} + \frac{1}{2x} \\
&= \frac{1+x}{x(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x+4} + \frac{1}{2x}\right)(1) + \left(\left(-\frac{1}{2(x+2)^2} - \frac{1}{2x^2}\right) + \left(\frac{1}{2x+4} + \frac{1}{2x}\right)^2 - \left(\frac{2x^2+4x-1}{(x^2+2x)^2}\right)\right) &= 0 \\
\frac{2 - 2a_0}{x(x+2)} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{1}{2x+4} + \frac{1}{2x}\right) dx} \\
&= (1+x)\, e^{\frac{\ln(x)}{2} + \frac{\ln(x+2)}{2}} \\
&= (1+x)\sqrt{x}\,\sqrt{x+2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+2}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x\,(x+2)}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1 + x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+2}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\ln(x)}{2} + \frac{1}{1+x} - \frac{\ln(x+2)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1+x) + c_2\left(1 + x\left(\frac{\ln(x)}{2} + \frac{1}{1+x} - \frac{\ln(x+2)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(1+x) + c_2\left(\frac{(-x-1)\ln(x+2)}{2} + 1 + \frac{(1+x)\ln(x)}{2}\right) \tag{1}
$$

Verification of solutions

$$y = c_1(1+x) + c_2\left(\frac{(-x-1)\ln(x+2)}{2} + 1 + \frac{(1+x)\ln(x)}{2}\right)$$

Verified OK.

2.740.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (2x+2)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x+2)} - \frac{2(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(1+x)y'}{x(x+2)} - \frac{2y}{x(x+2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2(1+x)}{x(x+2)}, P_3(x) = -\frac{2}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2} = 1$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2} = 0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (2x+2)\,y' - 2y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = x + 2$

  $[y = a_0(-x - 1)]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(x*(x+2)*diff(y(x),x$2)+2*(x+1)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2\left(\frac{x\ln(x)}{2} - \frac{\ln(x+2)\,x}{2} + \frac{\ln(x)}{2} - \frac{\ln(x+2)}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 37

```
DSolve[x*(x+2)*y''[x]+2*(x+1)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x+1) - \frac{1}{2}c_2((x+1)\log(-x) - (x+1)\log(x+2) + 2)$$

2.741 problem 757

2.741.1 Maple step by step solution 6960

Internal problem ID [8231]
Internal file name [`OUTPUT/7164_Sunday_June_05_2022_05_33_22_PM_62859810/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 757.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, `
   _with_symmetry_[0,F(x)]`]]
```

$$\boxed{x(x+2)\,y''+(1+x)\,y'-4y=0}$$

Writing the ode as

$$\left(x^{2}+2x\right)y''+(1+x)\,y'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^{2}+2x\\ B&=1+x\\ C&=-4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 + 30x - 3 \\ t &= 4\left(x^2 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{33}{16x} - \frac{33}{16(x+2)} - \frac{3}{16(x+2)^2} - \frac{3}{16x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 + 30x - 3}{4(x^2 + 2x)^2}$$

Since the gcd$(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15x^2 + 30x - 3}{4\left(x^2 + 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{3}{4}$	$\frac{1}{4}$
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} + (0) \\
&= \frac{3}{4\left(x+2\right)} + \frac{3}{4x} \\
&= \frac{\frac{3x}{2}+\frac{3}{2}}{x\left(x+2\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)(1) + \left(\left(-\frac{3}{4\left(x+2\right)^2} - \frac{3}{4x^2}\right) + \left(\frac{3}{4\left(x+2\right)} + \frac{3}{4x}\right)^2 - \left(\frac{15x^2+30x-3}{4\left(x^2+2x\right)^2}\right)\right) = \frac{3-3a_0}{x\left(x+2\right)} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 1 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1+x)\, e^{\int \left(\frac{3}{4(x+2)} + \frac{3}{4x}\right) dx} \\
&= (1+x)\, e^{\frac{3\ln(x)}{4} + \frac{3\ln(x+2)}{4}} \\
&= (1+x)\, x^{\frac{3}{4}} (x+2)^{\frac{3}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x^2+2x}\,dx} \\
&= z_1 e^{-\frac{\ln(x(x+2))}{4}} \\
&= z_1\left(\frac{1}{(x(x+2))^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)\,x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x(x+2))^{\frac{1}{4}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x(x+2))}{2}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{2\left(x^2+2x+\frac{1}{2}\right)}{\sqrt{x+2}\,\sqrt{x}\,(1+x)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)\,x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x(x+2))^{\frac{1}{4}}}\right) + c_2\left(\frac{(1+x)\,x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x(x+2))^{\frac{1}{4}}}\left(-\frac{2\left(x^2+2x+\frac{1}{2}\right)}{\sqrt{x+2}\,\sqrt{x}\,(1+x)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(1+x)\,x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x(x+2))^{\frac{1}{4}}} - \frac{2c_2 x^{\frac{1}{4}}(x+2)^{\frac{1}{4}}\left(x^2+2x+\frac{1}{2}\right)}{(x(x+2))^{\frac{1}{4}}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+x)\,x^{\frac{3}{4}}(x+2)^{\frac{3}{4}}}{(x\,(x+2))^{\frac{1}{4}}} - \frac{2c_2x^{\frac{1}{4}}(x+2)^{\frac{1}{4}}\left(x^2+2x+\frac{1}{2}\right)}{(x\,(x+2))^{\frac{1}{4}}}$$

Verified OK.

2.741.1 Maple step by step solution

Let's solve

$(x^2+2x)\,y''+(1+x)\,y'-4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{x(x+2)}-\frac{(1+x)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(1+x)y'}{x(x+2)}-\frac{4y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1+x}{x(x+2)}, P_3(x)=-\frac{4}{x(x+2)}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=\frac{1}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $y''x(x+2)+(1+x)\,y'-4y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

$$\left(u^2-2u\right)\left(\frac{d^2}{du^2}y(u)\right)+(-1+u)\left(\frac{d}{du}y(u)\right)-4y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

  - Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

  - Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-1+2r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k+r+2)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r+\frac{1}{2}\right)a_{k+1}+a_k(k+r+2)\,(k+r-2)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{a_k(k+r+2)(k+r-2)}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$; series terminates at $k=2$

$$a_{k+1}=\frac{a_k(k+2)(k-2)}{(k+1)(2k+1)}$$

- Apply recursion relation for $k=0$

$a_1 = -4a_0$

- Apply recursion relation for $k = 1$

$a_2 = -\frac{a_1}{2}$

- Express in terms of a_0

$a_2 = 2a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$y(u) = a_0 \cdot (2u^2 - 4u + 1)$

- Revert the change of variables $u = x + 2$

$[y = a_0(2x^2 + 4x + 1)]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = x + 2$

$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

$\left[y = a_0(2x^2 + 4x + 1) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+\frac{1}{2}}\right), b_{k+1} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   Solution is available but has compositions of trig with ln functions of radicals. Attempt
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x*(x+2)*diff(y(x),x$2)+(x+1)*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(2x^2 + 4x + 1) + c_2(x + 1)\sqrt{x(x + 2)}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 53

```
DSolve[x*(x+2)*y''[x]+(x+1)*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cosh\left(4\log\left(\sqrt{x+2} - \sqrt{x}\right)\right) - ic_2 \sinh\left(4\log\left(\sqrt{x+2} - \sqrt{x}\right)\right)$$

2.742 problem 758

2.742.1 Maple step by step solution 6970

Internal problem ID [8232]
Internal file name [`OUTPUT/7165_Sunday_June_05_2022_05_33_28_PM_75094995/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 758.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

Writing the ode as

$$(x-1)\,y''-xy'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2(x-1)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4(x-1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\,dx \\ &= y_1\left(-x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-x\,\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x - c_2 x$$

Verified OK.

2.742.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

- $x = 1$is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$x_0 = 1$

- Multiply by denominators

$y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x-1)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1x+e^x c_2$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

2.743 problem 759

Internal problem ID [8233]
Internal file name [`OUTPUT/7166_Sunday_June_05_2022_05_33_31_PM_68056643/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 759.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+1)\, y''-2xy'+2y=0}$$

Writing the ode as

$$(x^2+1)\, y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x-i\right)} + \frac{3}{2\left(x+i\right)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\frac{}{(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2+1} \, dx} \\
&= z_1 e^{\frac{\ln(x^2+1)}{2}} \\
&= z_1 \left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx$$

$$= y_1 \left(-\frac{x}{(x+i)^2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2 (x+i)^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2 (x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 (x^2 - 1)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 21

```
DSolve[(1+x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x - c_1 (x - i)^2$$

2.744 problem 760

Internal problem ID [8234]
Internal file name [`OUTPUT/7167_Sunday_June_05_2022_05_33_34_PM_87341802/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 760.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2 - 2x + 10)\, y'' + xy' - 4y = 0}$$

Writing the ode as

$$(x^2 - 2x + 10)\, y'' + xy' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 2x + 10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 32x + 180 \\ t &= 4\left(x^2 - 2x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1414: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12}
\end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\,(x^2 - 2x + 10)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{2}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$1+3i$	2	0	$\frac{3}{4}+\frac{i}{12}$	$\frac{1}{4}-\frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4}-\frac{i}{12}$	$\frac{1}{4}+\frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{+}+\alpha_{c_2}^{+}\right)\\
&= \frac{5}{2}-\left(\frac{3}{2}\right)\\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} + (0) \\
&= \frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i} \\
&= \frac{3x - 4}{2x^2 - 4x + 20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right)(1) + \left(\left(\frac{-\frac{3}{4} - \frac{i}{12}}{(x - 1 - 3i)^2} + \frac{-\frac{3}{4} + \frac{i}{12}}{(x - 1 + 3i)^2}\right) + \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right.\right.
$$

$$
-\frac{3}{(-
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) \mathrm{e}^{\int \left(\frac{\frac{3}{4} + \frac{i}{12}}{x - 1 - 3i} + \frac{\frac{3}{4} - \frac{i}{12}}{x - 1 + 3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) \mathrm{e}^{\frac{3 \ln\left(x^2 - 2x + 10\right)}{4} - \frac{\arctan\left(-\frac{1}{3} + \frac{x}{3}\right)}{6}} \\
&= \frac{(3x - 4)\left(x^2 - 2x + 10\right)^{\frac{3}{4}} \mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3} + \frac{x}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\right) \\
&\quad + c_2 \left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2 + \frac{6}{205}x - \frac{9}{82}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2 + \frac{6}{205}x - \frac{9}{82}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^2 - \frac{4}{3}x + 5\right) + c_2(3x-4)\sqrt{x^2-2x+10}\left(\frac{-x+1+3i}{x-1+3i}\right)^{\frac{i}{6}}$$

✓ Solution by Mathematica

Time used: 0.672 (sec). Leaf size: 92

```
DSolve[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2-2x+10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2\int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2\left(K[1]^2-2K[1]+10\right)^{3/2}}dK[1] + c_1\right)$$

2.745 problem 761

Internal problem ID [8235]
Internal file name [OUTPUT/7168_Sunday_June_05_2022_05_33_38_PM_66523082/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 761.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2-2x+10)\,y''+xy'-4y=0}$$

Writing the ode as

$$(x^2-2x+10)\,y''+xy'-4y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-2x+10 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15x^2 - 32x + 180 \\ t &= 4\left(x^2 - 2x + 10\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15x^2 - 32x + 180}{4\left(x^2 - 2x + 10\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x + 10)^2$. There is a pole at $x = 1 + 3i$ of order 2. There is a pole at $x = 1 - 3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{-\frac{7}{36} + \frac{i}{24}}{(x - 1 - 3i)^2} + \frac{-\frac{7}{36} - \frac{i}{24}}{(x - 1 + 3i)^2} - \frac{149i}{216\,(x - 1 - 3i)} + \frac{149i}{216\,(x - 1 + 3i)}$$

For the pole at $x = 1 + 3i$ let b be the coefficient of $\frac{1}{(x-1-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} + \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} + \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} - \frac{i}{12}
\end{aligned}$$

For the pole at $x = 1 - 3i$ let b be the coefficient of $\frac{1}{(x-1+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{7}{36} - \frac{i}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} - \frac{i}{12} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} + \frac{i}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15x^2 - 32x + 180}{4\,(x^2 - 2x + 10)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{15x^2-32x+180}{4\left(x^2-2x+10\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$1+3i$	2	0	$\frac{3}{4}+\frac{i}{12}$	$\frac{1}{4}-\frac{i}{12}$
$1-3i$	2	0	$\frac{3}{4}-\frac{i}{12}$	$\frac{1}{4}+\frac{i}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+-\left(\alpha_{c_1}^++\alpha_{c_2}^+\right) \\
&= \frac{5}{2}-\left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} + (0) \\
&= \frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i} \\
&= \frac{3x-4}{2x^2-4x+20}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right)(1) + \left(\left(\frac{-\frac{3}{4}-\frac{i}{12}}{(x-1-3i)^2} + \frac{-\frac{3}{4}+\frac{i}{12}}{(x-1+3i)^2}\right) + \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right.\right.
$$

$$
-\frac{3}{(-
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\left\{a_0 = -\frac{4}{3}\right\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - \frac{4}{3}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{4}{3}\right) e^{\int \left(\frac{\frac{3}{4}+\frac{i}{12}}{x-1-3i} + \frac{\frac{3}{4}-\frac{i}{12}}{x-1+3i}\right) dx} \\
&= \left(x - \frac{4}{3}\right) e^{\frac{3\ln\left(x^2-2x+10\right)}{4} - \frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= \frac{(3x-4)\left(x^2-2x+10\right)^{\frac{3}{4}} e^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{3}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2-2x+10}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2-2x+10\right)}{4}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}} \\
&= z_1\left(\frac{\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{6}}}{\left(x^2-2x+10\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2-2x+10}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln\left(x^2-2x+10\right)}{2}-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\right) \\
&\quad + c_2\left(\frac{(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3}\left(-\frac{27\,\mathrm{e}^{\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}\left(x^2-\frac{4}{3}x+5\right)}{\sqrt{x^2-2x+10}\,(1230x-1640)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2 + \frac{6}{205}x - \frac{9}{82}\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1(3x-4)\sqrt{x^2-2x+10}\,\mathrm{e}^{-\frac{\arctan\left(-\frac{1}{3}+\frac{x}{3}\right)}{3}}}{3} + c_2\left(-\frac{9}{410}x^2 + \frac{6}{205}x - \frac{9}{82}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve((x^2-2*x+10)*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^2 - \frac{4}{3}x + 5\right) + c_2(3x-4)\sqrt{x^2-2x+10}\left(\frac{-x+1+3i}{x-1+3i}\right)^{\frac{i}{6}}$$

✓ Solution by Mathematica

Time used: 0.579 (sec). Leaf size: 92

```
DSolve[(x^2-2*x+10)*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(3x - 4)\sqrt{x^2 - 2x + 10}e^{-\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)}\left(c_2\int_1^x \frac{9e^{\frac{1}{3}\arctan\left(\frac{1}{3}(K[1]-1)\right)}}{(4-3K[1])^2\left(K[1]^2-2K[1]+10\right)^{3/2}}dK[1] + c_1\right)$$

2.746 problem 762

2.746.1 Maple step by step solution 7003

Internal problem ID [8236]
Internal file name [OUTPUT/7169_Sunday_June_05_2022_05_33_41_PM_61456441/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 762.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Hermite]`

$$\boxed{y'' - xy' + 2y = 0}$$

Writing the ode as

$$y'' - xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 10}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 10 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{5}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1416: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{5}{2x} - \frac{25}{4x^3} - \frac{125}{4x^5} - \frac{3125}{16x^7} - \frac{21875}{16x^9} - \frac{328125}{32x^{11}} - \frac{2578125}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-10}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{5}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{5}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{5}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{5}{2}\right) - (0) \\
&= -\frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{5}{2}}{\frac{1}{2}} - 1\right) &&= 2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{5}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-3	2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 2$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (2) + 2\left(-\frac{x}{2}\right)(2x + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{5}{2}\right)\right) &= 0 \\ a_1 x + 2a_0 + 2 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{\int -\frac{x}{2} dx} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}} \\
&= \left(x^2 - 1\right) \mathrm{e}^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2 - 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2 - 1\right)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 - 1\right) + c_2 \left(x^2 - 1 \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{\left(x^2 - 1\right)^2} dx \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1(x^2-1)+c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right) \tag{1}$$

Verification of solutions

$$y = c_1(x^2-1)+c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x^2-1)^2}dx\right)$$

Verified OK.

2.746.1 Maple step by step solution

Let's solve

$y''-xy'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x\cdot y'$ to series expansion

    $x\cdot y'=\sum_{k=0}^{\infty} a_k k\, x^k$
  - Convert y'' to series expansion

    $y''=\sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$
  - Shift index using $k->k+2$

    $y''=\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)(k+1)-a_k(k-2))\, x^k=0$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right)a_{k+2}-a_k(k-2)=0$$

- Recursion relation; series terminates at $k=2$

$$a_{k+2}=\frac{a_k(k-2)}{k^2+3k+2}$$

- Apply recursion relation for $k=0$

$$a_2=-a_0$$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

$$y=A_2x^2+A_1x-a_0$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2-1) + c_2(x^2-1)\left(\int \frac{e^{\frac{x^2}{2}}}{(x-1)^2 (x+1)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 54

```
DSolve[y''[x]-x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}c_2\left(\sqrt{2\pi}(x^2-1)\operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) - 2e^{\frac{x^2}{2}}x\right) + c_1(x^2-1)$$

2.747 problem 763

2.747.1 Maple step by step solution 7013

Internal problem ID [8237]
Internal file name [`OUTPUT/7170_Sunday_June_05_2022_05_33_45_PM_44706727/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 763.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+2)\,y''+xy'-y=0}$$

Writing the ode as

$$(x+2)\,y''+xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x+2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{4(x+2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{4(x+2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1418: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{2}{(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{2}{x^2} - \frac{8}{x^3} + \frac{20}{x^4} - \frac{32}{x^5} + \frac{16}{x^6} + \frac{64}{x^7} - \frac{80}{x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{4x^2 + 16x + 16} \\ &= Q + \frac{R}{4x^2 + 16x + 16} \\ &= \left(\frac{1}{4}\right) + \left(\frac{8}{4x^2 + 16x + 16}\right) \\ &= \frac{1}{4} + \frac{8}{4x^2 + 16x + 16} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{4(x+2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+2} + (-)\left(\frac{1}{2}\right) \\
&= -\frac{1}{x+2} - \frac{1}{2} \\
&= -\frac{4+x}{2\,(x+2)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x+2} - \frac{1}{2}\right)(1) + \left(\left(\frac{1}{(x+2)^2}\right) + \left(-\frac{1}{x+2} - \frac{1}{2}\right)^2 - \left(\frac{x^2+4x+12}{4\,(x+2)^2}\right)\right) &= 0 \\
\frac{a_0 - 4}{x+2} &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 4\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 4 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (4+x)\, \mathrm{e}^{\int \left(-\frac{1}{x+2} - \frac{1}{2}\right) dx} \\
&= (4+x)\, \mathrm{e}^{-\frac{x}{2} - \ln(x+2)} \\
&= \frac{(4+x)\, \mathrm{e}^{-\frac{x}{2}}}{x+2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x+2}\,dx} \\
&= z_1 e^{-\frac{x}{2}+\ln(x+2)} \\
&= z_1\left((x+2)\,\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (4+x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x+2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+2\ln(x+2)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x\,\mathrm{e}^{x}}{4+x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((4+x)\,\mathrm{e}^{-x}\right) + c_2\left((4+x)\,\mathrm{e}^{-x}\left(\frac{x\,\mathrm{e}^{x}}{4+x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1(4+x)\,\mathrm{e}^{-x} + c_2 x \tag{1}$$

Verification of solutions

$$y = c_1(4+x)\,\mathrm{e}^{-x} + c_2 x$$

Verified OK.

2.747.1 Maple step by step solution

Let's solve

$(x+2)\, y''+x y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{x+2}-\frac{x y'}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{x y'}{x+2}-\frac{y}{x+2}=0$

□ Check to see if $x_0=-2$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{x}{x+2}, P_3(x)=-\frac{1}{x+2}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=-2$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if $x_0=-2$ is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $(x+2)\, y''+x y'-y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $u\left(\frac{d^2}{du^2}y(u)\right)+(u-2)\left(\frac{d}{du}y(u)\right)-y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) u^{k +r -1+m}$$

- Shift index using $k->k +1-m$

$$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right) = \sum_{k =-1+m}^{\infty} a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion

$$u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =0}^{\infty} a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k +1$

$$u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right) = \sum_{k =-1}^{\infty} a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r \left(-3+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty} \left(a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right)\right) u^{k +r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r \left(-3+r \right) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, 3\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k +1} \left(k +1+r \right) \left(k -2+r \right)+a_{k} \left(k +r -1\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1} = -\frac{a_{k} \left(k +r -1\right)}{\left(k +1+r \right) \left(k -2+r \right)}$$

- Recursion relation for $r =0$; series terminates at $k =1$

$$a_{k +1} = -\frac{a_{k} \left(k -1\right)}{\left(k +1\right) \left(k -2\right)}$$

- Apply recursion relation for $k =0$

$$a_{1} = -\frac{a_{0}}{2}$$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

$$y \left(u \right) = a_{0}\cdot \left(1-\frac{u}{2}\right)$$

- Revert the change of variables $u =x +2$

$$\left[y = -\frac{a_{0} x}{2}\right]$$

- Recursion relation for $r =3$

$$a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+2)^{k+3}, a_{k+1} = -\frac{a_k(k+2)}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = -\frac{a_0 x}{2} + \left(\sum_{k=0}^{\infty} b_k(x+2)^{k+3}\right), b_{k+1} = -\frac{b_k(k+2)}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x+2)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 e^{-x}(x+4)$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 72

```
DSolve[(x+2)*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2\sqrt{\frac{2}{\pi}}e^{-x-2}\sqrt{x+2}(c_1(e^{x+2}x+x+4)-ic_2((e^{x+2}-1)\,x-4))}{\sqrt{-i(x+2)}}$$

2.748 problem 764

Internal problem ID [8238]
Internal file name [`OUTPUT/7171_Sunday_June_05_2022_05_33_49_PM_42392033/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 764.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{(x^2+1)\,y''-6y=0}$$

Writing the ode as

$$(x^2+1)\,y''-6y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2 + 1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 + 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2 + 1}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1420: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2 + 1$. There is a pole at $x = i$ of order 1. There is a pole at $x = -i$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = i$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2 + 1}$$

Since the $\gcd(s, t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2 + 1}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	1	0	0	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 3$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\ &= 3 - (1) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x-i} + (0) \\ &= \frac{1}{x-i} \\ &= \frac{1}{x-i} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x-i}\right)(2x + a_1) + \left(\left(-\frac{1}{(x-i)^2}\right) + \left(\frac{1}{x-i}\right)^2 - \left(\frac{6}{x^2+1}\right) \right) = 0$$

$$2 + \frac{-4x - 2a_1}{-x+i} + \frac{-6x^2 - 6a_1 x - 6a_0}{x^2+1} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 + ix$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 + ix\right) e^{\int \frac{1}{x-i} dx} \\ &= \left(x^2 + ix\right) e^{\frac{\ln\left(x^2+1\right)}{2} + i \arctan(x)} \\ &= x(x+i)\,(ix+1) \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x(x+i)\,(ix+1) \end{aligned}$$

Which simplifies to

$$y_1 = i x^3 + ix$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= i x^3 + ix \int \frac{1}{\left(i x^3 + ix\right)^2} \, dx \\ &= i x^3 + ix \left(\frac{1}{x} + \frac{x}{2x^2 + 2} + \frac{3 \arctan(x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(i x^3 + i x) + c_2\left(i x^3 + i x\left(\frac{1}{x} + \frac{x}{2x^2+2} + \frac{3\arctan(x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(i x^3 + i x) + i c_2\left(1 + \frac{3(x^3+x)\arctan(x)}{2} + \frac{3x^2}{2}\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(i x^3 + i x) + i c_2\left(1 + \frac{3(x^3+x)\arctan(x)}{2} + \frac{3x^2}{2}\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((x^2+1)*diff(y(x),x$2)-6*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1(x^3 + x) + c_2\left(\frac{3\arctan(x)\,x^3}{2} + \frac{3\arctan(x)\,x}{2} + \frac{3x^2}{2} + 1\right)
$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 36

```
DSolve[(x^2+1)*y''[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x^3 + x) - \frac{1}{2}c_2(3(x^3 + x)\arctan(x) + 3x^2 + 2)$$

2.749 problem 765

Internal problem ID [8239]
Internal file name [`OUTPUT/7172_Sunday_June_05_2022_05_33_52_PM_30416257/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 765.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+2)\,y''+3xy'-y=0}$$

Writing the ode as

$$(x^2+2)\,y''+3xy'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+2 \\ B &= 3x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 20}{4\left(x^2 + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 20 \\ t &= 4\left(x^2 + 2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 20}{4\left(x^2 + 2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+2)^2$. There is a pole at $x = i\sqrt{2}$ of order 2. There is a pole at $x = -i\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\left(x - i\sqrt{2}\right)^2} - \frac{3}{16\left(x + i\sqrt{2}\right)^2} - \frac{17i\sqrt{2}}{32\left(x - i\sqrt{2}\right)} + \frac{17i\sqrt{2}}{32\left(x + i\sqrt{2}\right)}$$

For the pole at $x = i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x - i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

For the pole at $x = -i\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x + i\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{7x^2 + 20}{4\left(x^2+2\right)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{7}{4}$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
$i\sqrt{2}$	2	$\{1, 2, 3\}$
$-i\sqrt{2}$	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(2 - (1 + (1)))\\
&= 0
\end{aligned}$$

We now form the following rational function

$$\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c}\\
&= \frac{1}{2}\left(\frac{1}{\left(x-(i\sqrt{2})\right)} + \frac{1}{\left(x-(-i\sqrt{2})\right)}\right)\\
&= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}
\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{2x - 2i\sqrt{2}} + \frac{1}{2x + 2i\sqrt{2}}\right) w + \frac{7x^2 + 16}{4\left(\sqrt{2} + ix\right)^2 \left(x + i\sqrt{2}\right)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x + 2\sqrt{2x^2 + 4}}{2x^2 + 4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{x + 2\sqrt{2x^2+4}}{2x^2+4} dx}\\ &= \left(x^2 + 2\right)^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}\\ &= z_1 e^{-\int \frac{1}{2} \frac{3x}{x^2+2} \, dx}\\ &= z_1 e^{-\frac{3\ln\left(x^2+2\right)}{4}}\\ &= z_1 \left(\frac{1}{\left(x^2 + 2\right)^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2+2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln\left(x^2+2\right)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}}\right) + c_2 \left(\frac{e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} + \frac{c_2 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)}{\sqrt{x^2+2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} + \frac{c_2 e^{\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)} \left(\int \frac{e^{-2\sqrt{2}\, \operatorname{arcsinh}\left(\frac{\sqrt{2}\, x}{2}\right)}}{\sqrt{x^2+2}} dx\right)}{\sqrt{x^2+2}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 67

```
dsolve((x^2+2)*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1\left(\sqrt{2}\, x+\sqrt{2}\,\sqrt{x^2+2}\right)^{\sqrt{2}}}{\sqrt{x^2+2}}+\frac{c_2\left(\frac{\sqrt{2}}{2\sqrt{x^2+2}+2x}\right)^{\sqrt{2}}}{\sqrt{x^2+2}}$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 92

```
DSolve[(x^2+2)*y''[x]+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2^{3/4} c_1 \cos\left(2\sqrt{2}\arcsin\left(\frac{1}{2}\sqrt{2-i\sqrt{2}x}\right)\right)}{\sqrt{\pi}\sqrt{x^2+2}}+\frac{c_2 Q_{-\frac{1}{2}+\sqrt{2}}^{\frac{1}{2}}\left(\frac{ix}{\sqrt{2}}\right)}{\sqrt[4]{x^2+2}}$$

2.750 problem 766

2.750.1 Maple step by step solution 7037

Internal problem ID [8240]
Internal file name [OUTPUT/7173_Sunday_June_05_2022_05_33_56_PM_46166468/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 766.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\, y''-x y'+y=0}$$

Writing the ode as

$$(x-1)\, y''-x y'+y=0 \tag{1}$$
$$A y''+B y'+C y=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x-1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1422: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 2\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{0} a_i x^i \end{aligned}\tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x-1\right)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\left(x-1\right)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\left(x-1\right)^2}\right) + \left(-\frac{1}{2\left(x-1\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^2-4x+6}{4\left(x-1\right)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-x\, \mathrm{e}^{-x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x \left(-x\, \mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x - c_2 x$$

Verified OK.

2.750.1 Maple step by step solution

Let's solve

$y''(x-1) - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

□ Check to see if $x_0 = 1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = -1$

  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 0$

  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $y''(x-1) - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  - Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-1}$$

- Shift index using $k- > k+1$

  $$u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-2+r) u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_{k}(x-1)^{k}\right)+\left(\sum_{k=0}^{\infty} b_{k}(x-1)^{k+2}\right), a_{k+1}=\frac{a_{k}}{k+1}, b_{k+1}=\frac{b_{k}}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_{1} x+\mathrm{e}^{x} c_{2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_{1} e^{x}-c_{2} x$$

2.751 problem 767

Internal problem ID [8241]
Internal file name [`OUTPUT/7174_Sunday_June_05_2022_05_34_00_PM_2206756/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 767.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2xy' + 8y = 0}$$

Writing the ode as

$$y'' - 2xy' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 9)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1424: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - \frac{9}{2x} - \frac{81}{8x^3} - \frac{729}{16x^5} - \frac{32805}{128x^7} - \frac{413343}{256x^9} - \frac{11160261}{1024x^{11}} - \frac{157837977}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-9}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2-9) + (0) \\ &= x^2 - 9 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -9. Now b can be found.

$$\begin{aligned} b &= (-9) - (0) \\ &= -9 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-9}{1} - 1\right) &= -5 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-9}{1} - 1\right) &= 4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 - 9$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	-5	4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 4$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 4 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)(x) \\
&= -x \\
&= -x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 4$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(12x^2 + 6xa_3 + 2a_2) + 2(-x)\left(4x^3 + 3x^2 a_3 + 2xa_2 + a_1\right) + \left((-1) + (-x)^2 - (x^2 - 9)\right) = 0 \\
2a_3 x^3 + 4(3 + a_2)\, x^2 + 6(a_1 + a_3)\, x + 8a_0 + 2a_2 = 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = \frac{3}{4}, a_1 = 0, a_2 = -3, a_3 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^4 - 3x^2 + \frac{3}{4}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{\int -x dx} \\
&= \left(x^4 - 3x^2 + \frac{3}{4}\right) e^{-\frac{x^2}{2}} \\
&= \frac{(4x^4 - 12x^2 + 3)\, e^{-\frac{x^2}{2}}}{4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^4 - 3x^2 + \frac{3}{4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + c_2\left(x^4 - 3x^2 + \frac{3}{4}\left(\int \frac{16\,\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2\left(4x^4 - 12x^2 + 3\right)\left(\int \frac{\mathrm{e}^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = c_1\left(x^4 - 3x^2 + \frac{3}{4}\right) + 4c_2(4x^4 - 12x^2 + 3)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 54

```
dsolve(diff(y(x),x$2)-2*x*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{4}{3}x^4 - 4x^2 + 1\right) + c_2\left(\frac{4}{3}x^4 - 4x^2 + 1\right)\left(\int \frac{e^{x^2}}{(4x^4 - 12x^2 + 3)^2}dx\right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 49

```
DSolve[y''[x]-2*x*y''[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{4x-2}\left(c_1 \operatorname{BesselI}\left(1, 4\sqrt{x-\frac{1}{2}}\right) - c_2 K_1\left(4\sqrt{x-\frac{1}{2}}\right)\right)$$

2.752 problem 769

2.752.1 Maple step by step solution 7056

Internal problem ID [8242]
Internal file name [`OUTPUT/7175_Sunday_June_05_2022_05_34_03_PM_84194817/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 769.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0}$$

Writing the ode as

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= \frac{5}{3}x + x^2 \\ C &= -\frac{1}{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 30x + 7}{36x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9x^2 + 30x + 7$$
$$t = 36x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 30x + 7}{36x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{5}{6x} + \frac{7}{36x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{7}{36}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{6} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{6} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{5}{6x} - \frac{1}{2x^2} + \frac{5}{6x^3} - \frac{59}{36x^4} + \frac{385}{108x^5} - \frac{2681}{324x^6} + \frac{19525}{972x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 30x + 7}{36x^2} \\ &= Q + \frac{R}{36x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{30x + 7}{36x^2}\right) \\ &= \frac{1}{4} + \frac{30x + 7}{36x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 30. Dividing this by leading coefficient in t which is 36 gives $\frac{5}{6}$. Now b can be found.

$$\begin{aligned}
b &= \left(\frac{5}{6}\right) - (0) \\
&= \frac{5}{6}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= \frac{5}{6} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{6}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 30x + 7}{36x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= \frac{5}{6} - \left(-\frac{1}{6}\right) \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{6x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{6x} + \frac{1}{2} \\
&= -\frac{1}{6x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{6x} + \frac{1}{2}\right)(1) + \left(\left(\frac{1}{6x^2}\right) + \left(-\frac{1}{6x} + \frac{1}{2}\right)^2 - \left(\frac{9x^2 + 30x + 7}{36x^2}\right)\right) = 0$$

$$\frac{-1 - 3a_0}{3x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{1}{3}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{1}{3}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\int \left(-\frac{1}{6x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{1}{3}\right) \mathrm{e}^{\frac{x}{2} - \frac{\ln(x)}{6}} \\
&= \frac{(3x - 1)\, \mathrm{e}^{\frac{x}{2}}}{3x^{\frac{1}{6}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{5}{3}x + x^2}{x^2} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{5\ln(x)}{6}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{x^{\frac{5}{6}}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{3x - 1}{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{5}{3}x + x^2}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x - \frac{5\ln(x)}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{9x^{\frac{1}{3}} \mathrm{e}^{-x}}{(3x - 1)^2} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{3x-1}{3x}\right) + c_2\left(\frac{3x-1}{3x}\left(\int \frac{9x^{\frac{1}{3}}\mathrm{e}^{-x}}{\left(3x-1\right)^{2}}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(3x-1)}{3x} + \frac{c_2(9x-3)\left(\int \frac{x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(3x-1)}{3x} + \frac{c_2(9x-3)\left(\int \frac{x^{\frac{1}{3}}\mathrm{e}^{-x}}{(3x-1)^2}dx\right)}{x}
$$

Verified OK.

2.752.1 Maple step by step solution

Let's solve

$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{y}{3} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{3x^2} - \frac{(5+3x)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(5+3x)y'}{3x} - \frac{y}{3x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5+3x}{3x}, P_3(x) = -\frac{1}{3x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{5}{3}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{3}$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $3x^2y'' + x(5+3x)\,y' - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ○ Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ○ Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+3r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(3k+3r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{1}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $3(k+r+1)\left(k+r-\frac{1}{3}\right)a_k + 3a_{k-1}(k+r-1) = 0$

- Shift index using $k->k+1$

  $3(k+2+r)\left(k+\frac{2}{3}+r\right)a_{k+1} + 3a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = -\frac{3a_k(k+r)}{(k+2+r)(3k+2+3r)}$

- Recursion relation for $r = -1$; series terminates at $k = 1$

  $a_{k+1} = -\frac{3a_k(k-1)}{(k+1)(3k-1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -3a_0$

- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot (-3x + 1)$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = -\frac{3a_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (-3x + 1) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), b_{k+1} = -\frac{3b_k\left(k+\frac{1}{3}\right)}{\left(k+\frac{7}{3}\right)(3k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 42

```
dsolve(x^2*diff(y(x),x$2)+(5/3*x+x^2)*diff(y(x),x)-1/3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(3x-1)}{x} + \frac{c_2(3x-1)\left(\int \frac{x^{\frac{1}{3}} e^{-x}}{(3x-1)^2} dx\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 47

```
DSolve[x^2*y''[x]+(5/3*x+x^2)*y'[x]-1/3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-3c_1x + 3c_2e^{-x}\sqrt[3]{x} + c_2(1-3x)\Gamma\left(\frac{1}{3},x\right) + c_1}{3x}$$

2.753 problem 770

2.753.1 Maple step by step solution 7066

Internal problem ID [8243]
Internal file name [`OUTPUT/7176_Sunday_June_05_2022_05_34_07_PM_95097163/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 770.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{2xy'' - y' + 2y = 0}$$

Writing the ode as

$$2xy'' - y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -1 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\}\\ &= \{-1, 2, 5\}\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(1 - (-1))\\ &= 1\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)\\ &= -\frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2-12x+1}{64x^3+16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2x} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{1}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\right) \\
&\quad + c_2 \left(\frac{e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}}\left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{c_1 e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
& + \frac{c_2 e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{c_1 e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}} \\
& + \frac{c_2 e^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\, e^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}}
\end{aligned}
$$

Verified OK.

2.753.1 Maple step by step solution

Let's solve

$2xy'' - y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2x} + \frac{y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $2xy'' - y' + 2y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$
  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$

    Rewrite ODE with series expansions

$$a_0 r(-3+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-3+2r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, \frac{3}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $2\left(k-\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + 2a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{2a_k}{(2k-1+2r)(k+1+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}$
- Solution for $r = 0$
  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}\right]$
- Recursion relation for $r = \frac{3}{2}$
  $a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$
- Solution for $r = \frac{3}{2}$
  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$
- Combine solutions and rename parameters
  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+1} = -\frac{2a_k}{(2k-1)(k+1)}, b_{k+1} = -\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve(2*x*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(2i\sqrt{x}-1\right)}{1+2i\sqrt{x}}} + c_2 e^{-2i\sqrt{x}} \sqrt{\frac{(1+4x)\left(1+2i\sqrt{x}\right)}{2i\sqrt{x}-1}}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 59

```
DSolve[2*x*y''[x]-y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2i\sqrt{x}}\left(2\sqrt{x}+i\right) + \frac{1}{8} c_2 e^{-2i\sqrt{x}}\left(1+2i\sqrt{x}\right)$$

2.754 problem 771

2.754.1 Maple step by step solution 7077

Internal problem ID [8244]
Internal file name [`OUTPUT/7177_Sunday_June_05_2022_05_34_10_PM_40683000/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 771.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{2xy'' - (3 + 2x)\, y' + y = 0}$$

Writing the ode as

$$2xy'' + (-2x - 3)\, y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= -2x - 3 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x + 21}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x + 21 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x + 21}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{1}{4x} + \frac{21}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{7}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{4}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{1}{4x} + \frac{5}{4x^2} - \frac{5}{8x^3} - \frac{5}{4x^4} + \frac{35}{16x^5} + \frac{105}{64x^6} - \frac{1005}{128x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2+4x+21}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{4x+21}{16x^2}\right) \\ &= \frac{1}{4} + \frac{4x+21}{16x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &= \frac{1}{4} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{2}} - 0\right) &= -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 4x + 21}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = \frac{1}{4}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{4} - \left(-\frac{3}{4}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{4x} + \left(\frac{1}{2}\right) \\
&= -\frac{3}{4x} + \frac{1}{2} \\
&= -\frac{3}{4x} + \frac{1}{2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{4x} + \frac{1}{2}\right)(1) + \left(\left(\frac{3}{4x^2}\right) + \left(-\frac{3}{4x} + \frac{1}{2}\right)^2 - \left(\frac{4x^2 + 4x + 21}{16x^2}\right)\right) = 0$$

$$\frac{-3 - 2a_0}{2x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{3}{2}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - \frac{3}{2}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x - \frac{3}{2}\right) e^{\int \left(-\frac{3}{4x} + \frac{1}{2}\right) dx} \\
&= \left(x - \frac{3}{2}\right) e^{\frac{x}{2} - \frac{3 \ln(x)}{4}} \\
&= \frac{(2x - 3)\, e^{\frac{x}{2}}}{2x^{\frac{3}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x-3}{2x}\,dx} \\
&= z_1 e^{\frac{x}{2}+\frac{3\ln(x)}{4}} \\
&= z_1 \left(x^{\frac{3}{4}} \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(2x-3)\,\mathrm{e}^{x}}{2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-3}{2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x+\frac{3\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{4x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(2x-3)\,\mathrm{e}^{x}}{2}\right) + c_2 \left(\frac{(2x-3)\,\mathrm{e}^{x}}{2} \left(\int \frac{4x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1(2x-3)\,\mathrm{e}^{x}}{2} + c_2(4x-6)\,\mathrm{e}^{x} \left(\int \frac{x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^2}dx\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(2x-3)\,\mathrm{e}^{x}}{2} + c_2(4x-6)\,\mathrm{e}^{x}\left(\int \frac{x^{\frac{3}{2}}\mathrm{e}^{-x}}{(2x-3)^{2}}dx\right)$$

Verified OK.

2.754.1 Maple step by step solution

Let's solve

$2xy'' + (-2x-3)\,y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{2x} + \frac{(3+2x)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(3+2x)y'}{2x} + \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{3+2x}{2x}, P_3(x) = \frac{1}{2x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{3}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' + (-2x-3)\,y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-5+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k-3+2r) - a_k(2k+2r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{5}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+r-\frac{3}{2}\right)(k+1+r)\, a_{k+1} - 2\left(k-\frac{1}{2}+r\right) a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{(2k+2r-1)a_k}{(2k-3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{(2k-1)a_k}{(2k-3)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{(2k-1)a_k}{(2k-3)(k+1)}\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $$a_{k+1} = \frac{(2k+4)a_k}{(2k+2)\left(k+\frac{7}{2}\right)}$$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+1} = \frac{(2k+4) a_k}{(2k+2)\left(k+\frac{7}{2}\right)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+1} = \frac{(2k-1) a_k}{(2k-3)(k+1)}, b_{k+1} = \frac{(2k+4) b_k}{(2k+2)\left(k+\frac{7}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(2*x*diff(y(x),x$2)-(3+2*x)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \mathrm{e}^{x}(-3+2x)}{2} + c_2 \mathrm{e}^{x}(-3+2x)\left(\int \frac{x^{\frac{3}{2}}\mathrm{e}^{-x}}{(-3+2x)^{2}}dx\right)$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 54

```
DSolve[2*x*y''[x]-(3+2*x)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{\pi}c_2 e^{x}(2x-3)\mathrm{erf}\left(\sqrt{x}\right) + 2c_1 e^{x}(2x-3) - 6c_2\sqrt{x}\right)$$

2.755 problem 772

2.755.1 Maple step by step solution 7086

Internal problem ID [8245]
Internal file name [`OUTPUT/7178_Sunday_June_05_2022_05_34_14_PM_69163666/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 772.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + 3xy' + (2x - 1)\, y = 0}$$

Writing the ode as

$$2x^2y'' + 3xy' + (2x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5 - 16x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 5 - 16x \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5 - 16x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x} + \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (-1)) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))}\right)\\ &= -\frac{1}{2x}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{1-4a_0}{x^2} = 0$$

And solving for p gives

$$p = \frac{1}{4} + x$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{1}{\frac{1}{4}+x} - \frac{1}{2x}\right)w + \frac{64x^2-12x+1}{64x^3+16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{16x\sqrt{-x}+4x-1}{4\left(1+4x\right)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{16x\sqrt{-x}+4x-1}{4(1+4x)x} dx} \\
&= \frac{\sqrt{2\sqrt{-x}-1}\,\sqrt{1+4x}\,\mathrm{e}^{2\sqrt{-x}}}{\sqrt{2\sqrt{-x}+1}\,x^{\frac{1}{4}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\, dx} \\
&= z_1 e^{-\frac{3\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)} dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\right) \\
&\quad + c_2 \left(\frac{\mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {}& \frac{c_1 \mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x} \\
&+ \frac{c_2 \mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}\,x}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {}& \frac{c_1 \mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}}{\sqrt{2\sqrt{-x}+1}\,x} \\
&+ \frac{c_2 \mathrm{e}^{2\sqrt{-x}}\sqrt{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}\left(\int \frac{\sqrt{x}\,\mathrm{e}^{-4\sqrt{-x}}\left(2\sqrt{-x}+1\right)}{\left(2\sqrt{-x}-1\right)\left(1+4x\right)}dx\right)}{\sqrt{2\sqrt{-x}+1}\,x}
\end{aligned}
$$

Verified OK.

2.755.1 Maple step by step solution

Let's solve

$2x^2 y'' + 3xy' + (2x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2x} - \frac{(2x-1)y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} + \frac{(2x-1)y}{2x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{2x}, P_3(x) = \frac{2x-1}{2x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{3}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{1}{2}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2y'' + 3xy' + (2x-1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(2k+2r-1) + 2a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-1,\frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k-\frac{1}{2}+r\right)(k+r+1)\,a_k+2a_{k-1}=0$

- Shift index using $k->k+1$

  $2\left(k+\frac{1}{2}+r\right)(k+2+r)\,a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k}{(2k+1+2r)(k+2+r)}$

- Recursion relation for $r=-1$

  $a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k-1}, a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{2a_k}{(2k-1)(k+1)}, b_{k+1}=-\frac{2b_k}{(2k+2)\left(k+\frac{5}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 81

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+(2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{2i\sqrt{x}}\sqrt{\frac{(1+4x)(2i\sqrt{x}-1)}{1+2i\sqrt{x}}}}{x} + \frac{c_2 e^{-2i\sqrt{x}}\sqrt{\frac{(1+4x)(1+2i\sqrt{x})}{2i\sqrt{x}-1}}}{x}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 64

```
DSolve[2*x^2*y''[x]+3*x*y'[x]+(2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-2i\sqrt{x}}\left(8c_1 e^{4i\sqrt{x}}\left(2\sqrt{x}+i\right)+c_2\left(1+2i\sqrt{x}\right)\right)}{8x}$$

2.756 problem 773

2.756.1 Maple step by step solution 7093

Internal problem ID [8246]
Internal file name [OUTPUT/7179_Sunday_June_05_2022_05_34_18_PM_17766453/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 773.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' - yx = 0}$$

Writing the ode as

$$xy'' + 2y' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-x}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^{-x}}{x}\right) + c_2\left(\frac{e^{-x}}{x}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x}
$$

Verified OK.

2.756.1 Maple step by step solution

Let's solve

$xy'' + 2y' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' - yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- > k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) - a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)-a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)-a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sinh(x)}{x} + \frac{c_2 \cosh(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 28

```
DSolve[x*y''[x]+2*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-x} + c_2 e^x}{2x}$$

2.757 problem 774

2.757.1 Maple step by step solution 7100

Internal problem ID [8247]
Internal file name [OUTPUT/7180_Sunday_June_05_2022_05_34_20_PM_75796176/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 774.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.757.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.758 problem 775

2.758.1 Maple step by step solution 7111

Internal problem ID [8248]
Internal file name [OUTPUT/7181_Sunday_June_05_2022_05_34_23_PM_38825999/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 775.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (x-6)\,y' - 3y = 0}$$

Writing the ode as

$$xy'' + (x-6)\,y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= x-6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 48}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 48 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 48}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} + \frac{12}{x^2} - \frac{144}{x^4} + \frac{3456}{x^6} - \frac{103680}{x^8} + \frac{3483648}{x^{10}} - \frac{125411328}{x^{12}} + \frac{4729798656}{x^{14}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 48}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{12}{x^2}\right) \\
&= \frac{1}{4} + \frac{12}{x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 4 gives 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{1}{2}} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{1}{2}} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 48}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-3) \\
&= 3
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{x} + (-)\left(\frac{1}{2}\right) \\ &= -\frac{3}{x} - \frac{1}{2} \\ &= -\frac{x+6}{2x}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 3$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(6x + 2a_2) + 2\left(-\frac{3}{x} - \frac{1}{2}\right)(3x^2 + 2a_2 x + a_1) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x} - \frac{1}{2}\right)^2 - \left(\frac{x^2+48}{4x^2}\right)\right) = 0$$

$$\frac{(a_2 - 12)\, x^2 + 2(a_1 - 5a_2)\, x + 3a_0 - 6a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 120, a_1 = 60, a_2 = 12\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^3 + 12x^2 + 60x + 120$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= (x^3 + 12x^2 + 60x + 120)\, e^{\int \left(-\frac{3}{x} - \frac{1}{2}\right) dx} \\ &= (x^3 + 12x^2 + 60x + 120)\, e^{-\frac{x}{2} - 3\ln(x)} \\ &= \frac{(x^3 + 12x^2 + 60x + 120)\, e^{-\frac{x}{2}}}{x^3}\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x-6}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}+3\ln(x)} \\
&= z_1\left(x^3 \mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \left(x^3+12x^2+60x+120\right)\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x-6}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x+6\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{x}(x^3-12x^2+60x-120)}{x^3+12x^2+60x+120}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\left(x^3+12x^2+60x+120\right)\mathrm{e}^{-x}\right) \\
&\quad + c_2\left(\left(x^3+12x^2+60x+120\right)\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{x}(x^3-12x^2+60x-120)}{x^3+12x^2+60x+120}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(x^3+12x^2+60x+120\right)\mathrm{e}^{-x} + c_2\left(x^3-12x^2+60x-120\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(x^3+12x^2+60x+120\right)\mathrm{e}^{-x} + c_2\left(x^3-12x^2+60x-120\right)
$$

Verified OK.

2.758.1 Maple step by step solution

Let's solve

$xy'' + (x-6)\,y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x} - \frac{(x-6)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-6)y'}{x} - \frac{3y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x-6}{x}, P_3(x) = -\frac{3}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (x-6)\,y' - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-7+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 7\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k-6+r) + a_k(k+r-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k(k+r-3)}{(k+1+r)(k-6+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 3$

$$a_{k+1} = -\frac{a_k(k-3)}{(k+1)(k-6)}$$

- Apply recursion relation for $k = 0$

$$a_1 = -\frac{a_0}{2}$$

- Apply recursion relation for $k = 1$

$$a_2 = -\frac{a_1}{5}$$

- Express in terms of a_0

$$a_2 = \frac{a_0}{10}$$

- Apply recursion relation for $k = 2$

$$a_3 = -\frac{a_2}{12}$$

- Express in terms of a_0

$$a_3 = -\frac{a_0}{120}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)$

- Recursion relation for $r = 7$

  $a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}$

- Solution for $r = 7$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+7}, a_{k+1} = -\frac{a_k(k+4)}{(k+8)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+7}\right), b_{k+1} = -\frac{b_k(k+4)}{(k+8)(k+1)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(x*diff(y(x),x$2)+(x-6)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(x^3 - 12x^2 + 60x - 120\right) + c_2 e^{-x}\left(x^3 + 12x^2 + 60x + 120\right)$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 98

```
DSolve[x*y''[x]+(x-6)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^{-x/2}\sqrt{x}\left(\left(c_1x^3+12ic_2x^2+60c_1x+120ic_2\right)\cosh\left(\frac{x}{2}\right)-\left(12c_1(x^2+10)+ic_2x(x^2+60)\right)\sinh\left(\frac{x}{2}\right)\right)}{\sqrt{\pi}\sqrt{-ix}}$$

2.759 problem 776

Internal problem ID [8249]
Internal file name [`OUTPUT/7182_Sunday_June_05_2022_05_34_26_PM_71828463/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 776.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x^4y'' + \lambda y = 0}$$

Writing the ode as

$$x^4y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{\lambda}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$$

The partial fraction decomposition of r is

$$r = -\frac{\lambda}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i\sqrt{\lambda}}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i\sqrt{\lambda}}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i\sqrt{\lambda}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i\sqrt{\lambda}}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i\sqrt{\lambda}} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{\lambda}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i\sqrt{\lambda}}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x} \\
&= \frac{-i\sqrt{\lambda} + x}{x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i\sqrt{\lambda}}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{\lambda}{x^4}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{i\sqrt{\lambda}}{x^2} + \frac{1}{x}\right) dx} \\
&= x\, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \int \frac{1}{x^2 \mathrm{e}^{\frac{2i\sqrt{\lambda}}{x}}}\,dx \\ &= x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} \left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\right) + c_2\left(x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x\,\mathrm{e}^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}} \tag{1}$$

Verification of solutions

$$y = c_1 x \, \mathrm{e}^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x \, \mathrm{e}^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(x^4*diff(y(x),x$2)+lambda*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sinh\left(\frac{\sqrt{-\lambda}}{x}\right) + c_2 x \cosh\left(\frac{\sqrt{-\lambda}}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 52

```
DSolve[x^4*y''[x]+\[Lambda]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x e^{\frac{i\sqrt{\lambda}}{x}} - \frac{i c_2 x e^{-\frac{i\sqrt{\lambda}}{x}}}{2\sqrt{\lambda}}$$

2.760 problem 777

2.760.1 Maple step by step solution 7129

Internal problem ID [8250]
Internal file name [`OUTPUT/7183_Sunday_June_05_2022_05_34_30_PM_4233400/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 777.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4x^2y''+4xy'+\left(4x^2-25\right)y=0}$$

Writing the ode as

$$4x^2y''+4xy'+\left(4x^2-25\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x \\ C &= 4x^2-25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1440: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v=0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2+6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{i} - 0\right) = 0 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{i} - 0\right) = 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(i) \\ &= -\frac{2}{x} - i \\ &= -\frac{2}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) e^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) e^{-ix - 2\ln(x)} \\
&= \frac{\left(x^2 - 3ix - 3\right) e^{-ix}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4x}{4x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\right) + c_2\left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}}\left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2\mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^{\frac{5}{2}}} - \frac{c_2\mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^{\frac{5}{2}}}$$

Verified OK.

2.760.1 Maple step by step solution

Let's solve

$4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-25)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-25)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-25}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{25}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 25)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x\cdot y'$ to series expansion

  $$x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^{2}\cdot y''$ to series expansion

  $$x^{2}\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(5+2r)\,(-5+2r)\, x^{r}+a_1(7+2r)\,(-3+2r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+5)\,(2k+2r-5)+4a_{\cdot}$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(5+2r)\,(-5+2r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{-\frac{5}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(7+2r)\,(-3+2r)=0$$

- Solve for the dependent coefficient(s)

  $$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r+5)\,(2k+2r-5)+4a_{k-2}=0$$

- Shift index using $k- >k+2$

  $$a_{k+2}(2k+9+2r)\,(2k-1+2r)+4a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{4a_k}{(2k+9+2r)(2k-1+2r)}$$

- Recursion relation for $r=-\frac{5}{2}$

  $$a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}$$

- Solution for $r=-\frac{5}{2}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}, a_{k+2}=-\frac{4a_k}{(2k+4)(2k-6)}, a_1=0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}$
- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = -\frac{4a_k}{(2k+14)(2k+4)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{5}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = -\frac{4a_k}{(2k+4)(2k-6)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(2k+14)(2k+4)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(4*x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2-25)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{ix}(x^2 + 3ix - 3)}{x^{\frac{5}{2}}} + \frac{c_2 e^{-ix}(x^2 - 3ix - 3)}{x^{\frac{5}{2}}}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 59

```
DSolve[4*x^2*y''[x]+4*x*y'[x]+(4*x^2-25)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{2}{\pi}}((-c_2x^2 + 3c_1x + 3c_2)\cos(x) + (c_1(x^2 - 3) + 3c_2x)\sin(x))}{x^{5/2}}$$

2.761 problem 778

2.761.1 Maple step by step solution 7136

Internal problem ID [8251]
Internal file name [`OUTPUT/7184_Sunday_June_05_2022_05_34_34_PM_4795842/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 778.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$$

Writing the ode as

$$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 36x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -36$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -36z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1442: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(6x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(6x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(6x)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\cos(6x)}{\sqrt{x}}\right) + c_2\left(\frac{\cos(6x)}{\sqrt{x}}\left(\frac{\tan(6x)}{6}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1\cos(6x)}{\sqrt{x}} + \frac{c_2\sin(6x)}{6\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\cos(6x)}{\sqrt{x}} + \frac{c_2\sin(6x)}{6\sqrt{x}}$$

Verified OK.

2.761.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(144x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(144x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{144x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (144x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 144a\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 144a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 144a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{144a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{144a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{144a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{144a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{144b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(36*x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(6x)}{\sqrt{x}} + \frac{c_2 \cos(6x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(36*x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6ix}(12c_1 - ic_2e^{12ix})}{12\sqrt{x}}$$

2.762 problem 779

2.762.1 Maple step by step solution 7147

Internal problem ID [8252]
Internal file name [OUTPUT/7185_Sunday_June_05_2022_05_34_37_PM_94120071/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 779.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y(x^2 - 2) = 0}$$

Writing the ode as

$$x^2y'' + y(x^2 - 2) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1444: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{i}{x^2} - \frac{i}{2x^4} - \frac{i}{2x^6} - \frac{5i}{8x^8} - \frac{7i}{8x^{10}} - \frac{21i}{16x^{12}} - \frac{33i}{16x^{14}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{2}{x^2}\right) \\ &= -1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= i \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(i) \\ &= -\frac{1}{x} - i \\ &= -\frac{1}{x} - i \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x} - i\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - i\right)^2 - \left(\frac{-x^2 + 2}{x^2}\right)\right) = 0$$
$$\frac{2ia_0 - 2}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x - i$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (x - i)\, e^{\int \left(-\frac{1}{x} - i\right) dx} \\ &= (x - i)\, e^{-ix - \ln(x)} \\ &= \frac{(x - i)\, e^{-ix}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(x - i)\, e^{-ix}}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x-i)\,\mathrm{e}^{-ix}}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{(x-i)\,\mathrm{e}^{-ix}}{x} \int \frac{1}{\frac{(x-i)^2\mathrm{e}^{-2ix}}{x^2}}\,dx \\
&= \frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\right) + c_2\left(\frac{(x-i)\,\mathrm{e}^{-ix}}{x}\left(\frac{(ix-1)\,\mathrm{e}^{2ix}}{-2x+2i}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x-i)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x-i)\,\mathrm{e}^{-ix}}{x} - \frac{c_2(ix-1)\,\mathrm{e}^{ix}}{2x}$$

Verified OK.

2.762.1 Maple step by step solution

Let's solve

$x^2y'' + y(x^2 - 2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y(x^2-2)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y(x^2-2)}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{x^2-2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $x^2y'' + y(x^2 - 2) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- > k-m$

  $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{2} \cdot y''$ to series expansion

  $x^{2} \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)(-2+r) x^{r} + a_1(2+r)(-1+r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+1)(k+r-2) + a_{k-2}) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 2\}$

- Each term must be 0

  $a_1(2+r)(-1+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-2) + a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}(k+3+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-1)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+2)(k-1)}, a_1 = 0\right]$

- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{a_k}{(k+5)(k+2)}$

- Solution for $r = 2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+5)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+5)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(x^2*diff(y(x),x$2)+(x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1(-\sin(x)+\cos(x)\,x)}{x}+\frac{c_2(\cos(x)+x\sin(x))}{x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+(x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x(c_1 j_1(x)-c_2 y_1(x))$$

2.763 problem 780

2.763.1 Maple step by step solution 7156

Internal problem ID [8253]
Internal file name [`OUTPUT/7186_Sunday_June_05_2022_05_34_40_PM_57568519/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 780.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{xy'' + 3y' + yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1446: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx ix - \frac{3i}{8x^3} - \frac{9i}{128x^7} - \frac{27i}{1024x^{11}} - \frac{405i}{32768x^{15}} - \frac{1701i}{262144x^{19}} - \frac{15309i}{4194304x^{23}} - \frac{72171i}{33554432x^{27}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = -x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-4x^4+3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-x^2) + \left(\frac{3}{4x^2}\right) \\ &= -x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= ix \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{i} - 1\right) &= -\frac{1}{2} \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{i} - 1\right) &= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-4x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	ix	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(ix) \\ &= -\frac{1}{2x} - ix \\ &= -\frac{1}{2x} - ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - ix\right)(0) + \left(\left(\frac{1}{2x^2} - i\right) + \left(-\frac{1}{2x} - ix\right)^2 - \left(\frac{-4x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - ix\right) dx} \\ &= \frac{e^{-\frac{ix^2}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x}\, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{ix^2}{2}}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{i\,\mathrm{e}^{i x^2}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-\frac{i x^2}{2}}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{i x^2}{2}}}{x^2} \left(-\frac{i\,\mathrm{e}^{i x^2}}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{2}}}{x^2} - \frac{i c_2 \mathrm{e}^{\frac{i x^2}{2}}}{2 x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-\frac{i x^2}{2}}}{x^2} - \frac{i c_2 \mathrm{e}^{\frac{i x^2}{2}}}{2 x^2}
$$

Verified OK.

2.763.1 Maple step by step solution

Let's solve

$x y'' + 3 y' + y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3 y'}{x} - x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{3y'}{x} + x^2 y = 0$$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = \frac{3}{x}, P_3(x) = x^2\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$xy'' + 3y' + yx^3 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^3 \cdot y$ to series expansion

$$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- ○ Shift index using $k- >k-3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- ○ Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r) x^{-1+r} + a_1(1+r)(3+r) x^r + a_2(2+r)(4+r) x^{1+r} + a_3(3+r)(5+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r+3) + a_{k-3} = 0$$

- Shift index using $k->k+3$

$$a_{k+4}(k+4+r)(k+6+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$$a_{k+4} = -\frac{a_k}{(k+2)(k+4)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{a_k}{(k+4)(k+6)}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin\left(\frac{x^2}{2}\right)}{x^2}+\frac{c_2 \cos\left(\frac{x^2}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 43

```
DSolve[x*y''[x]+3*y'[x]+x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-\frac{ix^2}{2}}\left(2c_1-ic_2e^{ix^2}\right)}{2x^2}$$

2.764 problem 781

2.764.1 Maple step by step solution 7163

Internal problem ID [8254]
Internal file name [`OUTPUT/7187_Sunday_June_05_2022_05_34_44_PM_68432895/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 781.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + (x^2+2)\,y = 0}$$

Writing the ode as

$$x^2y'' + 4xy' + (x^2+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1448: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\cos(x)}{x^2} + \frac{c_2\sin(x)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\cos(x)}{x^2} + \frac{c_2\sin(x)}{x^2}
$$

Verified OK.

2.764.1 Maple step by step solution

Let's solve

$x^2y'' + 4xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 4xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (2 + r)\,(1 + r)\, x^r + a_1 (3 + r)\,(2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k + r + 2)\,(k + r + 1) + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1 (3 + r)\,(2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x^2} + \frac{c_2 \cos(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.765 problem 782

2.765.1 Maple step by step solution 7173

Internal problem ID [8255]
Internal file name [OUTPUT/7188_Sunday_June_05_2022_05_34_47_PM_1254305/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 782.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{16x^2y'' + 32xy' + (x^4 - 12)\, y = 0}$$

Writing the ode as

$$16x^2y'' + 32xy' + (x^4 - 12)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 16x^2 \\ B &= 32x \\ C &= x^4 - 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^4 + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^4 + 12 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^4 + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1450: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{x^2}{16} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{ix}{4} - \frac{3i}{2x^3} - \frac{9i}{2x^7} - \frac{27i}{x^{11}} - \frac{405i}{2x^{15}} - \frac{1701i}{x^{19}} - \frac{15309i}{x^{23}} - \frac{144342i}{x^{27}} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{i}{4}$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{ix}{4}
\end{aligned}
\tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = -\frac{x^2}{16}
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-x^4 + 12}{16x^2} \\
&= Q + \frac{R}{16x^2} \\
&= \left(-\frac{x^2}{16}\right) + \left(\frac{3}{4x^2}\right) \\
&= -\frac{x^2}{16} + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{ix}{4} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{\frac{i}{4}} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^4 + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{ix}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\left(\frac{ix}{4}\right) \\ &= -\frac{1}{2x} - \frac{ix}{4} \\ &= -\frac{1}{2x} - \frac{ix}{4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{ix}{4}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{i}{4}\right) + \left(-\frac{1}{2x} - \frac{ix}{4}\right)^2 - \left(\frac{-x^4 + 12}{16x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - \frac{ix}{4}\right) dx} \\ &= \frac{e^{-\frac{ix^2}{8}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{32x}{16x^2}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{32x}{16x^2}\,dx}}{(y_1)^2}\,dx$$

$$= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx$$

$$= y_1\left(-2i\mathrm{e}^{\frac{ix^2}{4}}\right)$$

Therefore the solution is

$$y = c_1y_1 + c_2y_2$$

$$= c_1\left(\frac{\mathrm{e}^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}}\right) + c_2\left(\frac{\mathrm{e}^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}}\left(-2i\mathrm{e}^{\frac{ix^2}{4}}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\mathrm{e}^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}} - \frac{2ic_2\mathrm{e}^{\frac{ix^2}{8}}}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\mathrm{e}^{-\frac{ix^2}{8}}}{x^{\frac{3}{2}}} - \frac{2ic_2\mathrm{e}^{\frac{ix^2}{8}}}{x^{\frac{3}{2}}}$$

Verified OK.

2.765.1 Maple step by step solution

Let's solve

$16x^2y'' + 32xy' + (x^4 - 12)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{(x^4-12)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{2y'}{x} + \frac{(x^4-12)y}{16x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = \frac{x^4-12}{16x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{3}{4}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + 32xy' + (x^4 - 12)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- ◦ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$4a_0(3+2r)\left(-1+2r\right)x^r+4a_1(5+2r)\left(1+2r\right)x^{1+r}+4a_2(7+2r)\left(3+2r\right)x^{2+r}+4a_3(9+2r)\left(5\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$4(3+2r)\left(-1+2r\right)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{-\frac{3}{2},\frac{1}{2}\right\}$$

- The coefficients of each power of x must be 0

  $$[4a_1(5+2r)\left(1+2r\right)=0,4a_2(7+2r)\left(3+2r\right)=0,4a_3(9+2r)\left(5+2r\right)=0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1=0,a_2=0,a_3=0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$16\left(k+r+\frac{3}{2}\right)\left(k-\frac{1}{2}+r\right)a_k+a_{k-4}=0$$

- Shift index using $k->k+4$

  $$16\left(k+\frac{11}{2}+r\right)\left(k+\frac{7}{2}+r\right)a_{k+4}+a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4}=-\frac{a_k}{4(2k+11+2r)(2k+7+2r)}$$

- Recursion relation for $r=-\frac{3}{2}$

  $$a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)}$$

- Solution for $r=-\frac{3}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}},a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)},a_1=0,a_2=0,a_3=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

  $$a_{k+4}=-\frac{a_k}{4(2k+12)(2k+8)}$$

- Solution for $r=\frac{1}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}},a_{k+4}=-\frac{a_k}{4(2k+12)(2k+8)},a_1=0,a_2=0,a_3=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k-\frac{3}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+4}=-\frac{a_k}{4(2k+8)(2k+4)},a_1=0,a_2=0,a_3=0,b_{k+4}=-\frac{}{4(2k\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(16*x^2*diff(y(x),x$2)+32*x*diff(y(x),x)+(x^4-12)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x^2}{8}\right)}{x^{\frac{3}{2}}} + \frac{c_2 \cos\left(\frac{x^2}{8}\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 42

```
DSolve[16*x^2*y''[x]+32*x*y'[x]+(x^4-12)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix^2}{8}}\left(c_1 - 2ic_2 e^{\frac{ix^2}{4}}\right)}{x^{3/2}}$$

2.766 problem 783

2.766.1 Maple step by step solution 7183

Internal problem ID [8256]
Internal file name [`OUTPUT/7189_Sunday_June_05_2022_05_34_51_PM_36856396/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 783.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - x^2y' + yx = 0}$$

Writing the ode as

$$y'' - x^2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x^2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x(x^3 - 8)}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x(x^3 - 8) \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x(x^3 - 8)}{4}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1452: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{2}{x} - \frac{4}{x^4} - \frac{16}{x^7} - \frac{80}{x^{10}} - \frac{448}{x^{13}} - \frac{2688}{x^{16}} - \frac{16896}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^1 = x$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of x in the above is 0. Now we need to find the coefficient of x in r. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of x in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3 - 8)}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^4 - 2x\right) + (0) \\
&= \frac{1}{4}x^4 - 2x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is -2. Now b can be found.

$$\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x^2}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{\frac{1}{2}} - 2\right) &&= -3 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{\frac{1}{2}} - 2\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x(x^3 - 8)}{4}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-4	$\frac{x^2}{2}$	-3	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{x^2}{2}\right)(1) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-8)}{4}\right)\right) = 0$$

$$xa_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x)\,\mathrm{e}^{\int -\frac{x^2}{2} dx} \\
&= (x)\,\mathrm{e}^{-\frac{x^3}{6}} \\
&= x\,\mathrm{e}^{-\frac{x^3}{6}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{1} \, dx} \\
&= z_1 e^{\frac{x^3}{6}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^3}{6}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^3}{3}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}} x^3 \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\,(-x^3)^{\frac{2}{3}}\,x} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}\,x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}} \tag{1}
$$

Verification of solutions

$$
y = c_1 x + \frac{c_2\left(-3(-x^3)^{\frac{2}{3}}\,\mathrm{e}^{\frac{x^3}{3}} + 3^{\frac{2}{3}}x^3\left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right)\right)\right)}{3\left(-x^3\right)^{\frac{2}{3}}}
$$

Verified OK.

2.766.1 Maple step by step solution

Let's solve

$y'' - x^2 y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+1}$

- Shift index using $k- > k-1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1) \, x^k$

- Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

- Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_{k-1}(k-2)) \, x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) \, a_{k+2} - a_{k-1}(k-2) = 0$

- Shift index using $k- > k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k(k-1) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k(k-1)}{k^2+5k+6}, 2a_2 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x - \frac{c_2 3^{\frac{1}{3}} \left(6(-x^3)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right) 3^{\frac{2}{3}} - 6(-x^3)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}, -\frac{x^3}{3}\right) 3^{\frac{2}{3}} + 18\, e^{\frac{x^3}{3}}\right)}{3\left(1+\sqrt{-3}\right)}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 41

```
DSolve[y''[x]-x^2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{c_2 \sqrt[3]{-x^3}\,\Gamma\left(-\frac{1}{3}, -\frac{x^3}{3}\right)}{3\sqrt[3]{3}}$$

2.767 problem 784

2.767.1 Maple step by step solution 7192

Internal problem ID [8257]
Internal file name [`OUTPUT/7190_Sunday_June_05_2022_05_34_55_PM_17810887/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 784.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (x+2)\,y' + 2y = 0}$$

Writing the ode as

$$xy'' + (-x-2)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 8$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1454: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 4x + 8}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-4x + 8}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{x} + \frac{1}{2} \\
&= \frac{x-2}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \ln(x)} \\
&= z_1 \left(x \, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\mathrm{e}^{-x}\left(x^2+2x+2\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2\left(-x^2-2x-2\right) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2\left(-x^2-2x-2\right)$$

Verified OK.

2.767.1 Maple step by step solution

Let's solve

$x y'' + (-x-2)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$$\left[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}\right]$$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$xy'' + (-x-2)\,y' + 2y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$$

- ○ Shift index using $k- >k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\,x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = \frac{a_k}{k+4}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)-(x+2)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^2 + 2x + 2) + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[x*y''[x]-(x+2)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2(x^2 + 2x + 2)$$

2.768 problem 785

2.768.1 Maple step by step solution 7202

Internal problem ID [8258]
Internal file name [`OUTPUT/7191_Sunday_June_05_2022_05_34_58_PM_2671751/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 785.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + xy' + 2y = 0}$$

Writing the ode as

$$y'' + xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1456: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-6}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\ &= \frac{x^2}{4} - \frac{3}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x) \, e^{\int -\frac{x}{2} dx} \\
&= (x) \, e^{-\frac{x^2}{4}} \\
&= x \, e^{-\frac{x^2}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(e^{-\frac{x^2}{4}} \right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, e^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{\left(y_1\right)^2} \, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{\left(y_1\right)^2} \, dx \\
&= y_1 \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\, e^{\frac{x^2}{2}}}{2x} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(x \,\mathrm{e}^{-\frac{x^2}{2}} \left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x \,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x \,\mathrm{e}^{-\frac{x^2}{2}} + c_2 \left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x \,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

2.768.1 Maple step by step solution

Let's solve

$y'' + xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k+2)\right)x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k+2)(ka_{k+2} + a_k + a_{k+2}) = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 50

```
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x\,\mathrm{e}^{-\frac{x^2}{2}} - \frac{c_2 \mathrm{e}^{-\frac{x^2}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x + 2\,\mathrm{e}^{\frac{x^2}{2}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 69

```
DSolve[y''[x]+x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

2.769 problem 786

2.769.1 Maple step by step solution 7211

Internal problem ID [8259]
Internal file name [`OUTPUT/7192_Sunday_June_05_2022_05_35_02_PM_56730584/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 786.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\, y''-2xy'+2y=0}$$

Writing the ode as

$$\begin{align*}(1-x^2)\, y''-2xy'+2y&=0 \tag{1}\\ Ay''+By'+Cy&=0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A&=1-x^2\\ B&=-2x \tag{3}\\ C&=2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^2 - 3}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2x^2 - 3$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^2 - 3}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1458: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{5}{4(x-1)} - \frac{5}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^2 - 3}{(x^2 - 1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^2-3}{(x^2-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=2$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\
&= 2 - (1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} + (0) \\
&= \frac{1}{2x-2} + \frac{1}{2x+2} \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)(1) + \left(\left(-\frac{1}{2(x-1)^2} - \frac{1}{2(1+x)^2}\right) + \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)^2 - \left(\frac{2x^2-3}{(x^2-1)^2}\right)\right.
$$

$$
-\frac{2a_0}{x^2-1} =
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= (x)\, e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= x\sqrt{x-1}\,\sqrt{1+x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2}-\frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\right) + c_2\left(\frac{x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}}\left(\frac{1}{x}-\frac{\ln(1+x)}{2}+\frac{\ln(x-1)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x\sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2\sqrt{x^2-1}\,(\ln(x-1)\,x-\ln(1+x)\,x+2)}{2\sqrt{x-1}\,\sqrt{1+x}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{x^2-1}}{\sqrt{x-1}\,\sqrt{1+x}} + \frac{c_2 \sqrt{x^2-1}\,(\ln(x-1)\,x - \ln(1+x)\,x + 2)}{2\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

2.769.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 2xy' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{2}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2xy' - 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (2u-2)\left(\frac{d}{du}y(u)\right) - 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(k+r+2)\,(k+r-1)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2+a_k(k+2)\,(k-1)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Recursion relation for $r=0$; series terminates at $k=1$

  $a_{k+1}=\frac{a_k(k+2)(k-1)}{2(k+1)^2}$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-u + 1)$
- Revert the change of variables $u = 1 + x$

  $[y = -a_0 x]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_2 \left(-\frac{\ln(x+1)\, x}{2} + \frac{\ln(x-1)\, x}{2} + 1\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - \frac{1}{2} c_2 (x \log(1-x) - x \log(x+1) + 2)$$

2.770 problem 787

2.770.1 Maple step by step solution 7217

Internal problem ID [8260]
Internal file name [OUTPUT/7193_Sunday_June_05_2022_05_35_05_PM_9079499/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 787.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1460: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\ &= z_1 e^{x^2} \\ &= z_1\left(\mathrm{e}^{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2}
$$

Verified OK.

2.770.1 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$

- Solve for the dependent coefficient(s)

$\{a_2 = a_0, a_3 = a_1\}$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{x^2} + c_2 x\, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2 x + c_1)$$

2.771 problem 788

2.771.1 Maple step by step solution 7226

Internal problem ID [8261]
Internal file name [`OUTPUT/7194_Sunday_June_05_2022_05_35_07_PM_6693030/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 788.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(1-x^2)\,y''-2xy'+30y=0}$$

Writing the ode as

$$(1-x^2)\,y''-2xy'+30y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= -2x \\ C &= 30 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{30x^2 - 31}{(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 30x^2 - 31$$
$$t = (x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{30x^2 - 31}{(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1462: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(1+x)^2} - \frac{1}{4(x-1)^2} + \frac{61}{4(x-1)} - \frac{61}{4(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{30x^2-31}{(x^2-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 30$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 6\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -5\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{30x^2-31}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{1}{2}$	$\frac{1}{2}$
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	6	-5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=6$ then

$$\begin{aligned}d&=\alpha_\infty^+-\left(\alpha_{c_1}^-+\alpha_{c_2}^-\right)\\&=6-(1)\\&=5\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+\left((-)[\sqrt{r}]_{c_2}+\frac{\alpha_{c_2}^-}{x-c_2}\right)+(+)[\sqrt{r}]_\infty\\&=\frac{1}{2x-2}+\frac{1}{2x+2}+(0)\\&=\frac{1}{2x-2}+\frac{1}{2x+2}\\&=\frac{x}{x^2-1}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\left(20x^3 + 12x^2 a_4 + 6x a_3 + 2a_2\right) + 2\left(\frac{1}{2x-2} + \frac{1}{2x+2}\right)\left(5x^4 + 4x^3 a_4 + 3x^2 a_3 + 2x a_2 + a_1\right) + \left(\left(-\frac{1}{2\,(x -} \right.\right.$$

$$-10a_4 x^4 + (-18a_3 - 20$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = 0, a_1 = \frac{5}{21}, a_2 = 0, a_3 = -\frac{10}{9}, a_4 = 0\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\int \left(\frac{1}{2x-2} + \frac{1}{2x+2}\right) dx} \\
&= \left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right) e^{\frac{\ln(x-1)}{2} + \frac{\ln(1+x)}{2}} \\
&= \frac{(63x^5 - 70x^3 + 15x)\sqrt{x-1}\,\sqrt{1+x}}{63}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{1-x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x-1)}{2} - \frac{\ln(1+x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x-1}\,\sqrt{1+x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-1)-\ln(1+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3087x(1449x^2-935)}{1600\,(63x^4-70x^2+15)} + \frac{441}{25x} - \frac{3969\ln(1+x)}{128} + \frac{3969\ln(x-1)}{128}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}}\right) \\
&\quad + c_2 \left(\frac{(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} \left(\frac{3087x(1449x^2-935)}{1600\,(63x^4-70x^2+15)} + \frac{441}{25x} \right.\right. \\
&\qquad \left.\left. - \frac{3969\ln(1+x)}{128} + \frac{3969\ln(x-1)}{128}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = {} & \frac{c_1(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} \\
& + \frac{3969c_2\sqrt{x^2-1}\left(\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)\ln(x-1) + \left(-x^5 + \frac{10}{9}x^3 - \frac{5}{21}x\right)\ln(1+x) + 2x^4 - \frac{14x^2}{9} + \frac{128}{945}\right)}{128\sqrt{x-1}\,\sqrt{1+x}}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(63x^5 - 70x^3 + 15x)\sqrt{x^2-1}}{63\sqrt{x-1}\,\sqrt{1+x}} + \frac{3969c_2\sqrt{x^2-1}\left(\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)\ln(x-1) + \left(-x^5 + \frac{10}{9}x^3 - \frac{5}{21}x\right)\ln(1+x) + 2x^4 - \frac{14x^2}{9} + \frac{128}{945}\right)}{128\sqrt{x-1}\,\sqrt{1+x}}$$

Verified OK.

2.771.1 Maple step by step solution

Let's solve

$(1-x^2)\,y'' - 2xy' + 30y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2xy'}{x^2-1} + \frac{30y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2xy'}{x^2-1} - \frac{30y}{x^2-1} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{30}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\,y'' + 2xy' - 30y = 0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $\left(u^{2}-2u\right)\left(\frac{d^{2}}{du^{2}}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)-30y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..2$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}r^{2}u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^{2}+a_{k}(k+r+6)\,(k+r-5)\right)u^{k+r}\right)=0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2r^{2}=0$

- Values of r that satisfy the indicial equation

  $r=0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^{2}+a_{k}(k+6)\,(k-5)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_{k}(k+6)(k-5)}{2(k+1)^{2}}$

- Recursion relation for $r=0$; series terminates at $k=5$

  $a_{k+1}=\frac{a_{k}(k+6)(k-5)}{2(k+1)^{2}}$

- Apply recursion relation for $k=0$

  $a_1 = -15a_0$
- Apply recursion relation for $k=1$

  $a_2 = -\frac{7a_1}{2}$
- Express in terms of a_0

  $a_2 = \frac{105a_0}{2}$
- Apply recursion relation for $k=2$

  $a_3 = -\frac{4a_2}{3}$
- Express in terms of a_0

  $a_3 = -70a_0$
- Apply recursion relation for $k=3$

  $a_4 = -\frac{9a_3}{16}$
- Express in terms of a_0

  $a_4 = \frac{315a_0}{8}$
- Apply recursion relation for $k=4$

  $a_5 = -\frac{a_4}{5}$
- Express in terms of a_0

  $a_5 = -\frac{63a_0}{8}$
- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - 15u + \frac{105}{2}u^2 - 70u^3 + \frac{315}{8}u^4 - \frac{63}{8}u^5\right)$
- Revert the change of variables $u = 1 + x$

  $\left[y = a_0\left(-\frac{15}{8}x + \frac{35}{4}x^3 - \frac{63}{8}x^5\right)\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 83

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+30*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\left(\frac{21}{5}x^5 - \frac{14}{3}x^3 + x\right) + c_2\left(-\frac{21\ln(x+1)\,x^5}{640} + \frac{21\ln(x-1)\,x^5}{640} + \frac{7\ln(x+1)\,x^3}{192} - \frac{7\ln(x-1)\,x^3}{192} + \frac{21x^4}{320} - \frac{\ln(x+1)\,x}{128} + \frac{\ln(x-1)\,x}{128} - \frac{49x^2}{960} + \frac{1}{225}\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 76

```
DSolve[(1-x^2)*y''[x]-2*x*y'[x]+30*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}c_1x(63x^4 - 70x^2 + 15) + c_2\left(-\frac{63x^4}{8} + \frac{49x^2}{8} - \frac{1}{16}(63x^4 - 70x^2 + 15)\,x(\log(1-x) - \log(x+1)) - \frac{8}{15}\right)$$

2.772 problem 789

2.772.1 Maple step by step solution 7233

Internal problem ID [8262]
Internal file name [`OUTPUT/7195_Sunday_June_05_2022_05_35_11_PM_64878954/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 789.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1464: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x}\right) + c_2\left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.772.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - ◦ Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - ◦ Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - ◦ Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - ◦ Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - ◦ Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 0\}$

- Each term must be 0

$a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

$a_{k+1}(k+r+1)(k+2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

$a_{k+2}(k+2+r)(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

$a_{k+2}=-\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

$a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica
Time used: 0.025 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.773 problem 790

2.773.1 Maple step by step solution 7242

Internal problem ID [8263]
Internal file name [`OUTPUT/7196_Sunday_June_05_2022_05_35_13_PM_21982793/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 790.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (2x+1)\,y' + (1+x)\,y = 0}$$

Writing the ode as

$$xy'' + (2x+1)\,y' + (1+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x+1 \\ C &= 1+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1466: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x+1}{x} \, dx} \\ &= z_1 e^{-x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{-x}}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x+1}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2x - \ln(x)}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\ln (x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \ln (x) \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \ln (x)
$$

Verified OK.

2.773.1 Maple step by step solution

Let's solve

$x y'' + (2x+1) y' + (1+x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(2x+1)y'}{x} - \frac{(1+x)y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x+1)y'}{x} + \frac{(1+x)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2x+1}{x}, P_3(x) = \frac{1+x}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (2x+1)\,y' + (1+x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- > k + 1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 + a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 + a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + 2a_k k + a_k + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2)^2 + 2a_{k+1}(k+1) + a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2ka_{k+1}+a_k+3a_{k+1}}{(k+2)^2}, a_1 + a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+(2*x+1)*diff(y(x),x)+(x+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}c_1 + c_2 e^{-x}\ln(x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2*x+1)*y'[x]+(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2 \log(x) + c_1)$$

2.774 problem 791

2.774.1 Maple step by step solution 7252

Internal problem ID [8264]
Internal file name [`OUTPUT/7197_Sunday_June_05_2022_05_35_19_PM_23791161/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 791.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{2x(x-1)\,y''-(1+x)\,y'+y=0}$$

Writing the ode as

$$\left(2x^{2}-2x\right)y''+(-x-1)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^{2}-2x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 + 18x - 3$$
$$t = 16\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1468: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x} + \frac{3}{4(x-1)^2} - \frac{3}{16x^2} - \frac{3}{4(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 + 18x - 3}{16(x^2 - x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 + 18x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\ &= \frac{1}{4} - \left(\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{3}{4x} - \frac{1}{2\,(x-1)} + (-)\,(0) \\
&= \frac{3}{4x} - \frac{1}{2\,(x-1)} \\
&= \frac{-3+x}{4x\,(x-1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{4x} - \frac{1}{2\,(x-1)}\right)(0) + \left(\left(-\frac{3}{4x^2} + \frac{1}{2\,(x-1)^2}\right) + \left(\frac{3}{4x} - \frac{1}{2\,(x-1)}\right)^2 - \left(\frac{-3x^2+18x-3}{16\,(x^2-x)^2}\right)\right) =
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{4x} - \frac{1}{2(x-1)}\right) dx} \\
&= \frac{x^{\frac{3}{4}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x-1}{2x^2-2x}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{4} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\frac{\sqrt{x-1}}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{2x^2-2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2x+2}{\sqrt{x}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\left(\frac{2x+2}{\sqrt{x}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} + c_2(2x+2) \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + c_2(2x+2)$$

Verified OK.

2.774.1 Maple step by step solution

Let's solve

$(2x^2 - 2x)\, y'' + (-x - 1)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(1+x)y'}{2x(x-1)} - \frac{y}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{2x(x-1)} + \frac{y}{2x(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1+x}{2x(x-1)}, P_3(x) = \frac{1}{2x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x(x-1) + (-x-1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

$$x^{m}\cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^{m}\cdot y''$ to series expansion for $m=1..2$

$$x^{m}\cdot y'' = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^{m}\cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0}r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_{k}(2k+2r-1)\,(k+r-1)\right) x^{k-}\right.$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 2(k+r-1)\left(k-\frac{1}{2}+r\right) a_{k} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{(k+r-1)(2k+2r-1)a_{k}}{(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r=0$; series terminates at $k=1$

$$a_{k+1} = \frac{(k-1)(2k-1)a_{k}}{(k+1)(2k+1)}$$

- Apply recursion relation for $k=0$

$$a_{1} = a_{0}$$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

$$y = a_{0}\cdot(1+x)$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1} = \frac{2\left(k-\frac{1}{2}\right)k a_{k}}{\left(k+\frac{3}{2}\right)(2k+2)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{2}}, a_{k+1} = \frac{2\left(k-\frac{1}{2}\right)k a_{k}}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0 \cdot (1+x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2\left(k-\frac{1}{2}\right) k b_k}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*x*(x-1)*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + c_2\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 21

```
DSolve[2*x*(x-1)*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt{x} - 2c_2(x+1)$$

2.775 problem 792

2.775.1 Maple step by step solution 7258

Internal problem ID [8265]
Internal file name [`OUTPUT/7198_Sunday_June_05_2022_05_35_22_PM_61720420/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 792.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + 2y' + 4yx = 0}$$

Writing the ode as

$$xy'' + 2y' + 4yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1470: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(2x)}{x}\right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x}
$$

Verified OK.

2.775.1 Maple step by step solution

Let's solve

$xy'' + 2y' + 4yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + 4y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 4\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + 4yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- > k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- > k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + 4a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + 4a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{4a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{4a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{4b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(2x)}{x} + \frac{c_2 \cos(2x)}{x}$$

✓ Solution by Mathematica
Time used: 0.029 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+4*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x}$$

2.776 problem 793

2.776.1 Maple step by step solution 7265

Internal problem ID [8266]
Internal file name [`OUTPUT/7199_Sunday_June_05_2022_05_35_25_PM_20896997/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 793.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' + (-2x+2)\,y' + (x-2)\,y = 0}$$

Writing the ode as

$$xy'' + (-2x+2)\,y' + (x-2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x+2 \\ C &= x-2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1472: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x+2}{x}\,dx} \\
&= z_1 e^{x-\ln(x)} \\
&= z_1 \left(\frac{e^x}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^x}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x+2}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^x}{x}\right) + c_2 \left(\frac{e^x}{x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 e^x}{x} + c_2 e^x \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 e^x}{x} + c_2 e^x
$$

Verified OK.

2.776.1 Maple step by step solution

Let's solve

$x y'' + (-2x + 2) y' + (x - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-2)y}{x} + \frac{2(x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x-1)y'}{x} + \frac{(x-2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2(x-1)}{x}, P_3(x) = \frac{x-2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$xy'' + (-2x+2)\,y' + (x-2)\,y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + \left(a_1(1+r)\,(2+r) - 2a_0(1+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) - 2a_k\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) - 2a_0(1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+2+r) - 2a_k k - 2a_k r - 2a_k + a_{k-1} = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+2+r)(k+3+r) - 2a_{k+1}(k+1) - 2ra_{k+1} - 2a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k+4a_{k+1}}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+4a_{k+1}}{(k+2)(k+3)}, 2a_1 - 2a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = \frac{2ka_{k+1}-a_k+2a_{k+1}}{(k+1)(k+2)}, 0 = 0, b_{k+2} = \frac{2kb_{k+1}-b_k+4b_{k+1}}{(k+2)(k+3)}, 2b_1 - 2b\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x$2)+(2-2*x)*diff(y(x),x)+(x-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x}{x} + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 19

```
DSolve[x*y''[x]+(2-2*x)*y'[x]+(x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x(c_2x + c_1)}{x}$$

2.777 problem 794

2.777.1 Maple step by step solution 7272

Internal problem ID [8267]
Internal file name [`OUTPUT/7200_Sunday_June_05_2022_05_35_28_PM_42643253/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 794.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 6xy' + (4x^2 + 6)\, y = 0}$$

Writing the ode as

$$x^2y'' + 6xy' + (4x^2 + 6)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 6x \\ C &= 4x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1474: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2}\, dx} \\ &= z_1 e^{-3\ln(x)} \\ &= z_1 \left(\frac{1}{x^3}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\cos(2x)}{x^3}\right) + c_2\left(\frac{\cos(2x)}{x^3}\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\cos(2x)}{x^3} + \frac{c_2\sin(2x)}{2x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\cos(2x)}{x^3} + \frac{c_2\sin(2x)}{2x^3}
$$

Verified OK.

2.777.1 Maple step by step solution

Let's solve

$x^2y'' + 6xy' + (4x^2 + 6)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(2x^2+3)y}{x^2} - \frac{6y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6y'}{x} + \frac{2(2x^2+3)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{6}{x}, P_3(x) = \frac{2(2x^2+3)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 6$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = 6$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2 y'' + 6xy' + (4x^2 + 6)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(3 + r)\,(2 + r)\, x^r + a_1(4 + r)\,(3 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 3)\,(k + r + 2) + 4a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(3 + r)\,(2 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-3, -2\}$

- Each term must be 0

  $a_1(4 + r)\,(3 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+3)(k+r+2) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+5+r)(k+4+r) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{(k+5+r)(k+4+r)}$

- Recursion relation for $r = -3$

  $a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}$

- Solution for $r = -3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-3}, a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{4a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-3}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-2}\right), a_{k+2} = -\frac{4a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{4b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+6*x*diff(y(x),x)+(4*x^2+6)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(2x)}{x^3} + \frac{c_2 \cos(2x)}{x^3}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+6*x*y'[x]+(4*x^2+6)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix}}{4x^3}$$

2.778 problem 795

2.778.1 Maple step by step solution 7281

Internal problem ID [8268]
Internal file name [`OUTPUT/7201_Sunday_June_05_2022_05_35_30_PM_94829251/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 795.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\,y' + (x-1)\,y = 0}$$

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x} \, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)+2x}}{(y_1)^2} \, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^x + c_2\mathrm{e}^x \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^x + c_2\mathrm{e}^x \ln(x)
$$

Verified OK.

2.778.1 Maple step by step solution

Let's solve

$xy'' + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (1 - 2x)\, y' + (x - 1)\, y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- > k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

  ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r-1}$

  ◦ Shift index using $k- > k + 1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k + 1 + r)\, (k + r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1 (1 + r)^2 - a_0 (1 + 2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1} (k + 1 + r)^2 - a_k (2k + 2r + 1) + a_{k-1}\right) x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\,a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\,a_{k+1} + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$

- Recursion relation for $r=0$

  $a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}$

- Solution for $r=0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1}-a_k+3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x \ln(x)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (c_2 \log(x) + c_1)$$

2.779 problem 796

2.779.1 Maple step by step solution 7291

Internal problem ID [8269]
Internal file name [`OUTPUT/7202_Sunday_June_05_2022_05_35_33_PM_40311267/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 796.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{x(1-x)\,y''+\left(2x+\frac{1}{2}\right)y'-2y=0}$$

Writing the ode as

$$\left(-x^{2}+x\right)y''+\left(2x+\frac{1}{2}\right)y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=-x^{2}+x\\ B&=2x+\frac{1}{2}\\ C&=-2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{48x - 3}{16\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 48x - 3$$
$$t = 16\left(x^2 - x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{48x - 3}{16\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{21}{8x} + \frac{45}{16(x-1)^2} - \frac{3}{16x^2} - \frac{21}{8(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{45}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{5}{4} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{48x - 3}{16\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
1	2	0	$\frac{9}{4}$	$-\frac{5}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} - \frac{5}{4\left(x - 1\right)} + (0) \\ &= \frac{1}{4x} - \frac{5}{4\left(x - 1\right)} \\ &= -\frac{1 + 4x}{4x\left(x - 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=x+a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{1}{4x}-\frac{5}{4(x-1)}\right)(1)+\left(\left(-\frac{1}{4x^2}+\frac{5}{4(x-1)^2}\right)+\left(\frac{1}{4x}-\frac{5}{4(x-1)}\right)^2-\left(\frac{48x-3}{16(x^2-x)^2}\right)\right)=0$$

$$\frac{-1+4a_0}{2x(x-1)}=0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0=\frac{1}{4}\right\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x)=\frac{1}{4}+x$$

Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int\omega\,dx}\\
&=\left(\frac{1}{4}+x\right)e^{\int\left(\frac{1}{4x}-\frac{5}{4(x-1)}\right)dx}\\
&=\left(\frac{1}{4}+x\right)e^{\frac{\ln(x)}{4}-\frac{5\ln(x-1)}{4}}\\
&=\frac{\left(\frac{1}{4}+x\right)x^{\frac{1}{4}}}{(x-1)^{\frac{5}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int-\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int\frac{1}{2}\frac{2x+\frac{1}{2}}{-x^2+x}\,dx}\\
&=z_1e^{-\frac{\ln(x)}{4}+\frac{5\ln(x-1)}{4}}\\
&=z_1\left(\frac{(x-1)^{\frac{5}{4}}}{x^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{4} + x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{2x+\frac{1}{2}}{-x^2+x}\, dx}}{(y_1)^2}\, dx$$

$$= y_1 \int \frac{e^{-\frac{\ln(x)}{2}+\frac{5\ln(x-1)}{2}}}{(y_1)^2}\, dx$$

$$= y_1 \left(\frac{\sqrt{x}\,\sqrt{x-1}\,\left(4\sqrt{x\,(x-1)}\,x - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 12x\ln(2) + 26\sqrt{x\,(x-1)} - 3\ln\right.}{\sqrt{x\,(x-1)}\,(1+4x)} \right.$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(\frac{1}{4}+x\right) + c_2\left(\frac{1}{4}\right.$$

$$\left. + x\left(\frac{\sqrt{x}\,\sqrt{x-1}\,\left(4\sqrt{x\,(x-1)}\,x - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 12x\ln(2) + 26\sqrt{x\,(x-1)} - 3\ln\right.}{\sqrt{x\,(x-1)}\,(1+4x)}\right.\right.$$

Summary

The solution(s) found are the following

$$y = c_1\left(\frac{1}{4}+x\right) \tag{1}$$

$$+ \frac{c_2\sqrt{x}\,\sqrt{x-1}\,\left(4\sqrt{x\,(x-1)}\,x - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 12x\ln(2) + 26\sqrt{x\,(x-1)} - 3\ln\left(\right.\right.}{4\sqrt{x\,(x-1)}}$$

Verification of solutions

$$y = c_1\left(\frac{1}{4}+x\right) + \frac{c_2\sqrt{x}\,\sqrt{x-1}\,\left(4\sqrt{x\,(x-1)}\,x - 12x\ln\left(2x-1+2\sqrt{x\,(x-1)}\right) + 12x\ln\left(2\right) + 26\sqrt{x\,(x-1)} - 3\ln\left(\right.\right.}{4\sqrt{x\,(x-1)}}$$

Verified OK.

2.779.1 Maple step by step solution

Let's solve

$\left(-x^2+x\right)y''+\left(2x+\frac{1}{2}\right)y'-2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x-1)}+\frac{(1+4x)y'}{2x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(1+4x)y'}{2x(x-1)}+\frac{2y}{x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{1+4x}{2x(x-1)}, P_3(x)=\frac{2}{x(x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2y''x(x-1)+(-1-4x)y'+4y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + 2a_k(k+r-1)\,(k+r-2)\right) x^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r+\frac{1}{2}\right) a_{k+1} + 2a_k(k+r-1)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r-1)(k+r-2)}{(k+1+r)(2k+1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{2a_k(k-1)(k-2)}{(k+1)(2k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 4a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot (1 + 4x)$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0 \cdot (1 + 4x) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), b_{k+1} = \frac{2b_k \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 61

```
dsolve(x*(1-x)*diff(y(x),x$2)+(1/2+2*x)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(1 + 4x) + c_2\left(4\sqrt{x(x-1)}\,x - 12\ln\left(x - \frac{1}{2} + \sqrt{x(x-1)}\right)x + 26\sqrt{x(x-1)} - 3\ln\left(x - \frac{1}{2} + \sqrt{x(x-1)}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 64

```
DSolve[x*(1-x)*y''[x]+(1/2+2*x)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_2\left(\sqrt{-((x-1)x)}(2x+13)-6(4x+1)\arctan\left(\frac{\sqrt{1-x}}{\sqrt{x}+1}\right)\right)+c_1\left(x+\frac{1}{4}\right)$$

2.780 problem 797

2.780.1 Maple step by step solution 7301

Internal problem ID [8270]
Internal file name [`OUTPUT/7203_Sunday_June_05_2022_05_35_44_PM_95833684/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 797.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4(t^2 - 3t + 2)\, y'' - 2y' + y = 0}$$

Writing the ode as

$$y''(4t^2 - 12t + 8) + y - 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4t^2 - 12t + 8 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4t^2 + 20t - 19 \\ t &= 16\left(t^2 - 3t + 2\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{-4t^2 + 20t - 19}{16\left(t^2 - 3t + 2\right)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 3t + 2)^2$. There is a pole at $t = 2$ of order 2. There is a pole at $t = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16(t-2)^2} + \frac{3}{8(t-1)} - \frac{3}{8(t-2)} - \frac{3}{16(t-1)^2}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $t = 1$ let b be the coefficient of $\frac{1}{(t-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-4t^2 + 20t - 19}{16(t^2 - 3t + 2)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-4t^2+20t-19}{16\left(t^2-3t+2\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
1	2	0	$\frac{3}{4}$	$\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{+}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t - c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t - c_2}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} + (-)(0) \\
&= -\frac{1}{4(t-2)} + \frac{3}{4(t-1)} \\
&= \frac{2t-5}{4(t-1)(t-2)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)(0) + \left(\left(\frac{1}{4(t-2)^2} - \frac{3}{4(t-1)^2}\right) + \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right)^2 - \left(\frac{-4t}{16(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= e^{\int \left(-\frac{1}{4(t-2)} + \frac{3}{4(t-1)}\right) dt} \\
&= \frac{(t-1)^{\frac{3}{4}}}{(t-2)^{\frac{1}{4}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{4t^2 - 12t + 8} \, dt} \\
&= z_1 e^{-\frac{\ln(t-1)}{4} + \frac{\ln(t-2)}{4}} \\
&= z_1 \left(\frac{(t-2)^{\frac{1}{4}}}{(t-1)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{t-1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{4t^2-12t+8}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(t-1)}{2}+\frac{\ln(t-2)}{2}}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4}{\sqrt{t-1}\,\sqrt{t-2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\sqrt{t-1}\right) \\ &\quad + c_2\left(\sqrt{t-1}\left(\frac{\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4}{\sqrt{t-1}\,\sqrt{t-2}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{t-1} \\ &\quad + \frac{c_2\left(\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4\right)}{\sqrt{t-2}} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1\sqrt{t-1} \\ &\quad + \frac{c_2\left(\sqrt{(t-1)(t-2)}\, \ln\left(2t-3+2\sqrt{(t-1)(t-2)}\right) - \sqrt{(t-1)(t-2)}\, \ln(2) - 2t + 4\right)}{\sqrt{t-2}} \end{aligned}$$

Verified OK.

2.780.1 Maple step by step solution

Let's solve

$y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4(t^2-3t+2)} + \frac{y'}{2(t^2-3t+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2(t^2-3t+2)} + \frac{y}{4(t^2-3t+2)} = 0$

□ Check to see if t_0 is a regular singular point

- ○ Define functions

  $\left[P_2(t) = -\frac{1}{2(t^2-3t+2)}, P_3(t) = \frac{1}{4(t^2-3t+2)}\right]$

- ○ $(t-1) \cdot P_2(t)$ is analytic at $t = 1$

  $\left((t-1) \cdot P_2(t)\right)\bigg|_{t=1} = \frac{1}{2}$

- ○ $(t-1)^2 \cdot P_3(t)$ is analytic at $t = 1$

  $\left((t-1)^2 \cdot P_3(t)\right)\bigg|_{t=1} = 0$

- ○ $t = 1$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 1$

- Multiply by denominators

  $y''(4t^2 - 12t + 8) + y - 2y' = 0$

- Change variables using $t = u + 1$ so that the regular singular point is at $u = 0$

  $(4u^2 - 4u)\left(\frac{d^2}{du^2}y(u)\right) - 2\frac{d}{du}y(u) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $$\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$$

- Shift index using $k->k+1$

  $$\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$-2a_0r(-1+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(2k+2r-1)^2\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-2r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(2k+2r-1)^2 - 4(k+1+r)\, a_{k+1}\left(k+r+\frac{1}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(2k+2r-1)^2}{2(k+1+r)(2k+1+2r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Revert the change of variables $u = t-1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(t-1)^k, a_{k+1} = \frac{a_k(2k-1)^2}{2(k+1)(2k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Revert the change of variables $u = t - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (t-1)^{k+\frac{1}{2}}, a_{k+1} = \frac{2a_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (t-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (t-1)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k (2k-1)^2}{2(k+1)(2k+1)}, b_{k+1} = \frac{2b_k k^2}{\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(4*(t^2-3*t+2)*diff(y(t),t$2)-2*diff(y(t),t)+y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\sqrt{t-1} + \frac{c_2\sqrt{t-2}\,(t-1)\left(\ln\left(t-\frac{3}{2}+\sqrt{t^2-3t+2}\right)\sqrt{t^2-3t+2}-2t+4\right)}{t^2-3t+2}$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 53

```
DSolve[4*(t^2-3*t+2)*y''[t]-2*y'[t]+y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \sqrt{1-t}\left(-2c_2\operatorname{arctanh}\left(\frac{1}{\sqrt{\frac{t-1}{t-2}}}\right)+\frac{2c_2}{\sqrt{\frac{t-1}{t-2}}}+c_1\right)$$

2.781 problem 798

2.781.1 Maple step by step solution 7311

Internal problem ID [8271]
Internal file name [`OUTPUT/7204_Sunday_June_05_2022_05_35_48_PM_9720514/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 798.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2(t^2-5t+6)\,y''+(2t-3)\,y'-8y=0}$$

Writing the ode as

$$y''(2t^2-10t+12)+(2t-3)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 2t^2-10t+12\\ B &= 2t-3\\ C &= -8\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{60t^2 - 308t + 381}{16\,(t^2 - 5t + 6)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 60t^2 - 308t + 381 \\ t &= 16\big(t^2 - 5t + 6\big)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{60t^2 - 308t + 381}{16\,(t^2 - 5t + 6)^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1482: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(t^2 - 5t + 6)^2$. There is a pole at $t = 3$ of order 2. There is a pole at $t = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{29}{8(t-3)} - \frac{3}{16(t-3)^2} + \frac{5}{16(t-2)^2} - \frac{29}{8(t-2)}$$

For the pole at $t = 3$ let b be the coefficient of $\frac{1}{(t-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{4}
\end{aligned}$$

For the pole at $t = 2$ let b be the coefficient of $\frac{1}{(t-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{60t^2 - 308t + 381}{16(t^2 - 5t + 6)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{60t^2 - 308t + 381}{16\left(t^2 - 5t + 6\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
3	2	0	$\frac{3}{4}$	$\frac{1}{4}$
2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{5}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= \frac{5}{2} - \left(\frac{3}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{t-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} + (0) \\
&= \frac{1}{4t-12} + \frac{5}{4(t-2)} \\
&= \frac{6t-17}{4(t-2)(t-3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 1$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = t + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)(1) + \left(\left(-\frac{1}{4(t-3)^2} - \frac{5}{4(t-2)^2}\right) + \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right)^2 - \left(\frac{60t^2 - \cdots}{16(t^2 \cdots}\right.\right.$$

$$\frac{-6\cdots}{2t^2 - \cdots}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\left\{a_0 = -\frac{17}{6}\right\}$$

Substituting these coefficients in $p(t)$ in eq. (2A) results in

$$p(t) = t - \frac{17}{6}$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \left(t - \frac{17}{6}\right) e^{\int \left(\frac{1}{4t-12} + \frac{5}{4(t-2)}\right) dt} \\
&= \left(t - \frac{17}{6}\right) e^{\frac{\ln(t-3)}{4} + \frac{5\ln(t-2)}{4}} \\
&= \left(t - \frac{17}{6}\right) (t-3)^{\frac{1}{4}} (t-2)^{\frac{5}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2t-3}{2t^2-10t+12}\,dt} \\
&= z_1 e^{\frac{\ln(t-2)}{4}-\frac{3\ln(t-3)}{4}} \\
&= z_1 \left(\frac{(t-2)^{\frac{1}{4}}}{(t-3)^{\frac{3}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t-3}{2t^2-10t+12}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\frac{\ln(t-2)}{2}-\frac{3\ln(t-3)}{2}}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{(576t^2-2496t+2664)\sqrt{t-3}}{(t-2)^{\frac{3}{2}}\,(30t-85)}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}}\right) + c_2 \left(\frac{(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}}\left(\frac{(576t^2-2496t+2664)\sqrt{t-3}}{(t-2)^{\frac{3}{2}}\,(30t-85)}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(6t-17)\,(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}} + c_2\left(\frac{96}{5}t^2 - \frac{416}{5}t + \frac{444}{5}\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(6t-17)(t-2)^{\frac{3}{2}}}{6\sqrt{t-3}} + c_2\left(\frac{96}{5}t^2 - \frac{416}{5}t + \frac{444}{5}\right)$$

Verified OK.

2.781.1 Maple step by step solution

Let's solve

$y''(2t^2-10t+12)+(2t-3)\,y'-8y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y}{t^2-5t+6} - \frac{(2t-3)y'}{2(t^2-5t+6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2t-3)y'}{2(t^2-5t+6)} - \frac{4y}{t^2-5t+6} = 0$

□ Check to see if t_0 is a regular singular point

  - Define functions

    $\left[P_2(t) = \frac{2t-3}{2(t^2-5t+6)}, P_3(t) = -\frac{4}{t^2-5t+6}\right]$

  - $(t-2)\cdot P_2(t)$ is analytic at $t=2$

    $\left((t-2)\cdot P_2(t)\right)\Big|_{t=2} = -\frac{1}{2}$

  - $(t-2)^2\cdot P_3(t)$ is analytic at $t=2$

    $\left((t-2)^2\cdot P_3(t)\right)\Big|_{t=2} = 0$

  - $t=2$is a regular singular point

  Check to see if t_0 is a regular singular point

  $t_0 = 2$

- Multiply by denominators

  $y''(2t^2-10t+12)+(2t-3)\,y'-8y=0$

- Change variables using $t=u+2$ so that the regular singular point is at $u=0$

  $(2u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u+1)\left(\frac{d}{du}y(u)\right)-8y(u)=0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-3+2r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k-1+2r) + 2a_k(k+r+2)\,(k+r-2)\right) u^{k+r}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + 2a_k(k+r+2)\,(k+r-2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{2a_k(k+r+2)(k+r-2)}{(k+1+r)(2k-1+2r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{2a_k(k+2)(k-2)}{(k+1)(2k-1)}$

- Apply recursion relation for $k = 0$

  $a_1 = 8a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -3a_1$

- Express in terms of a_0

  $a_2 = -24a_0$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot (-24u^2 + 8u + 1)$

- Revert the change of variables $u = t - 2$

  $[y = a_0(-24t^2 + 104t - 111)]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}$

- Solution for $r = \frac{3}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Revert the change of variables $u = t - 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (t-2)^{k+\frac{3}{2}}, a_{k+1} = \frac{2a_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

- Combine solutions and rename parameters

  $\left[y = a_0(-24t^2 + 104t - 111) + \left(\sum_{k=0}^{\infty} b_k (t-2)^{k+\frac{3}{2}}\right), b_{k+1} = \frac{2b_k\left(k+\frac{7}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{5}{2}\right)(2k+2)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(2*(t^2-5*t+6)*diff(y(t),t$2)+(2*t-3)*diff(y(t),t)-8*y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1\left(t^2 - \frac{13}{3}t + \frac{37}{8}\right) + \frac{c_2(6t-17)\,(t-2)^{\frac{3}{2}}}{\sqrt{t-3}}$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 84

```
DSolve[2*(t^2-5*t+6)*y''[t]+(2*t-3)*y'[t]-8*y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{\sqrt[4]{2-t}\left(5c_1\sqrt[4]{t-3}\sqrt{t-2}(6t^2-29t+34)+24c_2(t-3)^{3/4}\left(24t^2-104t+111\right)\right)}{30(3-t)^{3/4}\sqrt[4]{t-2}}$$

2.782 problem 799

2.782.1 Maple step by step solution 7321

Internal problem ID [8272]
Internal file name [`OUTPUT/7205_Sunday_June_05_2022_05_35_51_PM_84602373/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 799.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3t(t+1)\,y''+ty'-y=0}$$

Writing the ode as

$$\left(3t^2+3t\right)y''+ty'-y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3t^2+3t \\ B &= t \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t)=ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t)=rz(t) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{7t + 12}{36\,(t+1)^2\,t} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 7t + 12 \\
t &= 36(t+1)^2\,t
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{7t + 12}{36t\,(t+1)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$
y = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1484: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(t+1)^2 t$. There is a pole at $t = -1$ of order 2. There is a pole at $t = 0$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $t = 0$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{3t} - \frac{1}{3(t+1)} - \frac{5}{36(t+1)^2}
$$

For the pole at $t = -1$ let b be the coefficient of $\frac{1}{(t+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{36}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{6} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{6}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{7t+12}{36t(t+1)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{7}{36}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{7}{6}\\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{6}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{7t+12}{36t\,(t+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
-1	2	0	$\frac{5}{6}$	$\frac{1}{6}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{7}{6}$	$-\frac{1}{6}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+}=\frac{7}{6}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right)\\
&= \frac{7}{6}-\left(\frac{7}{6}\right)\\
&= 0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{t-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{t-c_2}\right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{t} + \frac{1}{6t+6} + (0) \\
&= \frac{1}{t} + \frac{1}{6t+6} \\
&= \frac{1}{t} + \frac{1}{6t+6}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{t} + \frac{1}{6t+6}\right)(0) + \left(\left(-\frac{1}{t^2} - \frac{1}{6\left(t+1\right)^2}\right) + \left(\frac{1}{t} + \frac{1}{6t+6}\right)^2 - \left(\frac{7t+12}{36t\left(t+1\right)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \left(\frac{1}{t} + \frac{1}{6t+6}\right) dt} \\
&= t(t+1)^{\frac{1}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{t}{3t^2+3t} \, dt} \\
&= z_1 e^{-\frac{\ln(t+1)}{6}} \\
&= z_1 \left(\frac{1}{(t+1)^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = t$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{t}{3t^2+3t}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-\frac{\ln(t+1)}{3}}}{(y_1)^2}\, dt \\ &= y_1 \left(\frac{-2\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t - 2\ln\left((t+1)^{\frac{1}{3}} - 1\right) t - 6(t+1)^{\frac{2}{3}} + \ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right.}{6\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \right. \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(t) \\ &\quad + c_2 \left(t \left(\frac{-2\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t - 2\ln\left((t+1)^{\frac{1}{3}} - 1\right) t - 6(t+1)^{\frac{2}{3}} + \ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}}\right.}{6\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \right.\right. \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 t - \frac{c_2\left(\sqrt{3}\, \arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right) t + \ln\left((t+1)^{\frac{1}{3}} - 1\right) t - \frac{\ln\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right) t}{2} + 3(t+1)^{\frac{2}{3}}\right) t}{3\left((t+1)^{\frac{2}{3}} + (t+1)^{\frac{1}{3}} + 1\right)\left((t+1)^{\frac{1}{3}} - 1\right)} \tag{1}$$

Verification of solutions

$$y = c_1 t - \frac{c_2\left(\sqrt{3}\,\arctan\left(\frac{\left(2(t+1)^{\frac{1}{3}}+1\right)\sqrt{3}}{3}\right)t + \ln\left((t+1)^{\frac{1}{3}}-1\right)t - \frac{\ln\left((t+1)^{\frac{2}{3}}+(t+1)^{\frac{1}{3}}+1\right)t}{2} + 3(t+1)^{\frac{2}{3}}\right)t}{3\left((t+1)^{\frac{2}{3}}+(t+1)^{\frac{1}{3}}+1\right)\left((t+1)^{\frac{1}{3}}-1\right)}$$

Verified OK.

2.782.1 Maple step by step solution

Let's solve

$(3t^2+3t)\,y''+ty'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{3t(t+1)}-\frac{y'}{3(t+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{y'}{3(t+1)}-\frac{y}{3t(t+1)}=0$

□ Check to see if t_0 is a regular singular point

  ◦ Define functions

    $\left[P_2(t)=\frac{1}{3(t+1)}, P_3(t)=-\frac{1}{3t(t+1)}\right]$

  ◦ $(t+1)\cdot P_2(t)$ is analytic at $t=-1$

    $\left((t+1)\cdot P_2(t)\right)\Big|_{t=-1}=\frac{1}{3}$

  ◦ $(t+1)^2\cdot P_3(t)$ is analytic at $t=-1$

    $\left((t+1)^2\cdot P_3(t)\right)\Big|_{t=-1}=0$

  ◦ $t=-1$is a regular singular point

    Check to see if t_0 is a regular singular point

    $t_0=-1$

- Multiply by denominators

  $3t(t+1)\,y''+ty'-y=0$

- Change variables using $t = u - 1$ so that the regular singular point is at $u = 0$

$$\left(3u^2 - 3u\right)\left(\frac{d^2}{du^2}y(u)\right) + (u-1)\left(\frac{d}{du}y(u)\right) - y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

  ◦ Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$$

  ◦ Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-a_0 r(-2+3r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)(3k+3r+1) + a_k(3k+3r+1)(k+r-1)\right) u^{k-}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{2}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$3\left(k+r+\frac{1}{3}\right)\left((-k-r-1)\, a_{k+1} + a_k(k+r-1)\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(k+r-1)}{k+1+r}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_k(k-1)}{k+1}$$

- Apply recursion relation for $k=0$

  $a_1=-a_0$

- Terminating series solution of the ODE for $r=0$. Use reduction of order to find the second li

  $y(u)=a_0\cdot(-u+1)$

- Revert the change of variables $u=t+1$

  $[y=-a_0 t]$

- Recursion relation for $r=\frac{2}{3}$

  $a_{k+1}=\frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}$

- Solution for $r=\frac{2}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Revert the change of variables $u=t+1$

  $\left[y=\sum_{k=0}^{\infty} a_k (t+1)^{k+\frac{2}{3}}, a_{k+1}=\frac{a_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

- Combine solutions and rename parameters

  $\left[y=-a_0 t+\left(\sum_{k=0}^{\infty} b_k (t+1)^{k+\frac{2}{3}}\right), b_{k+1}=\frac{b_k\left(k-\frac{1}{3}\right)}{k+\frac{5}{3}}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(3*t*(1+t)*diff(y(t),t$2)+t*diff(y(t),t)-y(t)=0,y(t), singsol=all)
```

$$y(t) = c_1 t + c_2 t \left(\int \frac{1}{(t+1)^{\frac{1}{3}} t^2} dt \right)$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 93

```
DSolve[3*t*(1+t)*y''[t]+t*y'[t]-y[t]==0,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow \frac{6c_1t - c_2\left(2\sqrt{3}t\arctan\left(\frac{2\sqrt[3]{t+1}+1}{\sqrt{3}}\right) + 6(t+1)^{2/3} + 2t\log\left(\sqrt[3]{t+1}-1\right) - t\log\left((t+1)^{2/3}+\sqrt[3]{t+1}\right.\right.}{6\sqrt[6]{3}}$$

2.783 problem 800

2.783.1 Maple step by step solution 7331

Internal problem ID [8273]
Internal file name [OUTPUT/7206_Sunday_June_05_2022_05_35_56_PM_74863823/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 800.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \frac{\left(\frac{3}{4} + x\right)y}{4} = 0}$$

Writing the ode as

$$x^2y'' + \left(\frac{3}{16} + \frac{x}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= \frac{3}{16} + \frac{x}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3 - 4x}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3 - 4x$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3 - 4x}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1486: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{1}{2}(1-(1)) \\
&= 0
\end{aligned}
$$

We now form the following rational function

$$
\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\
&= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$
p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}
$$

Since $d=0$, then letting

$$
p = 1 \tag{2A}
$$

Substituting p and θ into Eq. (1A) gives

$$
0 = 0
$$

And solving for p gives

$$
p = 1
$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{2x}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{2\sqrt{-x}+1}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{2\sqrt{-x}+1}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{2\sqrt{-x}}} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^{\frac{1}{4}}\mathrm{e}^{\sqrt{-x}}\right) + c_2\left(x^{\frac{1}{4}}\mathrm{e}^{\sqrt{-x}}\left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^{\frac{1}{4}}\mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{x^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = c_1x^{\frac{1}{4}}\mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{x^{\frac{1}{4}}}$$

Verified OK.

2.783.1 Maple step by step solution

Let's solve

$x^2y'' + \left(\frac{3}{16}+\frac{x}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3+4x)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3+4x)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{3+4x}{16x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2 y'' + (3 + 4x)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + 4r)\,(-3 + 4r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k + 4r - 1)\,(4k + 4r - 3) + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1 + 4r)\,(-3 + 4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k + r - \frac{3}{4}\right)\left(k + r - \frac{1}{4}\right) a_k + 4a_{k-1} = 0$

- Shift index using $k- >k + 1$

  $16\left(k + \frac{1}{4} + r\right)\left(k + \frac{3}{4} + r\right) a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{(4k+1+4r)(4k+3+4r)}$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}\right]$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}$

- Solution for $r = \frac{3}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = -\frac{4a_k}{(4k+4)(4k+6)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = -\frac{4a_k}{(4k+2)(4k+4)}, b_{k+1} = -\frac{4b_k}{(4k+4)(4k+6)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+1/4*(x+3/4)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\sqrt{x}\right) x^{\frac{1}{4}} + c_2 x^{\frac{1}{4}} \cos\left(\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 43

```
DSolve[x^2*y''[x]+1/4*(x+3/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-i\sqrt{x}} \sqrt[4]{x} \left(c_1 e^{2i\sqrt{x}} + ic_2\right)$$

2.784 problem 801

2.784.1 Maple step by step solution 7338

Internal problem ID [8274]
Internal file name [OUTPUT/7207_Sunday_June_05_2022_05_35_59_PM_28567150/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 801.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \frac{(x^2-1)\,y}{4} = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(\frac{x^2}{4} - \frac{1}{4}\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= \frac{x^2}{4} - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1488: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx}\\ &= z_1 e^{-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos\left(\frac{x}{2}\right)}{\sqrt{x}} \left(2\tan\left(\frac{x}{2}\right)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{2c_2 \sin\left(\frac{x}{2}\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{2c_2 \sin\left(\frac{x}{2}\right)}{\sqrt{x}}
$$

Verified OK.

2.784.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(\frac{x^2}{4} - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + a_{k-2}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

$a_1(3+2r)(1+2r)=0$

- Solve for the dependent coefficient(s)

$a_1=0$

- Each term in the series must be 0, giving the recursion relation

$a_k(4k^2+8kr+4r^2-1)+a_{k-2}=0$

- Shift index using $k->k+2$

$a_{k+2}\left(4(k+2)^2+8(k+2)r+4r^2-1\right)+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+2}=-\frac{a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r=-\frac{1}{2}$

$a_{k+2}=-\frac{a_k}{4k^2+12k+8}$

- Solution for $r=-\frac{1}{2}$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{a_k}{4k^2+12k+8}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

$a_{k+2}=-\frac{a_k}{4k^2+20k+24}$

- Solution for $r=\frac{1}{2}$

$\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{a_k}{4k^2+20k+24}, a_1=0\right]$

- Combine solutions and rename parameters

$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+1/4*(x^2-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{x}{2}\right)}{\sqrt{x}} + \frac{c_2 \cos\left(\frac{x}{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 36

```
DSolve[x^2*y''[x]+x*y'[x]+1/4*(x^2-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{ix}{2}}\left(c_1 - ic_2 e^{ix}\right)}{\sqrt{x}}$$

2.785 problem 802

2.785.1 Maple step by step solution 7347

Internal problem ID [8275]
Internal file name [`OUTPUT/7208_Sunday_June_05_2022_05_36_01_PM_23327978/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 802.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1-2x)\,y' + (x-1)\,y = 0}$$

Writing the ode as

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1-2x \\ C &= x-1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1490: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1-2x}{x}\, dx} \\ &= z_1 e^{x - \frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1-2x}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)+2x}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\ln(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^x \ln(x)$$

Verified OK.

2.785.1 Maple step by step solution

Let's solve

$xy'' + (1-2x)\,y' + (x-1)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x-1)y}{x} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(x-1)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{x-1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$xy'' + (1-2x)\,y' + (x-1)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r^2 x^{-1+r} + \left(a_1(1+r)^2 - a_0(1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k(2k+2r+1) + a_{k-1}\right) x^{k}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r^2 = 0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 - a_0(1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + (-2k-1)\, a_k + a_{k-1} = 0$
- Shift index using $k- > k+1$

  $a_{k+2}(k+2)^2 + (-2k-3)\, a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{2ka_{k+1} - a_k + 3a_{k+1}}{(k+2)^2}, a_1 - a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x$2)+(1-2*x)*diff(y(x),x)+(x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x \ln(x)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 17

```
DSolve[x*y''[x]+(1-2*x)*y'[x]+(x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 \log(x) + c_1)$$

2.786 problem 803

2.786.1 Maple step by step solution 7357

Internal problem ID [8276]
Internal file name [OUTPUT/7209_Sunday_June_05_2022_05_36_04_PM_7509697/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 803.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[_Laguerre]
```

$$\boxed{xy'' - (1+x)\,y' + y = 0}$$

Writing the ode as

$$xy'' + (-x-1)\,y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x-1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 2x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 2x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 2x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{2x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{2x^2} + \frac{1}{2x^3} + \frac{1}{4x^4} - \frac{1}{4x^5} - \frac{3}{4x^6} - \frac{3}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 2x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 3}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-2x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (0) \\
&= -\frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 2x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{+} - (\alpha_{c_1}^{-}) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2x} \\
&= \frac{x-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2} - \frac{1}{2x}\right)^2 - \left(\frac{x^2-2x+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-1}{x}\, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(-(1+x)\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(-(1+x)\,\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^x + c_2(-x-1) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x + c_2(-x-1)$$

Verified OK.

2.786.1 Maple step by step solution

Let's solve

$x y'' + (-x-1)\, y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x} + \frac{(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(1+x)y'}{x} + \frac{y}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1+x}{x}, P_3(x) = \frac{1}{x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (-x - 1)\, y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k}{k+3}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x$2)-(x+1)*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x+1) + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 19

```
DSolve[x*y''[x]-(x+1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2(x+1)$$

2.787 problem 804

2.787.1 Maple step by step solution 7367

Internal problem ID [8277]
Internal file name [OUTPUT/7210_Sunday_June_05_2022_05_36_08_PM_79351930/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 804.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + 3y' + 4yx^3 = 0}$$

Writing the ode as

$$xy'' + 3y' + 4yx^3 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 3 \\ C &= 4x^3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = -4x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4 + 3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{0}{2i} - 1\right) = -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{0}{2i} - 1\right) = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{x}\, dx} \\ &= z_1 e^{-\frac{3 \ln(x)}{2}} \\ &= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{e^{-ix^2}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2} dx \\
&= y_1 \left(-\frac{i \mathrm{e}^{2i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-i x^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-i x^2}}{x^2} \left(-\frac{i \mathrm{e}^{2i x^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \mathrm{e}^{-i x^2}}{x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4x^2}
$$

Verified OK.

2.787.1 Maple step by step solution

Let's solve

$x y'' + 3y' + 4y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{x} + 4x^2 y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 3y' + 4yx^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- >k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- >k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 0\}$$

- The coefficients of each power of x must be 0

$$[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$$

- Shift index using $k- >k+3$

$$a_{k+4}(k+4+r)\,(k+6+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$$

- Recursion relation for $r = -2$

$$a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$$

- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+3*diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sin(x^2)}{x^2}+\frac{c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 41

```
DSolve[x*y''[x]+3*y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{4c_1 e^{-ix^2}-ic_2 e^{ix^2}}{4x^2}$$

2.788 problem 805

Internal problem ID [8278]
Internal file name [`OUTPUT/7211_Sunday_June_05_2022_05_36_11_PM_74166679/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 805.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(1-x^2)\,y''+2x(1-x^2)\,y'-2y=0}$$

Writing the ode as

$$(-x^4+x^2)\,y''+(-2x^3+2x)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+x^2 \\ B &= -2x^3+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{x^2 (x^2 - 1)} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= x^2 (x^2 - 1) \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{2}{x^2 (x^2 - 1)}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2(x^2-1)$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 1. There is a pole at $x = -1$ of order 1. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2} - \frac{1}{x-1} + \frac{1}{1+x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{x^2(x^2-1)}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	1	0	0	1
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= 1 - (0) \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{x-1} - \frac{1}{x} + (-)(0) \\ &= \frac{1}{x-1} - \frac{1}{x} \\ &= \frac{1}{x^2-x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x-1} - \frac{1}{x}\right)(1) + \left(\left(-\frac{1}{(x-1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x-1} - \frac{1}{x}\right)^2 - \left(-\frac{2}{x^2\,(x^2-1)}\right)\right) = 0$$

$$\frac{-2a_0 + 2}{x^3 - x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (1+x)\, \mathrm{e}^{\int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx} \\ &= (1+x)\, \mathrm{e}^{\ln(x-1) - \ln(x)} \\ &= \frac{x^2-1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x^3+2x}{-x^4+x^2}\, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^2-1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^3+2x}{-x^4+x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2-1}{x^2}\right) + c_2 \left(\frac{x^2-1}{x^2} \left(-\frac{1}{4x+4} - \frac{\ln(1+x)}{4} - \frac{1}{4x-4} + \frac{\ln(x-1)}{4}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(-\ln(1+x)\, x^2 + \ln(x-1)\, x^2 + \ln(1+x) - \ln(x-1) - 2x)}{4x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-1)}{x^2} + \frac{c_2(-\ln(1+x)\, x^2 + \ln(x-1)\, x^2 + \ln(1+x) - \ln(x-1) - 2x)}{4x^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
dsolve(x^2*(1-x^2)*diff(y(x),x$2)+2*x*(1-x^2)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x^2-1)}{x^2} + \frac{c_2\left(-\frac{\ln(x+1)x^2}{4} + \frac{\ln(x-1)x^2}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} - \frac{x}{2}\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 56

```
DSolve[x^2*(1-x^2)*y''[x]+2*x*(1-x^2)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{-4c_1x^2 - c_2(x^2-1)\log(1-x) + c_2(x^2-1)\log(x+1) + 2c_2x + 4c_1}{4x^2}$$

2.789 problem 806

2.789.1 Maple step by step solution 7384

Internal problem ID [8279]
Internal file name [OUTPUT/7212_Sunday_June_05_2022_05_36_15_PM_13968250/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 806.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + (x-2)\,y' - y = 0}$$

Writing the ode as

$$2xy'' + (x-2)\,y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x \\ B &= x-2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x + 12}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x + 12$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 + 4x + 12}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{16} + \frac{1}{4x} + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{4} + \frac{1}{2x} + \frac{1}{x^2} - \frac{2}{x^3} + \frac{2}{x^4} + \frac{4}{x^5} - \frac{24}{x^6} + \frac{48}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{4}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{4}\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{16}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{x^2 + 4x + 12}{16x^2} \\ &= Q + \frac{R}{16x^2} \\ &= \left(\frac{1}{16}\right) + \left(\frac{4x + 12}{16x^2}\right) \\ &= \frac{1}{16} + \frac{4x + 12}{16x^2}\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 4. Dividing this by leading coefficient in t which is 16 gives $\frac{1}{4}$. Now b can be found.

$$\begin{aligned} b &= \left(\frac{1}{4}\right) - (0) \\ &= \frac{1}{4} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{4} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{4}}{\frac{1}{4}} - 0\right) &&= -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 + 4x + 12}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(\frac{1}{4}\right) \\
&= -\frac{1}{2x} - \frac{1}{4} \\
&= -\frac{x+2}{4x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - \frac{1}{4}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x} - \frac{1}{4}\right)^2 - \left(\frac{x^2 + 4x + 12}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{4}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x}{4}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x-2}{2x}\, dx} \\
&= z_1 e^{-\frac{x}{4} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x-2}{2x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{x}{2}+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1\left(2(x-2)\,e^{\frac{x}{2}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-\frac{x}{2}}\right) + c_2\left(e^{-\frac{x}{2}}\left(2(x-2)\,e^{\frac{x}{2}}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + c_2(2x-4) \tag{1}$$

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + c_2(2x-4)$$

Verified OK.

2.789.1 Maple step by step solution

Let's solve

$2xy'' + (x-2)\,y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{y}{2x} - \frac{(x-2)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-2)y'}{2x} - \frac{y}{2x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{x-2}{2x}, P_3(x) = -\frac{1}{2x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- Multiply by denominators

  $2xy'' + (x-2)\,y' - y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2(k+r-1)\left(a_{k+1}(k+1+r) + \frac{a_k}{2}\right) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{a_k}{2(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{a_k}{2(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)}\right]$$

- Recursion relation for $r = 2$

  $$a_{k+1} = -\frac{a_k}{2(k+3)}$$

- Solution for $r = 2$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+1} = -\frac{a_k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+1} = -\frac{a_k}{2(k+1)}, b_{k+1} = -\frac{b_k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(2*x*diff(y(x),x$2)+(x-2)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x-2) + c_2 e^{-\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 23

```
DSolve[2*x*y''[x]+(x-2)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} + 2c_2(x-2)$$

2.790 problem 807

2.790.1 Maple step by step solution 7391

Internal problem ID [8280]
Internal file name [OUTPUT/7213_Sunday_June_05_2022_05_36_19_PM_7084982/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 807.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.790.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 2$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$xy'' + 2y' + yx = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.791 problem 808

2.791.1 Maple step by step solution 7398

Internal problem ID [8281]
Internal file name [`OUTPUT/7214_Sunday_June_05_2022_05_36_21_PM_12578656/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 808.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0}$$

Writing the ode as

$$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= x^4 + 2x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\,dx} \\ &= z_1 e^{-\frac{x^3}{3}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^3}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x\left(x^2+3\right)}{3}} + \frac{c_2 \mathrm{e}^{-\frac{x\left(x^2-3\right)}{3}}}{2}
$$

Verified OK.

2.791.1 Maple step by step solution

Let's solve

$y'' + 2x^2 y' + \left(x^4 + 2x - 1\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite ODE with series expansions
  - ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..4$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - ∘ Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - ∘ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^{k+1}$$

- Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1) \, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1) \, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1 + 2a_0)\,x + (12a_4 - a_2 + 4a_1)\,x^2 + (20a_5 - a_3 + 6a_2)\,x^3 + \left(\sum_{k=4}^{\infty} (a_{k+2}(k+2)$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_0 = 0, 6a_3 - a_1 + 2a_0 = 0, 12a_4 - a_2 + 4a_1 = 0, 20a_5 - a_3 + 6a_2 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\,a_{k+2} + 2a_{k-1}k - a_k + a_{k-4} = 0$$

- Shift index using $k- >k+4$

$$\left((k+4)^2 + 3k + 14\right)a_{k+6} + 2a_{k+3}(k+4) - a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+6} = -\frac{2ka_{k+3} + a_k + 8a_{k+3} - a_{k+4}}{k^2 + 11k + 30}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6} - \frac{a_0}{3}, a_4 = \frac{a_0}{24} - \frac{a_1}{3}, a_5 = \frac{a_1}{120} - \frac{a_0}{6}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+(x^4+2*x-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{1}{3}x^3-x} + c_2 e^{-\frac{1}{3}x^3+x}$$

✓ Solution by Mathematica
Time used: 0.035 (sec). Leaf size: 34

```
DSolve[y''[x]+2*x^2*y'[x]+(x^4+2*x-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-\frac{1}{3}x(x^2+3)} \left(c_2 e^{2x} + 2c_1\right)$$

2.792 problem 809

2.792.1 Maple step by step solution 7405

Internal problem ID [8282]
Internal file name [`OUTPUT/7215_Sunday_June_05_2022_05_36_23_PM_40500981/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 809.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{u'' + 2u' + u = 0}$$

Writing the ode as

$$u'' + 2u' + u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-x}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(u_1)^2}\,dx \\
&= u_1 \int \frac{e^{-2x}}{(u_1)^2}\,dx \\
&= u_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
u = c_1 \mathrm{e}^{-x} + c_2 x \, \mathrm{e}^{-x} \tag{1}
$$

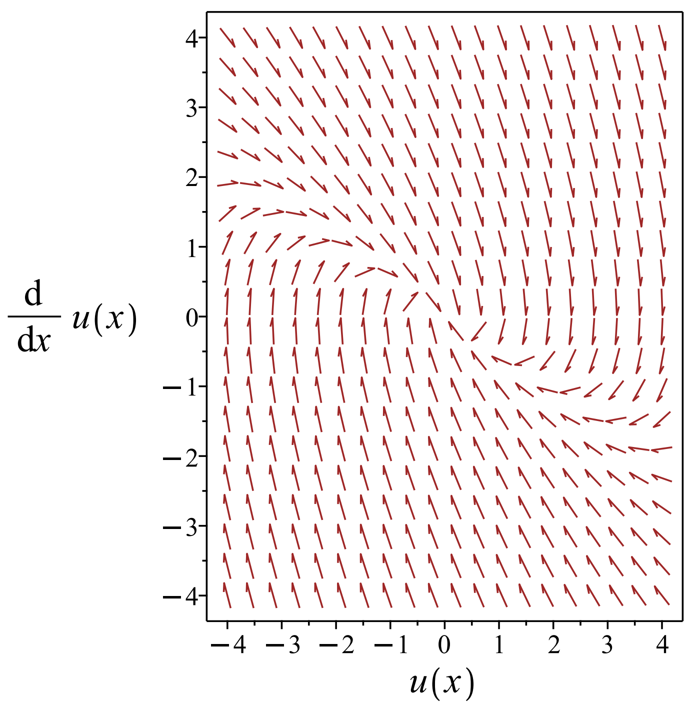

Figure 6: Slope field plot

Verification of solutions

$$
u = c_1 \mathrm{e}^{-x} + c_2 x \, \mathrm{e}^{-x}
$$

Verified OK.

2.792.1 Maple step by step solution

Let's solve

$u'' + 2u' + u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $u_1(x) = e^{-x}$

- Repeated root, multiply $u_1(x)$ by x to ensure linear independence

  $u_2(x) = x\,e^{-x}$

- General solution of the ODE

  $u = c_1 u_1(x) + c_2 u_2(x)$

- Substitute in solutions

  $u = c_1 e^{-x} + c_2 x\,e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(u(x),x$2)+2*diff(u(x),x)+u(x)=0,u(x), singsol=all)
```

$$u(x) = \mathrm{e}^{-x}c_1 + \mathrm{e}^{-x}c_2x$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[u''[x]+2*u'[x]+u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to e^{-x}(c_2x + c_1)$$

2.793 problem 810

2.793.1 Maple step by step solution 7410

Internal problem ID [8283]
Internal file name [`OUTPUT/7216_Sunday_June_05_2022_05_36_26_PM_46095909/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 810.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{u'' - (2x+1)\,u' + \left(x^2 + x - 1\right) u = 0}$$

Writing the ode as

$$u'' + (-2x-1)\,u' + \left(x^2 + x - 1\right) u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2x - 1 \\ C &= x^2 + x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{1}\,dx} \\ &= z_1 e^{\frac{1}{2}x^2+\frac{1}{2}x} \\ &= z_1 \left(\mathrm{e}^{\frac{x(1+x)}{2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{x^2}{2}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{u_1^2}\,dx$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{-2x-1}{1}\,dx}}{(u_1)^2}\,dx \\ &= u_1 \int \frac{e^{x^2+x}}{(u_1)^2}\,dx \\ &= u_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} u &= c_1 u_1 + c_2 u_2 \\ &= c_1 \left(\mathrm{e}^{\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{x^2}{2}} (\mathrm{e}^x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$u = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{x(x+2)}{2}} \tag{1}$$

Verification of solutions

$$u = c_1 \mathrm{e}^{\frac{x^2}{2}} + c_2 \mathrm{e}^{\frac{x(x+2)}{2}}$$

Verified OK.

2.793.1 Maple step by step solution

Let's solve

$u'' + (-2x - 1) u' + (x^2 + x - 1) u = 0$

- Highest derivative means the order of the ODE is 2

  u''

- Assume series solution for u

  $u = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot u$ to series expansion for $m = 0..2$

  $x^m \cdot u = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot u = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x^m \cdot u'$ to series expansion for $m = 0..1$

  $x^m \cdot u' = \sum_{k=\max(0,1-m)}^{\infty} a_k k\, x^{k-1+m}$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot u' = \sum_{k=\max(0,1-m)+m-1}^{\infty} a_{k+1-m}(k+1-m)\, x^k$$

- Convert u'' to series expansion

$$u'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$u'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - a_1 - a_0 + (6a_3 - 2a_2 - 3a_1 + a_0)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_{k+1}(k+1) - a_k(2k+1$$

- The coefficients of each power of x must be 0

$$[2a_2 - a_1 - a_0 = 0, 6a_3 - 2a_2 - 3a_1 + a_0 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$k^2 a_{k+2} + (-2a_k - a_{k+1} + 3a_{k+2})\, k - a_k + a_{k-2} + a_{k-1} - a_{k+1} + 2a_{k+2} = 0$$

- Shift index using $k- >k+2$

$$(k+2)^2\, a_{k+4} + (-2a_{k+2} - a_{k+3} + 3a_{k+4})\,(k+2) - a_{k+2} + a_k + a_{k+1} - a_{k+3} + 2a_{k+4} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[u = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2k a_{k+2} + k a_{k+3} - a_k - a_{k+1} + 5a_{k+2} + 3a_{k+3}}{k^2+7k+12}, a_2 = \frac{a_1}{2} + \frac{a_0}{2}, a_3 = \frac{2a_1}{3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(u(x),x$2)-(2*x+1)*diff(u(x),x)+(x^2+x-1)*u(x)=0,u(x), singsol=all)
```

$$u(x) = c_1 e^{\frac{x^2}{2}} + c_2 e^{\frac{1}{2}x^2+x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 24

```
DSolve[u''[x]-(2*x+1)*u'[x]+(x^2+x-1)*u[x]==0,u[x],x,IncludeSingularSolutions -> True]
```

$$u(x) \to e^{\frac{x^2}{2}}(c_2 e^x + c_1)$$

2.794 problem 811

2.794.1 Maple step by step solution 7418

Internal problem ID [8284]
Internal file name [`OUTPUT/7217_Sunday_June_05_2022_05_36_28_PM_18865124/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 811.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''+2y'+\left(1+\frac{2}{\left(3x+1\right)^{2}}\right)y=0}$$

Writing the ode as

$$y''+2y'+\left(1+\frac{2}{\left(3x+1\right)^{2}}\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=1\\ B&=2\\ C&=1+\frac{2}{\left(3x+1\right)^{2}} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-2}{(3x+1)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -2 \\
t &= (3x+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{2}{(3x+1)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (3x+1)^2$. There is a pole at $x = -\frac{1}{3}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9\left(x+\frac{1}{3}\right)^2}$$

For the pole at $x = -\frac{1}{3}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{3}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{2}{(3x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{2}{9}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{2}{3} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{3} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{2}{(3x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{3}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{2}{3}$	$\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{3}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{3} - \left(\frac{1}{3}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{3x+1} + (-)(0) \\
&= \frac{1}{3x+1} \\
&= \frac{1}{3x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{3x+1}\right)(0) + \left(\left(-\frac{1}{3\left(x+\frac{1}{3}\right)^2}\right) + \left(\frac{1}{3x+1}\right)^2 - \left(-\frac{2}{(3x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{3x+1} dx} \\ &= (3x+1)^{\frac{1}{3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2}{1} \, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (3x+1)^{\frac{1}{3}} \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left((3x+1)^{\frac{1}{3}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left((3x+1)^{\frac{1}{3}}\, \mathrm{e}^{-x}\right) + c_2 \left((3x+1)^{\frac{1}{3}}\, \mathrm{e}^{-x} \left((3x+1)^{\frac{1}{3}}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 (3x+1)^{\frac{1}{3}}\, \mathrm{e}^{-x} + c_2 (3x+1)^{\frac{2}{3}}\, \mathrm{e}^{-x} \tag{1}
$$

Verification of solutions

$$
y = c_1 (3x+1)^{\frac{1}{3}}\, \mathrm{e}^{-x} + c_2 (3x+1)^{\frac{2}{3}}\, \mathrm{e}^{-x}
$$

Verified OK.

2.794.1 Maple step by step solution

Let's solve

$y'' + 2y' + \left(1 + \frac{2}{(3x+1)^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3(3x^2+2x+1)y}{(3x+1)^2} - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{3(3x^2+2x+1)y}{(3x+1)^2} = 0$

□ Check to see if $x_0 = -\frac{1}{3}$ is a regular singular point

- Define functions

  $\left[P_2(x)=2, P_3(x)=\frac{3(3x^2+2x+1)}{(3x+1)^2}\right]$

- $\left(x+\frac{1}{3}\right)\cdot P_2(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)\cdot P_2(x)\right)\bigg|_{x=-\frac{1}{3}}=0$

- $\left(x+\frac{1}{3}\right)^2\cdot P_3(x)$ is analytic at $x=-\frac{1}{3}$

  $\left(\left(x+\frac{1}{3}\right)^2\cdot P_3(x)\right)\bigg|_{x=-\frac{1}{3}}=\frac{2}{9}$

- $x=-\frac{1}{3}$is a regular singular point

  Check to see if $x_0=-\frac{1}{3}$ is a regular singular point

  $x_0=-\frac{1}{3}$

- Multiply by denominators

  $y''(3x+1)^2+2y'(3x+1)^2+(9x^2+6x+3)\,y=0$

- Change variables using $x=u-\frac{1}{3}$ so that the regular singular point is at $u=0$

  $9u^2\left(\frac{d^2}{du^2}y(u)\right)+18u^2\left(\frac{d}{du}y(u)\right)+(9u^2+2)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..2$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$

- Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$

- Shift index using $k->k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$

- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $$u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k-1+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+3r)(-2+3r)\,u^r + (a_1(2+3r)(1+3r)+18a_0r)\,u^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(3k+3r-1)(3k+3r\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(-1+3r)(-2+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

- Each term must be 0

  $$a_1(2+3r)(1+3r) + 18a_0r = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = -\frac{18a_0r}{9r^2+9r+2}$$

- Each term in the series must be 0, giving the recursion relation

  $$9\left(k+r-\frac{2}{3}\right)\left(k+r-\frac{1}{3}\right)a_k + 18a_{k-1}k + 18a_{k-1}r + 9a_{k-2} - 18a_{k-1} = 0$$

- Shift index using $k- >k+2$

  $$9\left(k+\frac{4}{3}+r\right)\left(k+\frac{5}{3}+r\right)a_{k+2} + 18a_{k+1}(k+2) + 18a_{k+1}r + 9a_k - 18a_{k+1} = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = -\frac{9(2ka_{k+1}+2a_{k+1}r+a_k+2a_{k+1})}{(3k+4+3r)(3k+5+3r)}$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k\left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0\right]$$

- Recursion relation for $r = \frac{2}{3}$

  $$a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Revert the change of variables $u = x + \frac{1}{3}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}, a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{10}{3}a_{k+1}\right)}{(3k+6)(3k+7)}, a_1 = -a_0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{3}\right)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{3}\right)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{9\left(2ka_{k+1}+a_k+\frac{8}{3}a_{k+1}\right)}{(3k+5)(3k+6)}, a_1 = -a_0, b_{k+2} =\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+(1+2/(1+3*x)^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1(3x+1)^{\frac{1}{3}} \mathrm{e}^{-x} + c_2(3x+1)^{\frac{2}{3}} \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 35

```
DSolve[y''[x]+2*y'[x]+(1+2/(1+3*x)^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\sqrt[3]{3x+1}\left(c_2\sqrt[3]{3x+1}+c_1\right)$$

2.795 problem 812

2.795.1 Maple step by step solution 7426

Internal problem ID [8285]
Internal file name [OUTPUT/7218_Sunday_June_05_2022_05_36_31_PM_30055366/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 812.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + \left(x^2+2\right)y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2+2\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x\cos(x) + c_2 x\sin(x)$$

Verified OK.

2.795.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (x^2+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ○ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -2$

  ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 2$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-2xy'+(x^2+2)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+r)\,(-2+r)\,x^r+a_1r(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k+r-1)\,(k+r-2)+a_{k-2}\right)x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$

- Each term must be 0

  $a_1r(-1+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

2.796 problem 813

2.796.1 Maple step by step solution 7435

Internal problem ID [8286]
Internal file name [`OUTPUT/7219_Sunday_June_05_2022_05_36_34_PM_36785098/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 813.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0}$$

Writing the ode as

$$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -\frac{2}{(1+x)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{(1+x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = (1+x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{(1+x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+x)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{(1+x)^2}
$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{(1+x)^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{(1+x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{1+x} + (-)(0) \\ &= -\frac{1}{1+x} \\ &= -\frac{1}{1+x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(-\frac{1}{1+x}\right)(0) + \left(\left(\frac{1}{(1+x)^2}\right) + \left(-\frac{1}{1+x}\right)^2 - \left(\frac{2}{(1+x)^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{1+x} dx} \\
&= \frac{1}{1+x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{2}{x}}{1} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2 + x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(1+x)^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2+x}\right) + c_2 \left(\frac{1}{x^2+x}\left(\frac{(1+x)^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2+x} + \frac{c_2(1+x)^2}{3x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2+x} + \frac{c_2(1+x)^2}{3x}
$$

Verified OK.

2.796.1 Maple step by step solution

Let's solve

$y'' + \frac{2y'}{x} - \frac{2y}{(1+x)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = -\frac{2}{(1+x)^2}\right]$

- $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\bigg|_{x=-1}=0$
- $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\bigg|_{x=-1}=-2$
- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $x(1+x)^2 y''+2y'(1+x)^2-2yx=0$
- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-u^2)\left(\frac{d^2}{du^2}y(u)\right)+2u^2\left(\frac{d}{du}y(u)\right)+(-2u+2)\,y(u)=0$
- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $u^m\cdot y(u)=\sum_{k=0}^{\infty}a_k u^{k+r+m}$
- Shift index using $k->k-m$

  $u^m\cdot y(u)=\sum_{k=m}^{\infty}a_{k-m}u^{k+r}$
- Convert $u^2\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r+1}$
- Shift index using $k->k-1$

  $u^2\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=1}^{\infty}a_{k-1}(k-1+r)\,u^{k+r}$
- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=2..3$

  $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k-1+r)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $$u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_{0}(1+r)\,(-2+r)\,u^{r}+\left(\sum_{k=1}^{\infty}\left(-a_{k}(k+r+1)\,(k+r-2)+a_{k-1}(k+r+1)\,(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-(1+r)\,(-2+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{-1,2\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-(k+r+1)\,(k+r-2)\,(a_{k}-a_{k-1})=0$$

- Shift index using $k->k+1$

  $$-(k+r+2)\,(k-1+r)\,(a_{k+1}-a_{k})=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1}=a_{k}$$

- Recursion relation for $r=-1$

  $$a_{k+1}=a_{k}$$

- Solution for $r=-1$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k-1},a_{k+1}=a_{k}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(1+x)^{k-1},a_{k+1}=a_{k}\right]$$

- Recursion relation for $r=2$

  $$a_{k+1}=a_{k}$$

- Solution for $r=2$

  $$\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+2},a_{k+1}=a_{k}\right]$$

- Revert the change of variables $u=1+x$

  $$\left[y=\sum_{k=0}^{\infty}a_{k}(1+x)^{k+2},a_{k+1}=a_{k}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), a_{k+1} = a_k, b_{k+1} = b_k\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+2/x*diff(y(x),x)-2/(1+x)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x(x+1)} + \frac{c_2(x^3+3x^2+3x)}{x(x+1)}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 34

```
DSolve[y''[x]+2/x*y'[x]-2/(1+x)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x(x^2+3x+3) + 3c_1}{3x(x+1)}$$

2.797 problem 815

2.797.1 Maple step by step solution 7445

Internal problem ID [8287]
Internal file name [OUTPUT/7220_Sunday_June_05_2022_05_36_37_PM_99319963/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 815.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1513: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.797.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.163 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.798 problem 816

2.798.1 Maple step by step solution 7454

Internal problem ID [8288]
Internal file name [`OUTPUT/7221_Sunday_June_05_2022_05_36_40_PM_69941980/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 816.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1515: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2) \, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2) \, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i \mathrm{e}^{-2} \sqrt{2}\, (x+2) \sqrt{\pi}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\, \mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.798.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\, e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) x - i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{1}{2}x^2+2x} + 2\pi\, e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right) e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.799 problem 817

2.799.1 Maple step by step solution 7463

Internal problem ID [8289]
Internal file name [`OUTPUT/7222_Sunday_June_05_2022_05_36_44_PM_42203778/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 817.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1517: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.799.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= c_1 \mathrm{e}^{-x}(x+2) \\
&+ \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)
$$

2.800 problem 818

2.800.1 Maple step by step solution 7472

Internal problem ID [8290]
Internal file name [`OUTPUT/7223_Sunday_June_05_2022_05_36_47_PM_35213656/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 818.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1519: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.800.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ∘ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ∘ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ∘ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2+3k+2\right) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2+3k+5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.801 problem 819

2.801.1 Maple step by step solution 7481

Internal problem ID [8291]
Internal file name [`OUTPUT/7224_Sunday_June_05_2022_05_36_50_PM_89865814/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 819.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1521: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\left(\left(x+2\right) e^{-x}\right) + c_2\left(\left(x+2\right) e^{-x}\left(\frac{-i e^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,e^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.801.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.802 problem 820

2.802.1 Maple step by step solution 7490

Internal problem ID [8292]
Internal file name [`OUTPUT/7225_Sunday_June_05_2022_05_36_54_PM_43240515/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 820.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\,\mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.802.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.803 problem 821

2.803.1 Maple step by step solution 7499

Internal problem ID [8293]
Internal file name [`OUTPUT/7226_Sunday_June_05_2022_05_36_57_PM_53187689/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 821.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1525: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-) \left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.803.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ○ Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

- ○ Shift index using $k- >k-1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- > k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.804 problem 822

2.804.1 Maple step by step solution 7508

Internal problem ID [8294]
Internal file name [OUTPUT/7227_Sunday_June_05_2022_05_37_01_PM_92491953/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 822.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1527: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\; \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.804.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,e^{\frac{1}{2}x^2+2x} + 2\pi\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.805 problem 823

2.805.1 Maple step by step solution 7517

Internal problem ID [8295]
Internal file name [`OUTPUT/7228_Sunday_June_05_2022_05_37_04_PM_81520574/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 823.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1529: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left((x+2)\,e^{-x}\right) + c_2\left((x+2)\,e^{-x}\left(\frac{-i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,e^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}$$

Verification of solutions

$$y = c_1(x+2)\,e^{-x} - \frac{c_2 e^{-x}\left(i e^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,e^{\frac{x(4+x)}{2}}\right)}{2}$$

Verified OK.

2.805.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- > k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\, e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) x - i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{1}{2}x^2+2x} + 2\pi\, e^{-2} \operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right) e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.806 problem 824

2.806.1 Maple step by step solution 7526

Internal problem ID [8296]
Internal file name [OUTPUT/7229_Sunday_June_05_2022_05_37_08_PM_45379018/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 824.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1531: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2} + 1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x + 2$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)\, \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\big((x+2)\,\mathrm{e}^{-x}\big) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.806.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k k - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2 + 5k + 6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$
y(x) = c_1 \mathrm{e}^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}x^2+2x} + 2\pi\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)\mathrm{e}^{-x}}{2\sqrt{\pi}}
$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)
$$

2.807 problem 825

2.807.1 Maple step by step solution 7535

Internal problem ID [8297]
Internal file name [OUTPUT/7230_Sunday_June_05_2022_05_37_11_PM_13144909/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 825.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' - yx = 0}$$

Writing the ode as

$$y'' - xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1533: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q+\frac{R}{4} \\ &= \left(\frac{1}{4}x^2+x-\frac{1}{2}\right)+(0) \\ &= \frac{1}{4}x^2+x-\frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right)-(1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2}+1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}}-1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r=\frac{1}{4}x^2+x-\frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2}+1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x + 2
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x+2)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (x+2)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (x+2)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (x+2)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+2)\,\mathrm{e}^{-x}\right) + c_2\left((x+2)\,\mathrm{e}^{-x}\left(\frac{-i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}}{2x+4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1(x+2)\,\mathrm{e}^{-x} - \frac{c_2\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}\sqrt{2}\,(x+2)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

Verified OK.

2.807.1 Maple step by step solution

Let's solve

$y'' - xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  ◦ Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k->k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k$$

  Rewrite ODE with series expansions

  $$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k - a_{k-1}) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+3k+2) a_{k+2} - a_k k - a_{k-1} = 0$$

- Shift index using $k->k+1$

  $$\left((k+1)^2+3k+5\right) a_{k+3} - a_{k+1}(k+1) - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} + a_k + a_{k+1}}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)-x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x+2) + \frac{c_2\sqrt{2}\left(\pi\, e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)x - i\sqrt{2}\,\sqrt{\pi}\, e^{\frac{1}{2}x^2+2x} + 2\pi\, e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)\right)e^{-x}}{2\sqrt{\pi}}$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 78

```
DSolve[y''[x]-x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.808 problem 826

2.808.1 Maple step by step solution 7541

Internal problem ID [8298]
Internal file name [`OUTPUT/7231_Sunday_June_05_2022_05_37_17_PM_19248370/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 826.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

Writing the ode as

$$xy'' + 2y' + yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1535: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{x}\right) + c_2 \left(\frac{\cos(x)}{x} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x} + \frac{c_2 \sin(x)}{x}
$$

Verified OK.

2.808.1 Maple step by step solution

Let's solve

$xy'' + 2y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k-1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k+1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k+1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r)+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_k}{(k+1)(k+2)}$

- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0\right]$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+3)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+2}=-\frac{a_k}{(k+2)(k+3)}, 2a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, 0=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, 2b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x} + \frac{c_2 \cos(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 37

```
DSolve[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x}$$

2.809 problem 827

2.809.1 Maple step by step solution 7551

Internal problem ID [8299]
Internal file name [OUTPUT/7232_Sunday_June_05_2022_05_37_20_PM_74930806/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 827.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{2x^2y'' + 3xy' - yx = 0}$$

The ODE is

$$2x^2y'' + 3xy' - yx = 0$$

Or

$$x(2xy'' - y + 3y') = 0$$

For $x \neq 0$ the above simplifies to

$$2xy'' - y + 3y' = 0$$

Writing the ode as

$$2x^2y'' + 3xy' - yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1537: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{2x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{1}{(x - (0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{-8x + 1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{2}\,\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{2}\,\sqrt{x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{2}\,\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2}\,dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}}\right) + c_2 \left(\frac{e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} \left(-\frac{e^{-2\sqrt{2}\,\sqrt{x}}\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} - \frac{c_2 \sqrt{2}\, e^{-\sqrt{2}\,\sqrt{x}}}{2\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{\sqrt{2}\,\sqrt{x}}}{\sqrt{x}} - \frac{c_2 \sqrt{2}\, e^{-\sqrt{2}\,\sqrt{x}}}{2\sqrt{x}}$$

Verified OK. `{x <> 0}`

2.809.1 Maple step by step solution

Let's solve

$2x^2y'' + 3xy' - yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{2x} - \frac{3y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2x} - \frac{y}{2x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{3}{2x}, P_3(x) = -\frac{1}{2x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = \frac{3}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2xy'' - y + 3y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- ○ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- ○ Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+3+2r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+1} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{(2k+3)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(2k+3)(k+1)}\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+1} = \frac{a_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+1}=\frac{a_k}{(2k+3)(k+1)}, b_{k+1}=\frac{b_k}{(2k+2)\left(k+\frac{1}{2}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1 \sinh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}+\frac{c_2 \cosh\left(\sqrt{x}\,\sqrt{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 56

```
DSolve[2*x^2*y''[x]+3*x*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{e^{-\sqrt{2}\sqrt{x}}\left(2c_1 e^{2\sqrt{2}\sqrt{x}}-\sqrt{2}c_2\right)}{2\sqrt{x}}$$

2.810 problem 828

2.810.1 Maple step by step solution 7561

Internal problem ID [8300]
Internal file name [OUTPUT/7233_Sunday_June_05_2022_05_37_23_PM_2330560/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 828.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + (3x^2+2x)\,y' - 2y = 0}$$

Writing the ode as

$$x^2y'' + (3x^2+2x)\,y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x^2+2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 12x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1539: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} + \frac{1}{x} + \frac{1}{3x^2} - \frac{2}{9x^3} + \frac{1}{9x^4} - \frac{2}{81x^5} - \frac{2}{81x^6} + \frac{28}{729x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{9}{4}\right) + \left(\frac{12x + 8}{4x^2}\right) \\ &= \frac{9}{4} + \frac{12x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 12. Dividing this by leading coefficient in t which is 4 gives 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{\frac{3}{2}} - 0\right) &&= 1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{\frac{3}{2}} - 0\right) &&= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 12x + 8}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\left(\frac{3}{2}\right) \\
&= -\frac{1}{x} - \frac{3}{2} \\
&= -\frac{1}{x} - \frac{3}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x} - \frac{3}{2}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x} - \frac{3}{2}\right)^2 - \left(\frac{9x^2 + 12x + 8}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x} - \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{3x}{2}}}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{3x^2 + 2x}{x^2} \, dx} \\
&= z_1 e^{-\frac{3x}{2} - \ln(x)} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{3x}{2}}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-3x}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3x^2+2x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x-2\ln(x)}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}(9x^2-6x+2)}{27}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{\mathrm{e}^{-3x}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-3x}}{x^2}\left(\frac{\mathrm{e}^{3x}(9x^2-6x+2)}{27}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-3x}}{x^2} + \frac{c_2(9x^2-6x+2)}{27x^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 \mathrm{e}^{-3x}}{x^2} + \frac{c_2(9x^2-6x+2)}{27x^2}$$

Verified OK.

2.810.1 Maple step by step solution

Let's solve

$x^2y'' + (3x^2 + 2x)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2} - \frac{(3x+2)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(3x+2)y'}{x} - \frac{2y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3x+2}{x}, P_3(x) = -\frac{2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -2$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + x(3x + 2)\,y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\,x^{k+r-1+m}$

  - Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-2, 1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\,(a_k(k+r+2) + 3a_{k-1}) = 0$$

- Shift index using $k- >k+1$

$$(k+r)\,(a_{k+1}(k+3+r) + 3a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{3a_k}{k+3+r}$$

- Recursion relation for $r = -2$

$$a_{k+1} = -\frac{3a_k}{k+1}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{3a_k}{k+4}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x), x, x) + (2*x+3*x^2)*diff(y(x),x)-2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1(9x^2 - 6x + 2)}{x^2} + \frac{c_2 e^{-3x}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]+(2*x+3*x^2)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1(9x^2 - 6x + 2) + 27c_2 e^{-3x}}{27x^2}$$

2.811 problem 829

2.811.1 Maple step by step solution 7570

Internal problem ID [8301]
Internal file name [`OUTPUT/7234_Sunday_June_05_2022_05_37_27_PM_52350037/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 829.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^{2}\left(x^{2}+x+1\right)y''+x\left(11x^{2}+11x+9\right)y'+\left(7x^{2}+10x+6\right)y=0}$$

Writing the ode as

$$\left(2x^{4}+2x^{3}+2x^{2}\right)y''+\left(11x^{3}+11x^{2}+9x\right)y'+\left(7x^{2}+10x+6\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^{4}+2x^{3}+2x^{2} \\ B &= 11x^{3}+11x^{2}+9x \\ C &= 7x^{2}+10x+6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 21x^4 + 18x^3 + 27x^2 - 2x - 3$$
$$t = 16\left(x^3 + x^2 + x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1541: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^3 + x^2 + x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ of order 2. There is a pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} + \frac{1}{4x} + \frac{-\frac{5}{24} + \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{5}{24} - \frac{i\sqrt{3}}{24}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} + \frac{-\frac{1}{8} - \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{-\frac{1}{8} + \frac{43i\sqrt{3}}{72}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4} \end{aligned}$$

For the pole at $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{24} + \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - \frac{\sqrt{6 + 6i\sqrt{3}}}{12} \end{aligned}$$

For the pole at $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ let b be the coefficient of $\frac{1}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}$ in the partial fractions

decomposition of r given above. Therefore $b = -\frac{5}{24} - \frac{i\sqrt{3}}{24}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}\\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{21}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{7}{4}\\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{21x^4 + 18x^3 + 27x^2 - 2x - 3}{16\left(x^3 + x^2 + x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{4}$	$\frac{1}{4}$
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6+6i\sqrt{3}}}{12}$
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	2	0	$\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}$	$\frac{1}{2} - \frac{\sqrt{6-6i\sqrt{3}}}{12}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{7}{4}$	$-\frac{3}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{7}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+ + \alpha_{c_3}^+\right) \\ &= \frac{7}{4} - \left(\frac{7}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + \left((+)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^+}{x-c_3}\right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} + (0) \\ &= \frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}} \\ &= \frac{7x^2 + 3x + 1}{4x\left(x^2 + x + 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{4x} + \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{x + \frac{1}{2} - \frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{x + \frac{1}{2} + \frac{i\sqrt{3}}{2}}\right)(0) + \left(\left(-\frac{1}{4x^2} - \frac{\frac{1}{2} + \frac{\sqrt{6+6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{2} + \frac{\sqrt{6-6i\sqrt{3}}}{12}}{\left(x + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2}\right) + \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{4x} + \frac{\frac{1}{2}+\frac{\sqrt{6+6i\sqrt{3}}}{12}}{x+\frac{1}{2}-\frac{i\sqrt{3}}{2}} + \frac{\frac{1}{2}+\frac{\sqrt{6-6i\sqrt{3}}}{12}}{x+\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)dx} \\
&= 2\sqrt{2}\, x^{\frac{1}{4}} \left(x^2+x+1\right)^{\frac{3}{4}} \mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+x+1\right)}{4} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6} - \frac{9\ln(x)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{6}}}{x^{\frac{9}{4}}\left(x^2+x+1\right)^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{11x^3+11x^2+9x}{2x^4+2x^3+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{9\ln(x)}{2} - \frac{\ln\left(x^2+x+1\right)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\,\left(x^2+x+1\right)^{\frac{3}{2}}}dx\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2}\right) \\
&\quad + c_2 \left(\frac{2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{8\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx\right)}{4x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{2c_1\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{x^2} + \frac{c_2\sqrt{2}\,\sqrt{x^2+x+1}\,\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}} \left(\int \frac{\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}{3}}}{\sqrt{x}\,(x^2+x+1)^{\frac{3}{2}}} dx\right)}{4x^2}
$$

Verified OK.

2.811.1 Maple step by step solution

Let's solve

$(2x^4+2x^3+2x^2)\,y''+(11x^3+11x^2+9x)\,y'+(7x^2+10x+6)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(7x^2+10x+6)\,y}{2x^2(x^2+x+1)} - \frac{(11x^2+11x+9)\,y'}{2x(x^2+x+1)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(11x^2+11x+9)\,y'}{2x(x^2+x+1)} + \frac{(7x^2+10x+6)\,y}{2x^2(x^2+x+1)} = 0$$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{11x^2+11x+9}{2x(x^2+x+1)}, P_3(x) = \frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = \frac{9}{2}$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 3$$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2(x^2+x+1)\,y'' + x(11x^2+11x+9)\,y' + (7x^2+10x+6)\,y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$

- ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

$$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(3+2r)\, x^r + (a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(2+r)\,(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{-2, -\frac{3}{2}\right\}$$

- Each term must be 0

$$a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = -\frac{(2+r)a_0}{3+r}$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+\frac{3}{2}+r\right)\left((a_k+a_{k-2}+a_{k-1})\,k + (a_k+a_{k-2}+a_{k-1})\,r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k- >k+2$

$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2}+a_k+a_{k+1})\,(k+2) + (a_{k+2}+a_k+a_{k+1})\,r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k+ka_{k+1}+ra_k+ra_{k+1}+a_k+3a_{k+1}}{k+4+r}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}$$

- Solution for $r = -2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{ka_k+ka_{k+1}-a_k+a_{k+1}}{k+2}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2} = -\frac{k a_k + k a_{k+1} - \frac{1}{2} a_k + \frac{3}{2} a_{k+1}}{k + \frac{5}{2}}, a_1 = -\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{k a_k + k a_{k+1} - a_k + a_{k+1}}{k+2}, a_1 = 0, b_{k+2} = -\frac{k b_k + k b_{k+1} - \frac{1}{2} b_k +}{k + \frac{5}{2}}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <>
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 141

```
dsolve(2*x^2*(1+x+x^2)*diff(y(x), x$2) + x*(9+11*x+11*x^2)*diff(y(x), x) + (6+10*x+7*x^2)*y(
```

$$y(x) = \frac{c_1\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}}{x^2} + \frac{c_2\sqrt{x^2+x+1}\left(\frac{i\sqrt{3}+2x+1}{i\sqrt{3}-2x-1}\right)^{-\frac{i\sqrt{3}}{6}}\left(\int \frac{\left(\frac{i\sqrt{3}-2x-1}{i\sqrt{3}+2x+1}\right)^{-\frac{i\sqrt{3}}{6}}}{(x^2+x+1)^{\frac{3}{2}}\sqrt{x}}dx\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.718 (sec). Leaf size: 93

```
DSolve[2*x^2*(1+x+x^2)*y''[x] + x*(9+11*x+11*x^2)*y'[x] + (6+10*x+7*x^2)*y[x] == 0,y[x],x,In
```

$$y(x) \to \frac{\sqrt{x^2+x+1}e^{-\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}}\left(c_2\int_1^x \frac{e^{\frac{\arctan\left(\frac{2K[1]+1}{\sqrt{3}}\right)}{\sqrt{3}}}}{\sqrt{K[1]}(K[1]^2+K[1]+1)^{3/2}}dK[1]+c_1\right)}{x^2}$$

2.812 problem 830

2.812.1 Maple step by step solution 7582

Internal problem ID [8302]
Internal file name [`OUTPUT/7235_Sunday_June_05_2022_05_37_34_PM_70558197/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 830.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' + (1 + x) y' + 2y = 0}$$

Writing the ode as

$$xy'' + (1 + x) y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 + x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6x - 1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6x - 1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 6x - 1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1543: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{3}{2x} - \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{3}{2x} - \frac{5}{2x^2} - \frac{15}{2x^3} - \frac{115}{4x^4} - \frac{495}{4x^5} - \frac{2285}{4x^6} - \frac{11055}{4x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6x - 1}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-6x - 1}{4x^2}\right) \\
&= \frac{1}{4} + \frac{-6x - 1}{4x^2}
\end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= -\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 0\right) &= \frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 6x - 1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}\right) \\
&= \frac{3}{2} - \left(\frac{1}{2}\right) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\left(\frac{1}{2}\right) \\
&= \frac{1}{2x} - \frac{1}{2} \\
&= -\frac{x-1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x} - \frac{1}{2}\right)(1) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^2-6x-1}{4x^2}\right)\right) = 0
$$

$$
\frac{1+a_0}{x} = 0
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = -1\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x - 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x-1)\, e^{\int \left(\frac{1}{2x} - \frac{1}{2}\right) dx} \\
&= (x-1)\, e^{-\frac{x}{2} + \frac{\ln(x)}{2}} \\
&= (x-1)\sqrt{x}\, e^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+x}{x}\,dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{e^{-\frac{x}{2}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x-1)\,e^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+x}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - e^{x}}{x-1}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-1)\,e^{-x}\right) + c_2\left((x-1)\,e^{-x}\left(\frac{-\operatorname{expIntegral}_1(-x)\,x + \operatorname{expIntegral}_1(-x) - e^{x}}{x-1}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x-1)\,e^{-x} + c_2\left(-1 - (x-1)\,e^{-x}\operatorname{expIntegral}_1(-x)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(x-1)\,e^{-x} + c_2\left(-1 - (x-1)\,e^{-x}\operatorname{expIntegral}_1(-x)\right)
$$

Verified OK.

2.812.1 Maple step by step solution

Let's solve

$xy'' + (1+x)\, y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+x)y'}{x} - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1+x}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $xy'' + (1+x)\, y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  ◦ Shift index using $k- >k + 1 - m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_k(k+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(x*diff(y(x), x$2) +(1+x)*diff(y(x),x)+2*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x}(x-1) + c_2(\text{expIntegral}_1(-x)\, x - \text{expIntegral}_1(-x) + e^x)\, e^{-x}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 33

```
DSolve[x*y''[x] +(1+x)*y'[x]+2*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2(x-1)\,\text{ExpIntegralEi}(x) + c_1(x-1) - c_2 e^x)$$

2.813 problem 831

2.813.1 Maple step by step solution 7591

Internal problem ID [8303]
Internal file name [`OUTPUT/7236_Sunday_June_05_2022_05_37_37_PM_66087850/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 831.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2-2x+1)\, y''-x(x+3)\, y'+(4+x)\, y=0}$$

Writing the ode as

$$y''x^2(x-1)^2+(-x^2-3x)\, y'+(4+x)\, y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2(x-1)^2 \\ B &= -x^2-3x \\ C &= 4+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 7x^2 + 10x - 1 \\ t &= 4x^2(x - 1)^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{7x^2 + 10x - 1}{4x^2 (x - 1)^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1545: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2(x-1)^4$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{3}{2x} + \frac{7}{4(x-1)^2} - \frac{2}{(x-1)^3} - \frac{1}{4x^2} - \frac{3}{2(x-1)} + \frac{4}{(x-1)^4}$$

There is pole in r at $x=1$ of order 4, hence $v=2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c=1$ gives

$$[\sqrt{r}]_c \approx \frac{2}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{21}{32} - \frac{9x}{32} + \frac{53(x-1)^2}{256} - \frac{149(x-1)^3}{1024} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v=2$ the above becomes

$$[\sqrt{r}]_c = \frac{2}{(x-1)^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-1)^2}$ is

$$a = 2$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c=1$. This term becomes $\frac{1}{(x-1)^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -2. Therefore

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{2}{(x-1)^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{-2}{2} + 2\right) = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{-2}{2} + 2\right) = \frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{7x^2 + 10x - 1}{4x^2 (x-1)^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$
1	4	$\frac{2}{(x-1)^2}$	$\frac{1}{2}$	$\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} + (-)(0) \\
&= \frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2} \\
&= \frac{2x^2+x+1}{2x(x-1)^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right)(0) + \left(\left(-\frac{1}{2x^2} - \frac{4}{(x-1)^3} - \frac{1}{2(x-1)^2}\right) + \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x} + \frac{2}{(x-1)^2} + \frac{1}{2x-2}\right) dx} \\
&= \sqrt{x}\,\sqrt{x-1}\,\mathrm{e}^{-\frac{2}{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2-3x}{x^2(x-1)^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x)}{2} - \frac{2}{x-1} - \frac{3\ln(x-1)}{2}} \\
&= z_1 \left(\frac{x^{\frac{3}{2}}\mathrm{e}^{-\frac{2}{x-1}}}{(x-1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x^2-3x}{x^2(x-1)^2}\,dx}}{(y_1)^2}\,dx$$
$$= y_1 \int \frac{e^{3\ln(x)-\frac{4}{x-1}-3\ln(x-1)}}{(y_1)^2}\,dx$$
$$= y_1 \left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}}\left(\mathrm{e}^{-4}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}}\mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \mathrm{e}^{-\frac{4}{x-1}}\sqrt{x(x-1)}\,x^{\frac{3}{2}}}{(x-1)^{\frac{3}{2}}} + \frac{c_2 x^{\frac{3}{2}}\mathrm{e}^{-\frac{4x}{x-1}}\sqrt{x(x-1)}\,\mathrm{expIntegral}_1\left(-\frac{4x}{x-1}\right)}{(x-1)^{\frac{3}{2}}}$$

Verified OK.

2.813.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (-x^2-3x)\,y' + (4+x)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4+x)y}{x^2(x-1)^2} + \frac{(x+3)y'}{x(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x+3)y'}{x(x-1)^2}+\frac{(4+x)y}{x^2(x-1)^2}=0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{x+3}{x(x-1)^2}, P_3(x)=\frac{4+x}{x^2(x-1)^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-3$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=4$

  - $x=0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0=0$

- Multiply by denominators

  $y''x^2(x-1)^2-x(x+3)y'+(4+x)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m\cdot y$ to series expansion for $m=0..1$

    $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m\cdot y'$ to series expansion for $m=1..2$

    $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $x^m\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^{m} \cdot y''$ to series expansion for $m = 2..4$

  $x^{m} \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-2+m}$
- Shift index using $k- > k+2-m$

  $x^{m} \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (-2+r)^2 x^{r} + \left(a_1 (-1+r)^2 - a_0 (1+2r)(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-2)^2 - a_{k-1} (2k -\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 2$
- Each term must be 0

  $a_1 (-1+r)^2 - a_0 (1+2r)(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0 (1+2r)}{-1+r}$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1}) k + (a_k + a_{k-2} - 2a_{k-1}) r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$
- Shift index using $k- > k+2$

  $(k+r)\left((a_{k+2} + a_k - 2a_{k+1})(k+2) + (a_{k+2} + a_k - 2a_{k+1}) r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k a_k - 2k a_{k+1} + r a_k - 2r a_{k+1} - a_k - 3a_{k+1}}{k+r}$
- Recursion relation for $r = 2$

  $a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{k a_k - 2k a_{k+1} + a_k - 7a_{k+1}}{k+2}, a_1 = 5a_0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 52

```
dsolve(x^2*(1-2*x+x^2)*diff(y(x), x$2) -x*(3+x)*diff(y(x),x)+(4+x)*y(x) = 0,y(x), singsol=al
```

$$y(x) = \frac{c_1 x^2 \mathrm{e}^{-\frac{4}{x-1}}}{x-1} + \frac{c_2 x^2 \operatorname{expIntegral}_1\left(-\frac{4x}{x-1}\right) \mathrm{e}^{-\frac{4x}{x-1}}}{x-1}$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 54

```
DSolve[x^2*(1-2*x+x^2)*y''[x] -x*(3+x)*y'[x]+(4+x)*y[x] == 0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{e^{-\frac{4x}{x-1}} \sqrt{1-x} x^2 \left(c_2 \operatorname{ExpIntegralEi}\left(\frac{4x}{x-1}\right) + e^4 c_1\right)}{(x-1)^{3/2}}$$

2.814 problem 832

2.814.1 Maple step by step solution 7600

Internal problem ID [8304]
Internal file name [`OUTPUT/7237_Sunday_June_05_2022_05_37_41_PM_3977617/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 832.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{2x^2(x+2)\,y''+5x^2y'+(1+x)\,y=0}$$

Writing the ode as

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=2x^3+4x^2\\B&=5x^2\\C&=1+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -3x^2 - 24x - 16$$
$$t = 16\left(x^2 + 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-3x^2 - 24x - 16}{16\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1547: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{8x} + \frac{1}{8x + 16} + \frac{5}{16\,(x + 2)^2} - \frac{1}{4x^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-3x^2 - 24x - 16}{16\,(x^2 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-3x^2 - 24x - 16}{16 \left(x^2 + 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{4}$	$\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\
&= \frac{1}{4} - \left(\frac{1}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{4\left(x+2\right)} + \frac{1}{2x} + (-)\left(0\right) \\
&= -\frac{1}{4\left(x+2\right)} + \frac{1}{2x} \\
&= \frac{4+x}{4x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{4(x+2)^2} - \frac{1}{2x^2}\right) + \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right)^2 - \left(\frac{-3x^2 - 24x - 16}{16(x^2+2x)^2}\right)\right)$$
$$0 =$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{4(x+2)} + \frac{1}{2x}\right) dx} \\ &= \frac{\sqrt{x}}{(x+2)^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5x^2}{2x^3+4x^2}\, dx} \\ &= z_1 e^{-\frac{5\ln(x+2)}{4}} \\ &= z_1 \left(\frac{1}{(x+2)^{\frac{5}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x^2}{2x^3+4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{5\ln(x+2)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(2\sqrt{x+2} - 2\sqrt{2}\ \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x}}{(x+2)^{\frac{3}{2}}} \left(2\sqrt{x+2} - 2\sqrt{2}\ \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{2c_2\sqrt{x}\left(\sqrt{x+2} - \sqrt{2}\ \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)}{(x+2)^{\frac{3}{2}}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}} + \frac{2c_2\sqrt{x}\left(\sqrt{x+2} - \sqrt{2}\ \operatorname{arctanh}\left(\frac{\sqrt{2}\,\sqrt{x+2}}{2}\right)\right)}{(x+2)^{\frac{3}{2}}}
$$

Verified OK.

2.814.1 Maple step by step solution

Let's solve

$(2x^3 + 4x^2)\, y'' + 5x^2 y' + (1 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(1+x)y}{2x^2(x+2)} - \frac{5y'}{2(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{2(x+2)} + \frac{(1+x)y}{2x^2(x+2)} = 0$

☐ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{5}{2(x+2)}, P_3(x) = \frac{1+x}{2x^2(x+2)}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\bigg|_{x=-2} = \frac{5}{2}$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$

  - $x = -2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$

- Multiply by denominators

  $2x^2(x+2)\, y'' + 5x^2 y' + (1+x)\, y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(2u^3 - 8u^2 + 8u)\left(\frac{d^2}{du^2} y(u)\right) + (5u^2 - 20u + 20)\left(\frac{d}{du} y(u)\right) + (-1+u)\, y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

    $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k- >k - m$

    $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty}a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\,u^{-1+r}+\left(4a_1(1+r)\,(5+2r)-a_0(8r^2+12r+1)\right)u^r+\left(\sum_{k=1}^{\infty}(4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(3+2r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,-\frac{3}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+2r)-a_0(8r^2+12r+1)=0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k+a_{k-1}+4a_{k+1})\,k^2+\left(4(-4a_k+a_{k-1}+4a_{k+1})\,r-12a_k-a_{k-1}+28a_{k+1}\right)k+2(-4a_k+$$

- Shift index using $k->k+1$

$$2(-4a_{k+1}+a_k+4a_{k+2})\,(k+1)^2+\left(4(-4a_{k+1}+a_k+4a_{k+2})\,r-12a_{k+1}-a_k+28a_{k+2}\right)(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+4kra_k-16kra_{k+1}+2r^2a_k-8r^2a_{k+1}+3ka_k-28ka_{k+1}+3ra_k-28ra_{k+1}+a_k-21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+3ka_k-28ka_{k+1}+a_k-21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k,a_{k+2}=-\frac{2k^2a_k-8k^2a_{k+1}+3ka_k-28ka_{k+1}+a_k-21a_{k+1}}{4(2k^2+11k+14)},20a_1-a_0=0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2 + 5k + 2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2 + 11k + 14)}, 20a\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 53

```
dsolve(2*x^2*(2+x)*diff(y(x), x$2) +5*x^2*diff(y(x),x)+(1+x)*y(x) = 0,y(x), singsol=all)
```

$$y(x)=\frac{c_1\sqrt{x}}{(x+2)^{\frac{3}{2}}}-\frac{c_2\sqrt{2}\left(-2\sqrt{2}\sqrt{x+2}+4\operatorname{arctanh}\left(\frac{\sqrt{2}\sqrt{x+2}}{2}\right)\right)\sqrt{x}}{2(x+2)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 55

```
DSolve[2*x^2*(2+x)*y''[x] +5*x^2*y'[x]+(1+x)*y[x] == 0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x)\to\frac{\sqrt{x}\left(-2\sqrt{2}c_2\text{arctanh}\left(\frac{\sqrt{x+2}}{\sqrt{2}}\right)+2c_2\sqrt{x+2}+c_1\right)}{(x+2)^{3/2}}$$

2.815 problem 833

2.815.1 Maple step by step solution 7608

Internal problem ID [8305]
Internal file name [`OUTPUT/7238_Sunday_June_05_2022_05_37_44_PM_90641276/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 833.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 4xy' + (x^2 + 2) y = 0}$$

Writing the ode as

$$x^2y'' + 4xy' + (x^2 + 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1549: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\cos(x)}{x^2}\right) + c_2\left(\frac{\cos(x)}{x^2}(\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{x^2} + \frac{c_2 \sin(x)}{x^2}
$$

Verified OK.

2.815.1 Maple step by step solution

Let's solve

$x^2y'' + 4xy' + (x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} - \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{4}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + 4xy' + (x^2 + 2)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(2 + r)\,(1 + r)\, x^r + a_1(3 + r)\,(2 + r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r + 2)\,(k + r + 1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2 + r)\,(1 + r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, -1\}$

- Each term must be 0

  $a_1(3 + r)\,(2 + r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r+1) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = -2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{b_k}{(k+3)(k+2)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x), x, x) + 4*x*diff(y(x), x) + (x^2+2)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{x^2} + \frac{c_2 \cos(x)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+4*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2c_1 e^{-ix} - ic_2 e^{ix}}{2x^2}$$

2.816 problem 834

2.816.1 Maple step by step solution 7615

Internal problem ID [8306]
Internal file name [`OUTPUT/7239_Sunday_June_05_2022_05_37_47_PM_84870973/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 834.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1551: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}$$

Verified OK.

2.816.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.817 problem 835

2.817.1 Maple step by step solution 7626

Internal problem ID [8307]
Internal file name [`OUTPUT/7240_Sunday_June_05_2022_05_37_50_PM_16476048/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 835.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= -x^2 - \frac{5}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^2 + 2}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1553: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x^2} - \frac{1}{2x^4} + \frac{1}{2x^6} - \frac{5}{8x^8} + \frac{7}{8x^{10}} - \frac{21}{16x^{12}} + \frac{33}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 2}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (1) + \left(\frac{2}{x^2}\right) \\ &= 1 + \frac{2}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{1} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{1} - 0\right) &&= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 + 2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	1	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(1) \\
&= -1 - \frac{1}{x} \\
&= -\frac{1 + x}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-1 - \frac{1}{x}\right)(1) + \left(\left(\frac{1}{x^2}\right) + \left(-1 - \frac{1}{x}\right)^2 - \left(\frac{x^2 + 2}{x^2}\right)\right) &= 0 \\
\frac{-2 + 2a_0}{x} &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 1\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 1 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (1 + x)\, e^{\int \left(-1 - \frac{1}{x}\right) dx} \\
&= (1 + x)\, e^{-x - \ln(x)} \\
&= \frac{(1 + x)\, e^{-x}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{(x-1)\,\mathrm{e}^{2x}}{2x+2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\right) + c_2\left(\frac{(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}}\left(\frac{(x-1)\,\mathrm{e}^{2x}}{2x+2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1(1+x)\,\mathrm{e}^{-x}}{\sqrt{x}} + \frac{c_2(x-1)\,\mathrm{e}^{x}}{2\sqrt{x}} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1(1+x)\,e^{-x}}{\sqrt{x}} + \frac{c_2(x-1)\,e^{x}}{2\sqrt{x}}$$

Verified OK.

2.817.1 Maple step by step solution

Let's solve

$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(4x^2+5)y}{4x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{(4x^2+5)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = -\frac{4x^2+5}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{5}{4}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4xy' + (-4x^2 - 5)y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - ◦ Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - ◦ Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - ◦ Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-5+2r)\, x^r + a_1(3+2r)\,(-3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-5) - 4a\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-5+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$$

- Each term must be 0

  $$a_1(3+2r)\,(-3+2r) = 0$$

- Solve for the dependent coefficient(s)

  $$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(-\frac{5}{2}+k+r\right)\left(k+r+\frac{1}{2}\right) a_k - 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

  $$4\left(-\frac{1}{2}+k+r\right)\left(k+\frac{5}{2}+r\right) a_{k+2} - 4a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{4a_k}{(-1+2k+2r)(2k+5+2r)}$$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{(-2+2k)(2k+4)}$

- Solution for $r = -\frac{1}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4a_k}{(-2+2k)(2k+4)}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{5}{2}$

  $a_{k+2} = \frac{4a_k}{(2k+4)(2k+10)}$

- Solution for $r = \frac{5}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2} = \frac{4a_k}{(2k+4)(2k+10)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2} = \frac{4a_k}{(-2+2k)(2k+4)}, a_1 = 0, b_{k+2} = \frac{4b_k}{(2k+4)(2k+10)}, b_1 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-(x^2+5/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^x (x-1)}{\sqrt{x}} + \frac{c_2 e^{-x} (x+1)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 53

```
DSolve[x^2*y''[x]-x*y'[x]-(x^2+5/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{\frac{2}{\pi}}((ic_2x + c_1)\sinh(x) - (c_1x + ic_2)\cosh(x))}{\sqrt{-ix}}$$

2.818 problem 836

2.818.1 Maple step by step solution 7633

Internal problem ID [8308]
Internal file name [OUTPUT/7241_Sunday_June_05_2022_05_37_54_PM_17965468/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 836.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1555: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\frac{\cos\left(x\right)}{\sqrt{x}}\right) + c_2\left(\frac{\cos\left(x\right)}{\sqrt{x}}\left(\tan\left(x\right)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1\cos\left(x\right)}{\sqrt{x}} + \frac{c_2\sin\left(x\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\cos\left(x\right)}{\sqrt{x}} + \frac{c_2\sin\left(x\right)}{\sqrt{x}}
$$

Verified OK.

2.818.1 Maple step by step solution

Let's solve

$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\Bigg|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1 + 2r)\,(-1 + 2r)\, x^r + a_1(3 + 2r)\,(1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k + 2r + 1)\,(2k + 2r - 1) + 4a_{k-}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\,(-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3 + 2r)\,(1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.819 problem 837

2.819.1 Maple step by step solution 7644

Internal problem ID [8309]
Internal file name [OUTPUT/7242_Sunday_June_05_2022_05_37_56_PM_64974951/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 837.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 3xy' + 4yx^4 = 0}$$

The ODE is

$$x^2y'' + 3xy' + 4yx^4 = 0$$

Or

$$x\left(xy'' + 3y' + 4yx^3\right) = 0$$

For $x \neq 0$ the above simplifies to

$$xy'' + 3y' + 4yx^3 = 0$$

Writing the ode as

$$x^2y'' + 3xy' + 4yx^4 = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 4x^4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 1557: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{e^{-ix^2}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-ix^2}}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-ix^2}}{x^2}\right) + c_2 \left(\frac{\mathrm{e}^{-ix^2}}{x^2}\left(-\frac{i\mathrm{e}^{2ix^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\mathrm{e}^{-ix^2}}{x^2} - \frac{ic_2\mathrm{e}^{ix^2}}{4x^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1 e^{-i x^2}}{x^2} - \frac{i c_2 e^{i x^2}}{4 x^2}$$

Verified OK. `{x <> 0}`

2.819.1 Maple step by step solution

Let's solve

$x^2 y'' + 3 x y' + 4 y x^4 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4 x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + 4 x^2 y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 3$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x y'' + 3y' + 4 y x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

  $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$

- Shift index using $k- > k-3$

  $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(2+r)\, x^{-1+r} + a_1(1+r)\,(3+r)\, x^r + a_2(2+r)\,(4+r)\, x^{1+r} + a_3(3+r)\,(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(3+r) = 0, a_2(2+r)\,(4+r) = 0, a_3(3+r)\,(5+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r+3) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)(k+6+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r=-2$

  $a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k}, a_{k+4}=-\frac{4a_k}{(k+4)(k+6)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k}\right), a_{k+4}=-\frac{4a_k}{(k+2)(k+4)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+4)(k+6)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+4*x^4*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x^2)}{x^2} + \frac{c_2 \cos(x^2)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 41

```
DSolve[x^2*y''[x]+3*x*y'[x]+4*x^4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-ix^2} - ic_2 e^{ix^2}}{4x^2}$$

2.820 problem 838

2.820.1 Maple step by step solution 7654

Internal problem ID [8310]
Internal file name [`OUTPUT/7243_Sunday_June_05_2022_05_38_01_PM_11494398/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 838.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - (x^2 + 3)\, y = 0}$$

Writing the ode as

$$y'' + (-x^2 - 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1559: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+3}{1} \\
&= Q + \frac{R}{1} \\
&= (x^2+3) + (0) \\
&= x^2 + 3
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned}
b &= (3) - (0) \\
&= 3
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + (x) \\
&= x \\
&= x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\
-2a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (x)\, \mathrm{e}^{\int x dx} \\
&= (x)\, \mathrm{e}^{\frac{x^2}{2}} \\
&= x\, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^2}{2}}\left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right)$$

Verified OK.

2.820.1 Maple step by step solution

Let's solve

$y'' + (-x^2 - 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 3a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k- >k + 2$

$$\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+4}=\frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2=\frac{3a_0}{2}, a_3=\frac{a_1}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)=(x^2+3)*y(x),y(x), singsol=all)
```

$$y(x)=c_1\mathrm{e}^{\frac{x^2}{2}}x+c_2\mathrm{e}^{\frac{x^2}{2}}\left(\sqrt{\pi}\ \mathrm{erf}\left(x\right)x+\mathrm{e}^{-x^2}\right)$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 46

```
DSolve[y''[x]==(x^2+3)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\mathrm{erf}(x)+c_1e^{x^2}x-c_2\right)$$

2.821 problem 839

2.821.1 Maple step by step solution 7659

Internal problem ID [8311]
Internal file name [`OUTPUT/7244_Sunday_June_05_2022_05_38_04_PM_47335323/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 839.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 2xy' + (x^2+1)\, y = 0}$$

Writing the ode as

$$y'' + 2xy' + (x^2+1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1561: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x^2}}{(y_1)^2} \, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}} (x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x^2}{2}} + c_2 x\, \mathrm{e}^{-\frac{x^2}{2}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x^2}{2}} + c_2 x\, \mathrm{e}^{-\frac{x^2}{2}}
$$

Verified OK.

2.821.1 Maple step by step solution

Let's solve

$y'' + 2xy' + (x^2+1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ∘ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + 3a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k+2) + a_{k+2} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2} + a_k + 5a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^2}{2}} + c_2 \mathrm{e}^{-\frac{x^2}{2}} x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[y''[x]+2*x*y'[x]+(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{2}} (c_2 x + c_1)$$

2.822 problem 840

2.822.1 Maple step by step solution 7665

Internal problem ID [8312]
Internal file name [`OUTPUT/7245_Sunday_June_05_2022_05_38_07_PM_61539155/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 840.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0}$$

Writing the ode as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1563: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\cos(x)}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}}(\tan(x))\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

2.822.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2 y'' + 4xy' + (4x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 (1 + 2r)\, (-1 + 2r)\, x^r + a_1 (3 + 2r)\, (1 + 2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (2k + 2r + 1)\, (2k + 2r - 1) + 4a_{k-}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1 + 2r)\, (-1 + 2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1 (3 + 2r)\, (1 + 2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(diff(y(x),x),x)+x*diff(y(x),x)+(x^2-1/4)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(x)}{\sqrt{x}} + \frac{c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2 e^{2ix})}{2\sqrt{x}}$$

2.823 problem 841

2.823.1 Maple step by step solution 7672

Internal problem ID [8313]
Internal file name [`OUTPUT/7246_Sunday_June_05_2022_05_38_10_PM_4841825/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 841.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0}$$

Writing the ode as

$$4x^2y'' + (-8x^2 + 4x)\, y' + (4x^2 - 4x - 1)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1565: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx} \\
&= z_1 e^{x-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{e^x}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{e^x}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)+2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{e^x}{\sqrt{x}}\right) + c_2 \left(\frac{e^x}{\sqrt{x}}(x)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

Verified OK.

2.823.1 Maple step by step solution

Let's solve

$4x^2 y'' + (-8x^2 + 4x) y' + (4x^2 - 4x - 1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-4x-1)y}{4x^2} + \frac{(2x-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x-1)y'}{x} + \frac{(4x^2-4x-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2x-1}{x}, P_3(x) = \frac{4x^2-4x-1}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -\frac{1}{4}$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' - 4x(2x-1)\,y' + (4x^2 - 4x - 1)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^m \cdot y'$ to series expansion for $m = 1..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - Shift index using $k- > k + 1 - m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + (a_1(3+2r)\,(1+2r) - 4a_0(1+2r))\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) - 4a_0(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = \frac{4a_0}{3+2r}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + (-8k - 8r + 4)\, a_{k-1} + 4a_{k-2} = 0$
- Shift index using $k- > k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + (-8k - 12 - 8r)\, a_{k+1} + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4(2ka_{k+1}+2ra_{k+1}-a_k+3a_{k+1})}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{4(2ka_{k+1}-a_k+4a_{k+1})}{4k^2+20k+24}, a_1 = a_0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{4(2ka_{k+1}-a_k+2a_{k+1})}{4k^2+12k+8}, a_1 = 2a_0, b_{k+2} = \frac{4(2kb_{k+1}-b_k+4b_{k+1}}{4k^2+20k+24}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 0,y(x), sin
```

$$y(x) = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 21

```
DSolve[4*x^2*y''[x]+(-8*x^2+4*x)*y'[x]+(4*x^2-4*x-1)*y[x] == 0,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{e^x(c_2 x + c_1)}{\sqrt{x}}$$

2.824 problem 843

2.824.1 Maple step by step solution 7680

Internal problem ID [8314]
Internal file name [OUTPUT/7247_Sunday_June_05_2022_05_38_13_PM_10176488/index.tex]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 843.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 0}$$

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1567: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 \tag{1}$$

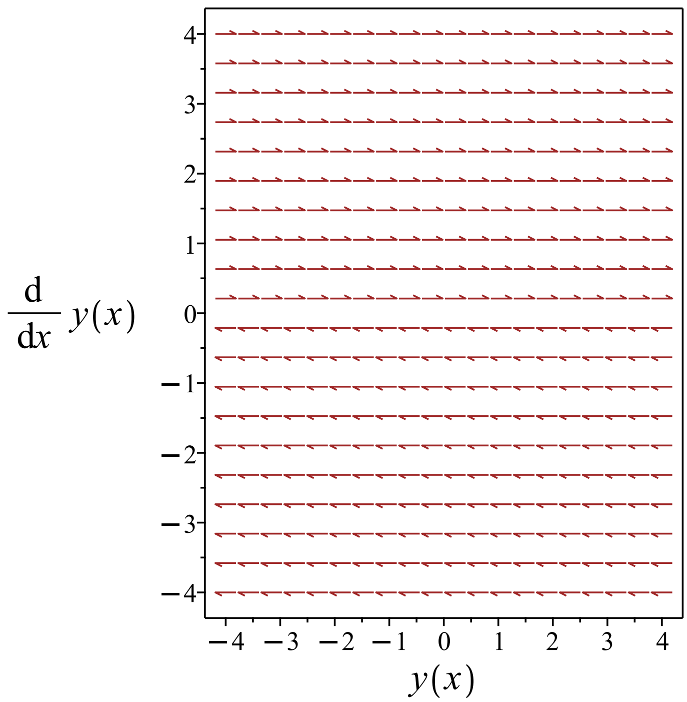

Figure 7: Slope field plot

Verification of solutions

$$y = c_2 x + c_1$$

Verified OK.

2.824.1 Maple step by step solution

Let's solve

$y'' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_2 x + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve(diff(y(x),x$2)=((4*(1/2)^2-1)/(4*x^2))*y(x),y(x), singsol=all)
```

$$y(x) = xc_2 + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[y''[x]==((4*(1/2)^2-1)/(4*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x + c_1$$

2.825 problem 844

2.825.1 Maple step by step solution 7687

Internal problem ID [8315]
Internal file name [`OUTPUT/7248_Sunday_June_05_2022_05_38_15_PM_2728613/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 844.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{y'' - \frac{2y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' - 2y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1569: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x} \left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2 x^2}{3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2 x^2}{3}
$$

Verified OK.

2.825.1 Maple step by step solution

Let's solve

$x^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$
- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 2)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{2t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x^2$
- Simplify

  $y = \frac{c_1}{x} + c_2 x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=((4*(3/2)^2-1)/(4*x^2))*y(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + c_2x^2$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]==((4*(3/2)^2-1)/(4*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2x^3 + c_1}{x}$$

2.826 problem 845

2.826.1 Maple step by step solution 7695

Internal problem ID [8316]
Internal file name [`OUTPUT/7249_Sunday_June_05_2022_05_38_19_PM_20278803/index.tex`]

Book: Collection of Kovacic problems
Section: section 1
Problem number: 845.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - \frac{6y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' - 6y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$x^2y'' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1571: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -2$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -2 - (-2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(0) \\ &= -\frac{2}{x} \\ &= -\frac{2}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x^2} \int \frac{1}{\frac{1}{x^4}} \, dx \\ &= \frac{1}{x^2}\left(\frac{x^5}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^5}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^3}{5} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^3}{5}
$$

Verified OK.

2.826.1 Maple step by step solution

Let's solve

$x^2 y'' - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 6y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_1e^{-2t} + c_2e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^2} + c_2x^3$

- Simplify

  $y = \frac{c_1}{x^2} + c_2x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=((4*(5/2)^2-1)/(4*x^2))*y(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^2} + c_2 x^3$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[y''[x]==((4*(5/2)^2-1)/(4*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^5 + c_1}{x^2}$$

3 section 2. Solution found using all possible Kovacic cases

3.1 problem 1 7699
3.2 problem 2 7712
3.3 problem 3 7732
3.4 problem 4 7752
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3.1 problem 1

3.1.1 Maple step by step solution 7707

Internal problem ID [8317]
Internal file name [OUTPUT/7250_Sunday_June_05_2022_05_38_22_PM_40023797/index.tex]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - \left(-\frac{3}{16x^2} - \frac{2}{9\left(x-1\right)^2} + \frac{3}{16x\left(x-1\right)}\right)y = 0}$$

Writing the ode as

$$y'' + \frac{(32x^2 - 27x + 27)\,y}{144x^2\left(x-1\right)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2\left(x-1\right)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -32x^2 + 27x - 27$$
$$t = 144(x^2 - x)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 27x - 27}{144 (x^2 - x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1573: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 2\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x} - \frac{2}{9(x-1)^2} - \frac{3}{16x^2} + \frac{3}{16(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm1, \pm2, \dots, \pm\frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{3, 6, 9\}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm1, \pm2, \dots, \pm\frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{ 6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \cdots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4, 5, 6, 7, 8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3, 6, 9\}$
1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4, 5, 6, 7, 8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 3,\ e_2 = 4,\ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{4}{12}(7 - (3 + (4))) \\
&= 0
\end{aligned}
$$

The following rational function is

$$
\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{4}{12}\left(\frac{3}{(x-(0))} + \frac{4}{(x-(1))}\right) \\
&= \frac{1}{x} + \frac{4}{3x-3}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c) \\
&= x(x-1)
\end{aligned}
$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$
p(x) = 1
$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$
\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned} \tag{1A}
$$

The coefficients a_i are solved for from

$$
P_{-1} = 0 \tag{2A}
$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r =$

$\frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$
\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{7x}{3} - 1 \\
P_2 &= -4x^2 + \frac{41}{12}x - \frac{3}{4} \\
P_1 &= \frac{40}{9}x^3 - \frac{409}{72}x^2 + \frac{5}{2}x - \frac{3}{8} \\
P_0 &= -\frac{64}{27}x^4 + \frac{871}{216}x^3 - \frac{257}{96}x^2 + \frac{13}{16}x - \frac{3}{32} \\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0 \\
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i &= 0
\end{aligned}
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\begin{aligned}
&-\frac{8x^4}{81} + \frac{871x^3}{5184} - \frac{257x^2}{2304} + \frac{13x}{384} - \frac{1}{256} + \frac{x(x-1)\left(320x^3 - 409x^2 + 180x - 27\right)\omega}{432} \\
&- \frac{x^2(x-1)^2\left(48x^2 - 41x + 9\right)\omega^2}{24} + x^3(x-1)^3\left(\frac{7x}{3} - 1\right)\omega^3 - x^4(x-1)^4\omega^4 = 0
\end{aligned}
$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)}\left(7x - 3 + \sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x} + \sqrt{-\frac{2\left(\left(-x^2 + x + \frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x} + x^2 (x-1)\right)}{\sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x}}}\right) \tag{4A}$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega \, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx \\ &= e^{\int \omega \, dx} \int \frac{e^{\int -\frac{B}{A} \, dx}}{\left(e^{\int \omega \, dx}\right)^2} \, dx \end{aligned}$$

Since $B = 0$ then the above reduces to

$$y_2 = e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx$$

Therefore for kovacic case $n=4$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{\int \omega \, dx}\right) + c_2 \left(e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx\right) \end{aligned}$$

Attempting to find a solution using $n=6$.

Unable to find solution using case three with $n=6$.

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1() + c_2() \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x+x^2(x-1)}\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}}{12x(x-1)} dx} \tag{1}$$

$$+ c_2 \mathrm{e}^{\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x+x^2(x-1)}\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}}{x(x-1)} dx\right)}{12}} \left(\int \mathrm{e}^{-\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-\right.}}{}\right.}{}} \right.$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}{12x(x-1)}dx}
$$

$$
+c_2 \mathrm{e}^{\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}{x(x-1)}dx\right)}{12}} \left(\int \mathrm{e}^{-\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-\right.}}{}\right.}{}}\right.
$$

Verified OK.

3.1.1 Maple step by step solution

Let's solve

$y'' + \frac{(32x^2-27x+27)y}{144x^2(x-1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $144y''x^2(x-1)^2 + (32x^2 - 27x + 27)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k + 2 - m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $9a_0(-1+4r)\,(-3+4r)\, x^r + (9a_1(3+4r)\,(1+4r) - 9a_0(32r^2 - 32r + 3))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9(-1+4r)\,(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term must be 0

  $9a_1(3+4r)\,(1+4r) - 9a_0(32r^2 - 32r + 3) = 0$

- Solve for the dependent coefficient(s)

$a_1 = \frac{a_0(32r^2-32r+3)}{16r^2+16r+3}$

- Each term in the series must be 0, giving the recursion relation

$144(a_k + a_{k-2} - 2a_{k-1}) k^2 + 144(2(a_k + a_{k-2} - 2a_{k-1}) r - a_k - 5a_{k-2} + 6a_{k-1}) k + 144(a_k + a_{k-}$

- Shift index using $k- >k+2$

$144(a_{k+2} + a_k - 2a_{k+1}) (k+2)^2 + 144(2(a_{k+2} + a_k - 2a_{k+1}) r - a_{k+2} - 5a_k + 6a_{k+1}) (k+2) + 1$

- Recursion relation that defines series solution to ODE

$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 288kra_k - 576kra_{k+1} + 144r^2a_k - 288r^2a_{k+1} - 144ka_k - 288ka_{k+1} - 144ra_k - 288ra_{k+1} + 32a_k - 27}{9(16k^2+32kr+16r^2+48k+48r+35)}$

- Recursion relation for $r = \frac{1}{4}$

$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}$

- Solution for $r = \frac{1}{4}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\frac{3a_0}{8}\right]$

- Recursion relation for $r = \frac{3}{4}$

$a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}$

- Solution for $r = \frac{3}{4}$

$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} + 72ka_k - 720ka_{k+1} + 5a_k - 405a_{k+1}}{9(16k^2+72k+80)}, a_1 = -\frac{a_0}{8}\right]$

- Combine solutions and rename parameters

$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{144k^2a_k - 288k^2a_{k+1} - 72ka_k - 432ka_{k+1} + 5a_k - 117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 209

```
dsolve(diff(y(x),x$2)= ( -3/(16*x^2)- 2/(9*(x-1)^2) + 3/(16*x*(x-1))) *y(x),y(x), singsol=al
```

$$y(x) = c_1\sqrt{(x-1)^{\frac{1}{3}}+1}\left((x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}(x-1)^{\frac{1}{3}}\left(\frac{\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}-\sqrt{3}}{-\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}+\sqrt{3}}\right)^{\frac{1}{8}} + \frac{c_2\sqrt{(x-1)^{\frac{1}{3}}+1}\left((x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}(x-1)^{\frac{1}{3}}}{\left(\frac{\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}-\sqrt{3}}{-\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}+\sqrt{3}}\right)^{\frac{1}{8}}}$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 550

```
DSolve[y''[x]== ( -3/(16*x^2)- 2/(9*(x-1)^2) + 3/(16*x*(x-1))) *y[x],y[x],x,IncludeSingularS
```

$$
\begin{aligned}
y(x) \to\; & c_1 \exp\Bigg(\int_1^x \operatorname{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1] \\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
& + 81\&, 1\big]\, dK[1]\Bigg) + c_2 \exp\Bigg(\int_1^x \operatorname{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1] \\
& + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 \\
& + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 \\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 \\
& + 81\&, 1\big]\, dK[1]\Bigg) \int_1^x \exp\Bigg(-2\int_1^{K[2]} \operatorname{Root}\big[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 \\
& - 702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 \\
& + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 11836\right. \\
& + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\big]\, dK[1]\Bigg)\, dK[2]
\end{aligned}
$$

3.2 problem 2

3.2.1 Maple step by step solution 7729

Internal problem ID [8318]
Internal file name [`OUTPUT/7251_Sunday_June_05_2022_05_38_31_PM_16419834/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - \frac{20y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' - 20y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$x^2y'' - 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{20}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 20$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{20}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1575: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{20}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 20$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 5 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -4 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{20}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	5	-4

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	5	-4

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -4$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -4 - (-4) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{4}{x} + (-)(0) \\
&= -\frac{4}{x} \\
&= -\frac{4}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{4}{x}\right)(0) + \left(\left(\frac{4}{x^2}\right) + \left(-\frac{4}{x}\right)^2 - \left(\frac{20}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{4}{x} dx} \\
&= \frac{1}{x^4}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{1}{x^4}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}} \, dx \\
&= \frac{1}{x^4} \left(\frac{x^9}{9}\right)
\end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{20}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 20$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-16, 2, 20\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{20}{x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 20$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-16, 2, 20\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-16, 2, 20\}$

Order of r at ∞	E_∞
2	$\{-16, 2, 20\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{20}{x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{4}{x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{4}{x} dx} \\ &= \frac{1}{x^4} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^4} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}}\, dx \\
&= \frac{1}{x^4}\left(\frac{x^9}{9}\right)
\end{aligned}
$$

Therefore for kovacic case $n = 2$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right)
\end{aligned}
$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{20}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 20$. Hence

$$
E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$
E_c = \{-48, -21, 6, 33, 60\}
$$

Let

$$
E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \middle| k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}
$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$
r \approx \frac{20}{x^2} + \cdots
$$

The above shows that

$$b = 20$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-48, -21, 6, 33, 60\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-48, -21, 6, 33, 60\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-48, -21, 6, 33, 60\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -48,\ e_\infty = -48$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(-48 - (-48)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{-48}{(x-(0))}\right)\\ &= -\frac{16}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i = n, n-1, \ldots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{20}{x^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= -16\\ P_2 &= -192\\ P_1 &= -1536\\ P_0 &= -6144\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 4)^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-4)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{4}{x} dx} \\ &= \frac{1}{x^4} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^4} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}}\,dx \\ &= \frac{1}{x^4}\left(\frac{x^9}{9}\right) \end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right) \end{aligned}$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{20}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 20$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \,|\, k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{-48, -30, -12, 6, 24, 42, 60\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \,|\, k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{20}{x^2} + \cdots$$

The above shows that

$$b = 20$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-48, -30, -12, 6, 24, 42, 60\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-48, -30, -12, 6, 24, 42, 60\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-48, -30, -12, 6, 24, 42, 60\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -48,\ e_\infty = -48$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{6}{12}(-48 - (-48)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{6}{12}\left(\frac{-48}{(x-(0))}\right)\\ &= -\frac{24}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x)=1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1}=0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=6$ and $r=\frac{20}{x^2}$).

$$\begin{aligned}P_6 &= -p\\ &= -1\\ P_5 &= -24\\ P_4 &= -480\\ P_3 &= -7680\\ P_2 &= -92160\\ P_1 &= -737280\\ P_0 &= -2949120\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 4)^6 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-4)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{4}{x} dx} \\ &= \frac{1}{x^4} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^4} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{1}{x^4} \int \frac{1}{\frac{1}{x^8}}\, dx \\ &= \frac{1}{x^4}\left(\frac{x^9}{9}\right) \end{aligned}$$

Therefore for kovacic case $n = 6$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^4}\right) + c_2\left(\frac{1}{x^4}\left(\frac{x^9}{9}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2 x^5}{9} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2 x^5}{9}$$

Verified OK.

3.2.1 Maple step by step solution

Let's solve

$x^2y'' - 20y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{20y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{20y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 20y = 0$

- Make a change of variables

  $t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - ○ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 20y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 20y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 20 = 0$

- Factor the characteristic polynomial

  $(r+4)(r-5)=0$
- Roots of the characteristic polynomial

  $r=(-4,5)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-4t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{5t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{5t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^4}+c_2x^5$
- Simplify

  $y=\frac{c_1}{x^4}+c_2x^5$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=((4*(9/2)^2-1)/(4*x^2))*y(x),y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x^4}+c_2x^5$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[y''[x]==((4*(9/2)^2-1)/(4*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^9 + c_1}{x^4}$$

3.3 problem 3

3.3.1 Maple step by step solution 7749

Internal problem ID [8319]
Internal file name [`OUTPUT/7252_Sunday_June_05_2022_05_38_50_PM_53496081/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - \frac{12y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' - 12y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$x^2y'' - 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 12$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1577: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -3 - (-3) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{x} + (-)(0) \\ &= -\frac{3}{x} \\ &= -\frac{3}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3}{x} dx} \\
&= \frac{1}{x^3}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \frac{1}{x^3}
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}} \, dx \\
&= \frac{1}{x^3} \left(\frac{x^7}{7}\right)
\end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3} \left(\frac{x^7}{7}\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{12}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-12, 2, 16\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{12}{x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 12$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-12, 2, 16\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-12, 2, 16\}$

Order of r at ∞	E_∞
2	$\{-12, 2, 16\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{12}{x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{3}{x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^3} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}} \, dx \\ &= \frac{1}{x^3} \left(\frac{x^7}{7} \right) \end{aligned}$$

Therefore for kovacic case $n = 2$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x^3} \right) + c_2 \left(\frac{1}{x^3} \left(\frac{x^7}{7} \right) \right) \end{aligned}$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 12$. Hence

$$E_c = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{-36, -15, 6, 27, 48\}$$

Let

$$E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{12}{x^2} + \cdots$$

The above shows that

$$b = 12$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-36, -15, 6, 27, 48\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-36, -15, 6, 27, 48\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-36, -15, 6, 27, 48\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -36, \; e_\infty = -36$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(-36 - (-36)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{-36}{(x-(0))}\right)\\ &= -\frac{12}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=\frac{12}{x^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= -12\\ P_2 &= -108\\ P_1 &= -648\\ P_0 &= -1944\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 3)^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-3)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^3} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}}\,dx \\
&= \frac{1}{x^3}\left(\frac{x^7}{7}\right)
\end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^7}{7}\right)\right)
\end{aligned}$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 12$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{-36, -22, -8, 6, 20, 34, 48\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{12}{x^2} + \cdots$$

The above shows that

$$b = 12$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-36, -22, -8, 6, 20, 34, 48\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-36, -22, -8, 6, 20, 34, 48\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-36, -22, -8, 6, 20, 34, 48\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = -36,\ e_\infty = -36$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{6}{12}(-36 - (-36)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{6}{12}\left(\frac{-36}{(x-(0))}\right)\\ &= -\frac{18}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=6$ and $r=\frac{12}{x^2}$).

$$\begin{aligned}P_6 &= -p\\ &= -1\\ P_5 &= -18\\ P_4 &= -270\\ P_3 &= -3240\\ P_2 &= -29160\\ P_1 &= -174960\\ P_0 &= -524880\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 3)^6 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-3)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^3} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \frac{1}{x^3} \int \frac{1}{\frac{1}{x^6}}\,dx \\
&= \frac{1}{x^3}\left(\frac{x^7}{7}\right)
\end{aligned}$$

Therefore for kovacic case $n = 6$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^7}{7}\right)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^7}{7}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} + \frac{c_2 x^4}{7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + \frac{c_2 x^4}{7}$$

Verified OK.

3.3.1 Maple step by step solution

Let's solve

$x^2y'' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{12y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{12y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 12y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 12y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 12y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-4)=0$
- Roots of the characteristic polynomial

  $r=(-3,4)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-3t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{4t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{4t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^3}+c_2x^4$
- Simplify

  $y=\frac{c_1}{x^3}+c_2x^4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=((4*(7/2)^2-1)/(4*x^2))*y(x),y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x^3}+x^4c_2$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[y''[x]==((4*(7/2)^2-1)/(4*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^7 + c_1}{x^3}$$

3.4 problem 4

3.4.1 Maple step by step solution 7769

Internal problem ID [8320]
Internal file name [`OUTPUT/7253_Sunday_June_05_2022_05_39_05_PM_48791029/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - \frac{y}{4x^2} = 0}$$

The ode can be written as

$$4x^2y'' - y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$4x^2y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1579: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{\sqrt{2}}{2} + \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{\sqrt{2}}{2} + \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{\sqrt{2}}{2} + \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{\sqrt{2}}{2} + \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{\sqrt{2}}{2}+\frac{1}{2}$	$-\frac{\sqrt{2}}{2}+\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{\sqrt{2}}{2}+\frac{1}{2}$	$-\frac{\sqrt{2}}{2}+\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{\sqrt{2}}{2}+\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{\sqrt{2}}{2}+\frac{1}{2} - \left(-\frac{\sqrt{2}}{2}+\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{-\frac{\sqrt{2}}{2}+\frac{1}{2}}{x} + (-)(0) \\
&= \frac{-\frac{\sqrt{2}}{2}+\frac{1}{2}}{x} \\
&= -\frac{\sqrt{2}-1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{-\frac{\sqrt{2}}{2} + \frac{1}{2}}{x}\right)(0) + \left(\left(-\frac{-\frac{\sqrt{2}}{2} + \frac{1}{2}}{x^2}\right) + \left(\frac{-\frac{\sqrt{2}}{2} + \frac{1}{2}}{x}\right)^2 - \left(\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{-\frac{\sqrt{2}}{2} + \frac{1}{2}}{x} dx} \\ &= x^{-\frac{\sqrt{2}}{2} + \frac{1}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{-\frac{\sqrt{2}}{2} + \frac{1}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{-\frac{\sqrt{2}}{2} + \frac{1}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \int \frac{1}{x^{1-\sqrt{2}}}\,dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)
\end{aligned}
$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\right) + c_2\left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{1}{4}$. Hence

$$
\begin{aligned}
E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \left\{2, 2+2\sqrt{2}, -2\sqrt{2}+2\right\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{1}{4}$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\
&= \{2\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, 2+2\sqrt{2}, -2\sqrt{2}+2\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2,\; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{1}{4x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + \sqrt{2}}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+\sqrt{2}}{2x} dx} \\ &= x^{\frac{\sqrt{2}}{2}+\frac{1}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= x^{\frac{\sqrt{2}}{2}+\frac{1}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{\sqrt{2}}{2}+\frac{1}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{\frac{\sqrt{2}}{2}+\frac{1}{2}} \int \frac{1}{x^{1+\sqrt{2}}} \, dx \\
&= x^{\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(-\frac{x^{-\sqrt{2}}\sqrt{2}}{2} \right)
\end{aligned}
$$

Therefore for kovacic case $n = 2$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{\sqrt{2}}{2}+\frac{1}{2}} \right) + c_2 \left(x^{\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(-\frac{x^{-\sqrt{2}}\sqrt{2}}{2} \right) \right)
\end{aligned}
$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{1}{4}$. Hence

$$
E_c = \left\{ 6 + \frac{12k}{n} \sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$
E_c = \{6\}
$$

Let

$$
E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}
$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$
r \approx \frac{1}{4x^2} + \cdots
$$

The above shows that

$$b = \frac{1}{4}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{6\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 6, \; e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(6 - (6)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{6}{(x-(0))}\right)\\ &= \frac{2}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=\frac{1}{4x^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= 2\\ P_2 &= -1\\ P_1 &= -3\\ P_0 &= -\frac{3}{2}\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\frac{1}{16}\left(-\left(4x^2\omega^2 - 4x\omega - 1\right)^2\right) = 0 \tag{3A}$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x}\left(1 - \sqrt{2}\right)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1-\sqrt{2}}{2x} dx} \\ &= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \int \frac{1}{x^{1-\sqrt{2}}}\,dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)
\end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\right) + c_2 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)\right)
\end{aligned}$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{1}{4}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{6\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{1}{4x^2} + \cdots$$

The above shows that

$$b = \frac{1}{4}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{6\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 6,\ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{6}{12}(6 - (6)) \\ &= 0 \end{aligned}$$

The following rational function is

$$
\begin{aligned}
\theta &= \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x - c} \\
&= \frac{6}{12} \left(\frac{6}{(x - (0))} \right) \\
&= \frac{3}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c \in \Gamma} (x - c) \\
&= x
\end{aligned}
$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$
p(x) = 1
$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$
\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n - i)S' - S\theta)P_i - (n - 1)(i + 1)S^2 r P_{i+1} \qquad i = n, n - 1, \ldots, 0
\end{aligned}
\tag{1A}
$$

The coefficients a_i are solved for from

$$
P_{-1} = 0 \tag{2A}
$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r = \frac{1}{4x^2}$).

$$
\begin{aligned}
P_6 &= -p \\
&= -1 \\
P_5 &= 3 \\
P_4 &= -\frac{9}{2} \\
P_3 &= -3 \\
P_2 &= \frac{27}{2}
\end{aligned}
$$

$$P_1 = \frac{45}{2}$$
$$P_0 = \frac{45}{4}$$
$$P_{-1} = 0$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\frac{1}{64}\left(-\left(4x^2\omega^2 - 4x\omega - 1\right)^3\right) = 0 \tag{3A}$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x}\left(1 - \sqrt{2}\right)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1-\sqrt{2}}{2x} dx} \\ &= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \int \frac{1}{x^{1-\sqrt{2}}} \, dx \\
&= x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)
\end{aligned}
$$

Therefore for kovacic case $n = 6$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\right) + c_2 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}}\right) + c_2 \left(x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} \left(\frac{x^{\sqrt{2}}\sqrt{2}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} + \frac{c_2\sqrt{2}\, x^{\frac{\sqrt{2}}{2}+\frac{1}{2}}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x^{-\frac{\sqrt{2}}{2}+\frac{1}{2}} + \frac{c_2\sqrt{2}\, x^{\frac{\sqrt{2}}{2}+\frac{1}{2}}}{2}$$

Verified OK.

3.4.1 Maple step by step solution

Let's solve

$4x^2 y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2 y'' - y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$4x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) - y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) + \frac{y(t)}{4}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - \frac{y(t)}{4} = 0$
- Characteristic polynomial of ODE

  $r^2 - r - \frac{1}{4} = 0$
- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{2})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{\sqrt{2}}{2} + \frac{1}{2}, \frac{\sqrt{2}}{2} + \frac{1}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = e^{\left(-\frac{\sqrt{2}}{2} + \frac{1}{2}\right)t}$
- 2nd solution of the ODE

  $y_2(t) = e^{\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{\left(-\frac{\sqrt{2}}{2} + \frac{1}{2}\right)t} + c_2 e^{\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 e^{\left(-\frac{\sqrt{2}}{2} + \frac{1}{2}\right)\ln(x)} + c_2 e^{\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)\ln(x)}$
- Simplify

  $y = \sqrt{x}\left(c_1 x^{-\frac{\sqrt{2}}{2}} + c_2 x^{\frac{\sqrt{2}}{2}}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-1/(4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\,x^{\frac{\sqrt{2}}{2}} + c_2\sqrt{x}\,x^{-\frac{\sqrt{2}}{2}}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 32

```
DSolve[y''[x]-1/(4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{\frac{1}{2}-\frac{1}{\sqrt{2}}}\left(c_2x^{\sqrt{2}} + c_1\right)$$

3.5 problem 5

3.5.1 Maple step by step solution 7789

Internal problem ID [8321]
Internal file name [`OUTPUT/7254_Sunday_June_05_2022_05_39_21_PM_38062785/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - (2x+2)\,y' + (x+2)\,y = 0}$$

Writing the ode as

$$xy'' + (-2x-2)\,y' + (x+2)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1581: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)\,(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, e^x) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-4, 2, 8\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-4, 2, 8\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-4, 2, 8\}$

Order of r at ∞	E_∞
2	$\{-4, 2, 8\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 2, \; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{2}{x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{1}{x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 2$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 2$. Hence

$$
E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$
E_c = \{-12, -3, 6, 15, 24\}
$$

Let

$$
E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}
$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$
r \approx \frac{2}{x^2} + \cdots
$$

The above shows that

$$
b = 2
$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$
E_\infty = \{-12, -3, 6, 15, 24\}
$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-12, -3, 6, 15, 24\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-12, -3, 6, 15, 24\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -12,\ e_\infty = -12$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(-12 - (-12)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{4}{12}\left(\frac{-12}{(x-(0))}\right) \\ &= -\frac{4}{x} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= x \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{2}{x^2}$).

$$\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= -4 \\
P_2 &= -12 \\
P_1 &= -24 \\
P_0 &= -24 \\
P_{-1} &= 0
\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i &= 0 \\
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i &= 0
\end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 1)^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-1)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x^3}{3} \right) \right)
\end{aligned}
$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 2$. Hence

$$
E_c = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$
E_c = \{-12, -6, 0, 6, 12, 18, 24\}
$$

Let

$$
E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}
$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$
r \approx \frac{2}{x^2} + \cdots
$$

The above shows that

$$
b = 2
$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$
E_\infty = \{-12, -6, 0, 6, 12, 18, 24\}
$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-12, -6, 0, 6, 12, 18, 24\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-12, -6, 0, 6, 12, 18, 24\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 0,\ e_\infty = 0$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{6}{12}(0 - (0)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{6}{12}\left(\frac{0}{(x-(0))}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= x \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned} P_n &= -p(x) \\ &= -1 \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r = \frac{2}{x^2}$).

$$\begin{aligned} P_6 &= -p \\ &= -1 \\ P_5 &= 0 \\ P_4 &= 12 \\ P_3 &= 24 \\ P_2 &= -216 \\ P_1 &= -1440 \\ P_0 &= -2880 \\ P_{-1} &= 0 \end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned} \sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i &= 0 \\ \sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i &= 0 \end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega - 2)^2 (x\omega + 1)^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-1)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x+\ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx$$

$$= y_1 \int \frac{e^{2x+2\ln(x)}}{(y_1)^2} \, dx$$

$$= y_1 \left(\frac{x^3}{3} \right)$$

Therefore for kovacic case $n = 6$ the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x^3}{3} \right) \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x^3}{3} \right) \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{3}$$

Verified OK.

3.5.1 Maple step by step solution

Let's solve

$xy'' + (-2x - 2)\, y' + (x + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(1+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(1+x)y'}{x} + \frac{(x+2)y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2(1+x)}{x}, P_3(x) = \frac{x+2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $xy'' + (-2x - 2)\, y' + (x + 2)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$

- Convert $x^{m}\cdot y'$ to series expansion for $m=0..1$

  $$x^{m}\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x\cdot y''$ to series expansion

  $$x\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_{0}r(-3+r)\, x^{-1+r}+\left(a_{1}(1+r)\,(-2+r)-2a_{0}(-1+r)\right)x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(k-2+r)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\{0,3\}$$

- Each term must be 0

  $$a_{1}(1+r)\,(-2+r)-2a_{0}(-1+r)=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)\,(k-2+r)-2a_{k}k-2a_{k}r+2a_{k}+a_{k-1}=0$$

- Shift index using $k->k+1$

  $$a_{k+2}(k+2+r)\,(k+r-1)-2a_{k+1}(k+1)-2ra_{k+1}+2a_{k+1}+a_{k}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_{k}}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r=0$

  $$a_{k+2}=\frac{2ka_{k+1}-a_{k}}{(k+2)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $a_{k+2} = \frac{2ka_{k+1} - a_k}{(k+2)(k-1)}$

- Recursion relation for $r = 3$

  $a_{k+2} = \frac{2ka_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}$

- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{2ka_{k+1} - a_k + 6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+2)*diff(y(x),x)+(2+x)*y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^x x^3$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[x*y''[x]-(2*x+2)*y'[x]+(2+x)*y[x] ==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3} e^x \left(c_2 x^3 + 3c_1\right)$$

3.6 problem 6

3.6.1 Maple step by step solution 7809

Internal problem ID [8322]
Internal file name [`OUTPUT/7255_Sunday_June_05_2022_05_39_39_PM_24727576/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + \frac{y}{x^2} = 0}$$

The ode can be written as

$$x^2y'' + y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$x^2y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1583: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+\frac{i\sqrt{3}}{2}$	$\frac{1}{2}-\frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\ &= \frac{1 - i\sqrt{3}}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)^2 - \left(-\frac{1}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}
\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}} \, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right) \end{aligned}$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \left\{2, -2i\sqrt{3} + 2, 2i\sqrt{3} + 2\right\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{2, -2i\sqrt{3}+2, 2i\sqrt{3}+2\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2,\; e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2-(2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{1}{x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} + \frac{1}{x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + i\sqrt{3}}{2x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{1+i\sqrt{3}}{2x} dx}\\ &= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1+i\sqrt{3}}} \, dx \\
&= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{i x^{-i\sqrt{3}} \sqrt{3}}{3} \right)
\end{aligned}
$$

Therefore for kovacic case $n = 2$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \right) + c_2 \left(x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(\frac{i x^{-i\sqrt{3}} \sqrt{3}}{3} \right) \right)
\end{aligned}
$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -1$. Hence

$$
E_c = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}
$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$
E_c = \{6\}
$$

Let

$$
E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}
$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$
r \approx -\frac{1}{x^2} + \cdots
$$

The above shows that

$$b = -1$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{6\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 6, \; e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(6 - (6)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{4}{12}\left(\frac{6}{(x-(0))}\right)\\ &= \frac{2}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d=0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n=4$ and $r=-\frac{1}{x^2}$).

$$\begin{aligned}P_4 &= -p\\ &= -1\\ P_3 &= 2\\ P_2 &= -6\\ P_1 &= 12\\ P_0 &= -24\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-\left(x^2\omega^2 - x\omega + 1\right)^2 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x}\left(1 - i\sqrt{3}\right)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1-i\sqrt{3}}{2x} dx} \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right) \end{aligned}$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -1$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{6\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{1}{x^2} + \cdots$$

The above shows that

$$b = -1$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{6\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 6,\ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{6}{12}(6 - (6)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{6}{12}\left(\frac{6}{(x-(0))}\right)\\ &= \frac{3}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 rP_{i+1} \qquad i = n, n-1, \dots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r = -\frac{1}{x^2}$).

$$\begin{aligned}P_6 &= -p\\ &= -1\\ P_5 &= 3\\ P_4 &= -12\\ P_3 &= 42\\ P_2 &= -144\\ P_1 &= 360\\ P_0 &= -720\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-\left(x^2\omega^2 - x\omega + 1\right)^3 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x}\left(1 - i\sqrt{3}\right)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= e^{\int \frac{1-i\sqrt{3}}{2x} dx} \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 6$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}}\sqrt{3}}{3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}$$

Verified OK.

3.6.1 Maple step by step solution

Let's solve

$x^2 y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-3})}{2}$
- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{3}}{2}, \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)$
- 2nd solution of the ODE

  $y_2(t) = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)$
- Change variables back using $t = \ln(x)$

  $y = c_1 \sqrt{x}\, \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) + c_2 \sqrt{x}\, \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)$
- Simplify

  $y = \sqrt{x}\left(c_1 \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+1/x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}\, x^{\frac{\sqrt{-3}}{2}} + c_2\sqrt{x}\, x^{-\frac{\sqrt{-3}}{2}}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 42

```
DSolve[y''[x]+1/x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\left(c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

3.7 problem 7

3.7.1 Maple step by step solution 7830

Internal problem ID [8323]
Internal file name [OUTPUT/7256_Sunday_June_05_2022_05_39_54_PM_73652588/index.tex]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1-x^2)\, y''+y'+y=0}$$

Writing the ode as

$$(1-x^2)\, y''+y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1-x^2 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 4x - 3}{4(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 4x - 3 \\ t &= 4(x^2 - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 4x - 3}{4(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1585: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2-1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16(1+x)^2} + \frac{5}{16(x-1)^2} + \frac{7}{16(x-1)} - \frac{7}{16(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4x^2+4x-3}{4(x^2-1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{2\} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
1	2	$\{-1, 2, 5\}$
-1	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = 1,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (-1 + (1))) \\ &= 1 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c} \\ &= \frac{1}{2}\left(\frac{-1}{(x - (1))} + \frac{1}{(x - (-1))}\right) \\ &= -\frac{1}{2(x-1)} + \frac{1}{2x+2} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 1$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 1$, then letting

$$p = x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$\frac{4a_0 + 6}{(x-1)^2 (1+x)} = 0$$

And solving for p gives

$$p = x - \frac{3}{2}$$

Now that $p(x)$ is found let

$$
\begin{aligned}
\phi &= \theta + \frac{p'}{p} \\
&= \frac{1}{x-\frac{3}{2}} - \frac{1}{2\left(x-1\right)} + \frac{1}{2x+2}
\end{aligned}
$$

Let ω be the solution of

$$
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0
$$

Substituting the values for ϕ and r into the above equation gives

$$
w^2 - \left(\frac{1}{x-\frac{3}{2}} - \frac{1}{2\left(x-1\right)} + \frac{1}{2x+2}\right) w + \frac{-8x^3+4x^2+10x-7}{4\left(x^2-1\right)^2\left(2x-3\right)} = 0
$$

Solving for ω gives

$$
\omega = \frac{2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)}\,x - 2\sqrt{5}\,\sqrt{\left(x-1\right)\left(1+x\right)} + 2x^2 - 2x + 1}{2\left(2x-3\right)\left(x-1\right)\left(1+x\right)}
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \frac{2\sqrt{5}\,\sqrt{(x-1)(1+x)}\,x - 2\sqrt{5}\,\sqrt{(x-1)(1+x)} + 2x^2 - 2x + 1}{2(2x-3)(x-1)(1+x)}dx} \\
&= \frac{\sqrt{2x-3}\,\left(1+x\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} 5^{\frac{1}{4}}}{\left(x-1\right)^{\frac{1}{4}}\sqrt{\frac{5\sqrt{x^2-1}+(3x-2)\sqrt{5}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{2x-3}\,(5+5x)^{\frac{1}{4}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}\right)}{2x-3}}\,(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}\right)\sqrt{x-1}}{(2x-3)^2\sqrt{5+5x}}dx \right)
\end{aligned}$$

Therefore for kovacic case $n=2$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{2x-3}\,(5+5x)^{\frac{1}{4}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}\right)}{2x-3}}\,(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right) \\
&\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{2}} \sqrt{2x-3}\,(5+5x)^{\frac{1}{4}}}{\sqrt{\frac{i\left(3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}\right)}{2x-3}}\,(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \left(\int \frac{i\left(x+\sqrt{x^2-1}\right)^{-\sqrt{5}} \left(3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}\right)\sqrt{}}{(2x-3)^2\sqrt{5+5x}} \right.\right.
\end{aligned}$$

Attempting to find a solution using $n=4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(1+x)^2} + \frac{5}{16\,(x-1)^2} + \frac{7}{16\,(x-1)} - \frac{7}{16\,(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{5}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{-3, 6, 15\}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{3, 6, 9\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{1}{x^2} + \frac{1}{x^3} + \frac{5}{4x^4} + \frac{2}{x^5} + \frac{3}{2x^6} + \frac{3}{x^7} + \dots$$

The above shows that

$$b = 1$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
1	2	$\{-3, 6, 15\}$
-1	2	$\{3, 6, 9\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -3,\ e_2 = 3,\ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(6 - (-3 + (3))) \\ &= 2 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{4}{12}\left(\frac{-3}{(x-(1))} + \frac{3}{(x-(-1))}\right) \\ &= -\frac{2}{x^2-1} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= (x-1)(1+x) \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 2$, then let

$$p(x) = x^2 + a_1 x + a_0$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}
P_n &= -p(x) \\
&= -\left(x^2 + a_1 x + a_0\right) \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0
\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{4x^2+4x-3}{4(x^2-1)^2}$).

$$\begin{aligned}
P_4 &= -p \\
&= -x^2 - a_1 x - a_0 \\
P_3 &= 2x^3 + (a_1 - 2)\, x^2 + (-2a_1 - 2)\, x - 2a_0 - a_1 \\
P_2 &= 2x^4 + (4a_1 + 8)\, x^3 + (4a_0 + 4a_1 - 3)\, x^2 + (-7a_1 - 8)\, x - 7a_0 - 4a_1 - 2 \\
P_1 &= -12 - 12x^5 + 2(6 - a_1)\, x^4 + (51 + 8a_0 + 22a_1)\, x^3 \\
&\quad + \frac{(40a_0 + 43a_1 - 6)\, x^2}{2} + (-39 - 8a_0 - 25a_1)\, x - 23a_0 - \frac{39a_1}{2} \\
P_0 &= -72 - 24x^6 + 4(-15 - 7a_1)\, x^5 + 2(48 + 7a_1)\, x^4 + 2(120 + 36a_0 + 79a_1)\, x^3 \\
&\quad + \frac{(168a_0 + 136a_1 - 3)\, x^2}{2} + \frac{(-144a_0 - 263a_1 - 360)\, x}{2} - \frac{171a_0}{2} - 82a_1 \\
P_{-1} &= -2(x-1)\,(1+x)^3 \left(38x^2 a_1 + 114x^2 + 16x a_0 - 36a_1 x - 144x - 156a_0 - 177a_1 - 180\right)
\end{aligned}$$

Using method of undetermined coefficient to solve for a_i from the last equation above $P_{-1} = 0$ gives the following solution for the coefficients a_i

$$\begin{aligned}
a_0 &= \frac{9}{4} \\
a_1 &= -3
\end{aligned}$$

Substituting these in the the polynomial $p(x)$ results in

$$P(x) = x^2 - 3x + \frac{9}{4}$$

ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\begin{aligned}
&-3 - x^6 + \frac{(-15-7a_1)\,x^5}{6} + \frac{(48+7a_1)\,x^4}{12} + \frac{(120+36a_0+79a_1)\,x^3}{12} \\
&+ \frac{(168a_0+136a_1-3)\,x^2}{48} + \frac{(-144a_0-263a_1-360)\,x}{48} - \frac{57a_0}{16} - \frac{41a_1}{12} \\
&- 2\omega(1+x)\left(x^5 + \left(\frac{a_1}{6}-1\right)x^4 + \left(-\frac{2a_0}{3} - \frac{11a_1}{6} - \frac{17}{4}\right)x^3 + \left(-\frac{5a_0}{3} - \frac{43a_1}{24} + \frac{1}{4}\right)x^2 \right. \\
&\left. + \left(\frac{2a_0}{3} + \frac{25a_1}{12} + \frac{13}{4}\right)x + \frac{23a_0}{12} + \frac{13a_1}{8} + 1\right)(x-1) + \omega^2(1+x)^2\left(x^4 \right. \\
&\left. + (2a_1+4)\,x^3 + \left(2a_0+2a_1-\frac{3}{2}\right)x^2 + \left(-\frac{7a_1}{2}-4\right)x - \frac{7a_0}{2} - 2a_1 - 1\right)(x-1)^2 \\
&+ (x-1)^3(1+x)^3\left(2x^3 + (a_1-2)\,x^2 + (-2a_1-2)\,x - 2a_0 - a_1\right)\omega^3 \\
&- (1+x)^4(x-1)^4\left(x^2 + a_1 x + a_0\right)\omega^4 = 0
\end{aligned}$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{4x^3-6x^2-4x+6}\left(-2\sqrt{5}\sqrt{(1+x)(x-1)^3} + 2x^2 - 2x + 1\right)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega\, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned}
z_1(x) &= e^{\int \omega\, dx} \\
&= e^{\int \frac{-2\sqrt{5}\sqrt{(1+x)(x-1)^3}+2x^2-2x+1}{4x^3-6x^2-4x+6} dx} \\
&= \frac{(1+x)^{\frac{1}{4}}\sqrt{2x-3}\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}\sqrt{(1+x)(x-1)^3}}{\sqrt{x^2-1}\,(2x-2)}} \mathrm{e}^{\frac{\operatorname{arctanh}\left(\frac{(3x-2)\sqrt{5}}{5\sqrt{x^2-1}}\right)\sqrt{(1+x)(x-1)^3}}{\sqrt{x^2-1}\,(2x-2)}}}{(x-1)^{\frac{1}{4}}}
\end{aligned}$$

Which simplifies to

$$z_1(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1-x^2}\,dx} \\
&= z_1 e^{-\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}}\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\,\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)}dx\right)
\end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right)$$

$$+c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}}\,\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3}{x^2-1}}}{\sqrt{1+x}\,\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)\,(} \right.\right.$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16\,(1+x)^2} + \frac{5}{16\,(x-1)^2} + \frac{7}{16\,(x-1)} - \frac{7}{16\,(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{5}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{-3, 0, 3, 6, 9, 12, 15\}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{3, 4, 5, 6, 7, 8, 9\}$$

Let

$$E_\infty = \left\{ 6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{1}{x^2} + \frac{1}{x^3} + \frac{5}{4x^4} + \frac{2}{x^5} + \frac{3}{2x^6} + \frac{3}{x^7} + \cdots$$

The above shows that

$$b = 1$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
1	2	$\{-3, 0, 3, 6, 9, 12, 15\}$
-1	2	$\{3, 4, 5, 6, 7, 8, 9\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -3, \; e_2 = 3, \; e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_{\infty} - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{6}{12}(6 - (-3 + (3))) \\ &= 3 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{6}{12}\left(\frac{-3}{(x-(1))} + \frac{3}{(x-(-1))}\right) \\ &= -\frac{3}{x^2-1} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= (x-1)(1+x) \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 3$, then let

$$p(x) = x^3 + a_2x^2 + a_1x + a_0$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned} P_n &= -p(x) \\ &= -\left(x^3 + a_2x^2 + a_1x + a_0\right) \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r =$

$\frac{4x^2+4x-3}{4(x^2-1)^2}$).

$$
\begin{aligned}
P_6 &= -p \\
&= -x^3 - a_2x^2 - a_1x - a_0 \\
P_5 &= 3x^4 + (2a_2-3)\,x^3 + (a_1-3a_2-3)\,x^2 + (-3a_1-2a_2)\,x - 3a_0 - a_1 \\
P_4 &= 2(9+2a_2)\,x^4 + \frac{3(-1+4a_1+8a_2)\,x^3}{2} + \frac{(12a_0+12a_1-19a_2-36)\,x^2}{2} \\
&\quad + \frac{3(-4-9a_1-8a_2)\,x}{2} - \frac{27a_0}{2} - 6a_1 - 2a_2 \\
P_3 &= -30x^6 - 20x^5a_2 + (-4a_1+34a_2+135)\,x^4 + (12a_0+50a_1+98a_2+39)\,x^3 \\
&\quad + (48a_0+43a_1-31a_2-99)\,x^2 + (-12a_0-65a_1-78a_2-54)\,x - 63a_0 - 39a_1 - 18a_2 - 6 \\
P_2 &= -72 + 34(-9-2a_2)\,x^6 + 4(-27-20a_1-46a_2)\,x^5 + 2(459-18a_0-3a_1+151a_2)\,x^4 \\
&\quad + \frac{3(357+104a_0+300a_1+448a_2)\,x^3}{2} + \frac{(672a_0+484a_1-213a_2-1080)\,x^2}{2} \\
&\quad + \frac{(-312a_0-785a_1-976a_2-900)\,x}{2} - \frac{645a_0}{2} - 236a_1 - 150a_2 \\
P_1 &= -720 + 360x^8 + 4(-63+26a_2)\,x^7 + 12(-261-16a_1-77a_2)\,x^6 \\
&\quad + 4(-144-72a_0-204a_1-379a_2)\,x^5 + \frac{(-96a_0+1360a_1+5696a_2+13653)\,x^4}{2} \\
&\quad + \frac{(4032a_0+6984a_1+9550a_2+8451)\,x^3}{2} \\
&\quad + \frac{(3432a_0+1927a_1-1693a_2-6669)\,x^2}{2} \\
&\quad + \frac{3(-2280-1152a_0-1799a_1-2242a_2)\,x}{2} - \frac{3381a_0}{2} - \frac{2903a_1}{2} - 1100a_2 \\
P_0 &= -6120 + 720x^9 + 8(423+124a_2)\,x^8 + 8(-783+4a_1-108a_2)\,x^7 \\
&\quad + (-24435-1080a_0-3796a_1-11314a_2)\,x^6 \\
&\quad + 2(2547-1176a_0-2280a_1-2062a_2)\,x^5 + (54108+7338a_0+15439a_1+30999a_2)\,x^4 \\
&\quad + \frac{(75816a_0+103136a_1+119720a_2+108891)\,x^3}{4} \\
&\quad + \frac{(11796a_0-11280a_1-51601a_2-107748)\,x^2}{4} \\
&\quad + \frac{(-66408a_0-85069a_1-99768a_2-107136)\,x}{4} - \frac{36873a_0}{4} - 8823a_1 - 7788a_2
\end{aligned}
$$

$$
P_{-1} = 3344(x-1)\left(\left(a_2+\frac{9}{2}\right)x^4+\left(\frac{82a_1}{209}-\frac{807a_2}{418}-\frac{9477}{836}\right)x^3 + \left(-\frac{27a_0}{19}-\frac{2571a_1}{418}-\frac{2172a_2}{209}-\frac{8397}{836}\right)x^2+\left(-\frac{39a_0}{38}+\frac{3987a_1}{836}+\frac{2678a_2}{209}+\frac{4599}{209}\right)x + \frac{6483a_0}{418}+\frac{13567a_1}{836}+\frac{6657a_2}{418}+\frac{3024}{209}\right)(1+x)^4
$$

Using method of undetermined coefficient to solve for a_i from the last equation above $P_{-1}=0$ gives the following solution for the coefficients a_i

$$
\begin{aligned}
a_0 &= -\frac{27}{8} \\
a_1 &= \frac{27}{4} \\
a_2 &= -\frac{9}{2}
\end{aligned}
$$

Substituting these in the the polynomial $p(x)$ results in

$$
P(x) = x^3 - \frac{9}{2}x^2 + \frac{27}{4}x - \frac{27}{8}
$$

ω is found by finding a solution to the equation generated by the following sum

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0 \\
\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!}\omega^i &= 0
\end{aligned}
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\text{Expression too large to display} = 0
$$

The solution ω of eq. 3A is found as

$$
\omega = \frac{1}{4x^3-6x^2-4x+6}\left(-2\sqrt{5}\sqrt{(1+x)(x-1)^3}+2x^2-2x+1\right)
$$

This ω is used to find a solution to $z'' = rz$.

$$
z_1(x) = e^{\int \omega \, dx} \tag{5A}
$$

Doing the integration gives in eq. (4A) gives

$$
\begin{aligned}
z_1(x) &= e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \frac{-2\sqrt{5}\,\sqrt{(1+x)(x-1)^3}+2x^2-2x+1}{4x^3-6x^2-4x+6} dx} \\
&= \frac{(1+x)^{\frac{1}{4}}\sqrt{2x-3}\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}\,\sqrt{(1+x)(x-1)^3}}{\sqrt{x^2-1}\,(2x-2)}}\,\mathrm{e}^{\frac{\operatorname{arctanh}\left(\frac{(3x-2)\sqrt{5}}{5\sqrt{x^2-1}}\right)\sqrt{(1+x)(x-1)^3}}{\sqrt{x^2-1}\,(2x-2)}}}{(x-1)^{\frac{1}{4}}}
\end{aligned}
$$

Which simplifies to

$$
z_1(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}}\sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1-x^2}\, dx} \\
&= z_1 e^{-\frac{\operatorname{arctanh}(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}}\sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1-x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\operatorname{arctanh}(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}} \sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\,\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)} dx \right)
\end{aligned}
$$

Therefore for kovacic case $n=6$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right) \\
&\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right) \left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}} \sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\,\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)} dx \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right) \\
&\quad + c_2 \left(\frac{\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\,\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\,\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \right) \left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}} \sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\,\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)} dx \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} + \frac{c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}\left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}}\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)}dx\right)}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}} + \frac{c_2 \left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{2}} \sqrt{\frac{3x-2+\sqrt{5}\,\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}}}\,(1+x)^{\frac{1}{4}}\sqrt{2x-3}\left(\int \frac{\left(x+\sqrt{x^2-1}\right)^{\sqrt{5}}\sqrt{x^2-1}\,\sqrt{-\frac{(2x-3)^2}{x^2-1}}\,\sqrt{x-1}}{\sqrt{1+x}\left(3x-2+\sqrt{5}\,\sqrt{x^2-1}\right)(2x-3)}dx\right)}{(x-1)^{\frac{1}{4}}\sqrt{\frac{1+x}{\sqrt{1-x^2}}}}$$

Verified OK.

3.7.1 Maple step by step solution

Let's solve

$(1-x^2)\,y''+y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y'}{x^2-1}+\frac{y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{x^2-1}-\frac{y}{x^2-1}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{1}{x^2-1}, P_3(x) = -\frac{1}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $((1+x) \cdot P_2(x))\bigg|_{x=-1} = \frac{1}{2}$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $((1+x)^2 \cdot P_3(x))\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $(x^2-1)\, y'' - y' - y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) - \frac{d}{du}y(u) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $\frac{d}{du}y(u)$ to series expansion

  $\frac{d}{du}y(u) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $\frac{d}{du}y(u) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(2r-1)\, u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-a_{k+1}(k+1+r)\,(2k+1+2r)+a_k(k^2+2kr+r^2-k-r-1)\right) u^{k}\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(2r-1)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{0,\frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1}+a_k(k^2+(2r-1)\,k+r^2-r-1)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=\frac{a_k\,(k^2+2kr+r^2-k-r-1)}{(2k+1+2r)(k+1+r)}$
- Recursion relation for $r=0$

  $a_{k+1}=\frac{a_k\,(k^2-k-1)}{(2k+1)(k+1)}$
- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+1}=\frac{a_k\,(k^2-k-1)}{(2k+1)(k+1)}\right]$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^k\,, a_{k+1}=\frac{a_k\,(k^2-k-1)}{(2k+1)(k+1)}\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=\frac{a_k\,(k^2-\frac{5}{4})}{(2k+2)(k+\frac{3}{2})}$
- Solution for $r=\frac{1}{2}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{1}{2}}, a_{k+1}=\frac{a_k\,(k^2-\frac{5}{4})}{(2k+2)(k+\frac{3}{2})}\right]$
- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_k(1+x)^{k+\frac{1}{2}}\,, a_{k+1}=\frac{a_k\,(k^2-\frac{5}{4})}{(2k+2)(k+\frac{3}{2})}\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(1+x)^k\right)+\left(\sum_{k=0}^{\infty}b_k(1+x)^{k+\frac{1}{2}}\right), a_{k+1}=\frac{a_k\,(k^2-k-1)}{(2k+1)(k+1)}, b_{k+1}=\frac{b_k\,(k^2-\frac{5}{4})}{(2k+2)(k+\frac{3}{2})}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 177

```
dsolve((1-x^2)*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{-3+2x}\left(\frac{3\sqrt{5}\,x-2\sqrt{5}-5\sqrt{x^2-1}}{3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{\frac{\sqrt{5}}{5}} + c_2\sqrt{-3+2x}\left(\frac{3\sqrt{5}\,x-2\sqrt{5}+5\sqrt{x^2-1}}{3\sqrt{5}\,x-2\sqrt{5}-5\sqrt{x^2-1}}\right)^{\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{-\frac{3\sqrt{5}}{10}}\left(x+\sqrt{x^2-1}\right)^{-\frac{\sqrt{5}}{5}}$$

✓ Solution by Mathematica

Time used: 36.335 (sec). Leaf size: 171

```
DSolve[(1-x^2)*y''[x]+y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[4]{1-x}\left(\sqrt{5}\sqrt{x-1}-\sqrt{x+1}\right)e^{2\sqrt{5}\operatorname{arctanh}\left(\frac{\sqrt{x+1}+\sqrt{2}}{\sqrt{x-1}}\right)}\left(c_2\int_1^x \frac{2e^{-4\sqrt{5}\operatorname{arctanh}\left(\frac{\sqrt{K[1]+1}+\sqrt{2}}{\sqrt{K[1]-1}}\right)}\sqrt{\frac{K[1]-1}{K[1]+1}}}{\left(\sqrt{K[1]+1}-\sqrt{5}\sqrt{K[1]-1}\right)^2}dK[1]+c_1\right)}{\sqrt{2}\sqrt[4]{x-1}}$$

3.8 problem 8

Internal problem ID [8324]
Internal file name [`OUTPUT/7257_Sunday_June_05_2022_05_40_07_PM_33549532/index.tex`]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - x) y'' - xy' + y = 0}$$

Writing the ode as

$$(x^2 - x) y'' - xy' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - x \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x + 4}{4x\left(x - 1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x + 4$$
$$t = 4x(x - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x + 4}{4x\left(x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1587: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x(x-1)^2$. There is a pole at $x = 0$ of order 1. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x(x-1)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{-x+4}{4x\left(x-1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	1	0	0	1
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-}-\left(\alpha_{c_1}^{-}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2}-\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$\begin{aligned}\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{x} - \frac{1}{2\,(x-1)} + (-)\,(0) \\ &= \frac{1}{x} - \frac{1}{2\,(x-1)} \\ &= \frac{x-2}{2x\,(x-1)}\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x} - \frac{1}{2\,(x-1)}\right)(0) + \left(\left(-\frac{1}{x^2} + \frac{1}{2\,(x-1)^2}\right) + \left(\frac{1}{x} - \frac{1}{2\,(x-1)}\right)^2 - \left(\frac{-x+4}{4x\,(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(\frac{1}{x} - \frac{1}{2(x-1)}\right) dx} \\ &= \frac{x}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2-x}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(\ln(x) + \frac{1}{x}\right)
\end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\ln(x) + \frac{1}{x}\right)\right)
\end{aligned}$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$E_c = \{4\}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\
&= \{-2, 2, 6\}
\end{aligned}$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-x+4}{4x\,(x-1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}E_\infty &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\}\\ &= \{2\}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	1	$\{4\}$
1	2	$\{-2, 2, 6\}$

Order of r at ∞	E_∞
2	$\{2\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 4,\ e_2 = -2,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\ &= \frac{1}{2}(2-(4+(-2)))\\ &= 0\end{aligned}$$

We now form the following rational function

$$\begin{aligned}\theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{1}{2}\left(\frac{4}{(x-(0))} + \frac{-2}{(x-(1))}\right)\\ &= \frac{2}{x} - \frac{1}{x-1}\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p}\\ &= \frac{2}{x} - \frac{1}{x-1}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \left(\frac{2}{x} - \frac{1}{x-1}\right) w + \frac{(x-2)^2}{4x^2 (x-1)^2} = 0$$

Solving for ω gives

$$\omega = \frac{x-2}{2x(x-1)}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \,dx}\\ &= \mathrm{e}^{\int \frac{x-2}{2x(x-1)}dx}\\ &= \frac{x}{\sqrt{x-1}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2-x}\,dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2}} \\
&= z_1\left(\sqrt{x-1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1\left(\ln(x) + \frac{1}{x}\right)
\end{aligned}$$

Therefore for kovacic case $n = 2$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\ln(x) + \frac{1}{x}\right)\right)
\end{aligned}$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$E_c = \{12\}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{3}{4}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm\frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{-6, 0, 6, 12, 18\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm\frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{1}{4x^2} + \frac{1}{2x^3} + \frac{5}{4x^4} + \frac{2}{x^5} + \frac{11}{4x^6} + \frac{7}{2x^7} + \cdots$$

The above shows that

$$b = -\frac{1}{4}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	1	$\{12\}$
1	2	$\{-6, 0, 6, 12, 18\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 12,\ e_2 = -6,\ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(6 - (12 + (-6))) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{4}{12}\left(\frac{12}{(x-(0))} + \frac{-6}{(x-(1))}\right) \\ &= \frac{2x-4}{x(x-1)} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= x(x-1) \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned} P_n &= -p(x) \\ &= -1 \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \dots, 0 \end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{-x+4}{4x(x-1)^2}$).

$$\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= 2x - 4 \\
P_2 &= -3(x-2)^2 \\
P_1 &= 3(x-2)^3 \\
P_0 &= -\frac{3(x-2)^4}{2} \\
P_{-1} &= 0
\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i &= 0 \\
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!} \omega^i &= 0
\end{aligned}$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\frac{1}{16}\left(-\left(2\omega\, x^2 - 2x\omega - x + 2\right)^4\right) = 0 \tag{3A}$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x\,(x-1)}(x-2)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega\, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{x-2}{2x(x-1)} dx} \\
&= \frac{x}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2-x} \, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\ln(x) + \frac{1}{x}\right)
\end{aligned}
$$

Therefore for kovacic case $n = 4$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2 \left(x \left(\ln(x) + \frac{1}{x}\right)\right)
\end{aligned}
$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 1. For the pole at $x = 0$ of order 1 then

$$E_c = \{12\}$$

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} + \frac{3}{4(x-1)^2} - \frac{1}{x-1}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = \frac{3}{4}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{-6, -2, 2, 6, 10, 14, 18\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{1}{4x^2} + \frac{1}{2x^3} + \frac{5}{4x^4} + \frac{2}{x^5} + \frac{11}{4x^6} + \frac{7}{2x^7} + \dots$$

The above shows that

$$b = -\frac{1}{4}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{6\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	1	$\{12\}$
1	2	$\{-6, -2, 2, 6, 10, 14, 18\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{6\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 12,\ e_2 = -6,\ e_\infty = 6$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{6}{12}(6 - (12 + (-6))) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{6}{12}\left(\frac{12}{(x-(0))} + \frac{-6}{(x-(1))}\right) \\ &= \frac{-6+3x}{x\,(x-1)} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= x(x-1) \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned} P_n &= -p(x) \\ &= -1 \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r = \frac{-x+4}{4x(x-1)^2}$).

$$\begin{aligned} P_6 &= -p \\ &= -1 \\ P_5 &= -6 + 3x \\ P_4 &= -\frac{15(x-2)^2}{2} \\ P_3 &= 15(x-2)^3 \\ P_2 &= -\frac{45(x-2)^4}{2} \\ P_1 &= \frac{45(x-2)^5}{2} \\ P_0 &= -\frac{45(x-2)^6}{4} \\ P_{-1} &= 0 \end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!} \omega^i = 0$$
$$\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!} \omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$\frac{1}{64}\left(-\left(2\omega\, x^2 - 2x\omega - x + 2\right)^6\right) = 0 \tag{3A}$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{2x\,(x-1)}(x-2)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega\, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{x-2}{2x(x-1)} dx} \\ &= \frac{x}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2-x}\, dx} \\ &= z_1 e^{\frac{\ln(x-1)}{2}} \\ &= z_1\left(\sqrt{x-1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2-x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x-1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\ln\left(x\right)+\frac{1}{x}\right)
\end{aligned}
$$

Therefore for kovacic case $n=6$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\ln\left(x\right)+\frac{1}{x}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\ln\left(x\right)+\frac{1}{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + c_2(x\ln\left(x\right)+1) \tag{1}
$$

Verification of solutions

$$
y = c_1 x + c_2(x\ln\left(x\right)+1)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((x^2-x)*diff(y(x), x$2)-x*diff(y(x), x)+y(x) = 0,y(x), singsol=all)
```

$$y(x) = c_1x + c_2(x\ln(x) + 1)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 20

```
DSolve[(x^2-x)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x - c_2(x\log(x) + 1)$$

3.9 problem 9

3.9.1 Maple step by step solution 7863

Internal problem ID [8325]
Internal file name [OUTPUT/7258_Sunday_June_05_2022_05_40_16_PM_90399427/index.tex]

Book: Collection of Kovacic problems
Section: section 2. Solution found using all possible Kovacic cases
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(-x^2+2)\,y''-x(4x^2+3)\,y'+(-2x^2+2)\,y=0}$$

Writing the ode as

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-3x\right)y'+\left(-2x^2+2\right)y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^4+2x^2 \\ B &= -4x^3-3x \\ C &= -2x^2+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 14x^2 + 5$$
$$t = 4\left(x^3 - 2x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 6 - 2 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^3 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = \sqrt{2}$ of order 2. There is a pole at $x = -\sqrt{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} + \frac{33}{64\left(x - \sqrt{2}\right)^2} + \frac{33}{64\left(x + \sqrt{2}\right)^2} - \frac{43\sqrt{2}}{128\left(x - \sqrt{2}\right)} + \frac{43\sqrt{2}}{128\left(x + \sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{11}{8} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{8}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{14x^2 + 5}{4\left(x^3 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$
$\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$
$-\sqrt{2}$	2	0	$\frac{11}{8}$	$-\frac{3}{8}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-} + \alpha_{c_3}^{-}\right) \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + \left((-)[\sqrt{r}]_{c_3} + \frac{\alpha_{c_3}^{-}}{x-c_3}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} + (-)\left(0\right) \\
&= -\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)} \\
&= \frac{-2x^2+1}{2x^3-4x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x^2 + a_1 x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(2) + 2\left(-\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)}\right)(2x+a_1) + \left(\left(\frac{1}{4x^2} + \frac{3}{8\left(x-\sqrt{2}\right)^2} + \frac{3}{8\left(x+\sqrt{2}\right)^2}\right) + \left(-\right.\right.
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 1, a_1 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x^2 + 1
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2+1\right) e^{\int \left(-\frac{1}{4x} - \frac{3}{8\left(x-\sqrt{2}\right)} - \frac{3}{8\left(x+\sqrt{2}\right)}\right) dx} \\
&= \left(x^2+1\right) e^{-\frac{\ln(x)}{4} - \frac{3\ln\left(x-\sqrt{2}\right)}{8} - \frac{3\ln\left(x+\sqrt{2}\right)}{8}} \\
&= \frac{x^2+1}{x^{\frac{1}{4}} \left(x-\sqrt{2}\right)^{\frac{3}{8}} \left(x+\sqrt{2}\right)^{\frac{3}{8}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-3x}{-x^4+2x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{4}-\frac{11\ln\left(x^2-2\right)}{8}} \\
&= z_1 \left(\frac{x^{\frac{3}{4}}}{\left(x^2-2\right)^{\frac{11}{8}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{5}{2}}+\sqrt{x}}{\left(x-\sqrt{2}\right)^{\frac{3}{8}}\left(x+\sqrt{2}\right)^{\frac{3}{8}}\left(x^2-2\right)^{\frac{11}{8}}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-3x}{-x^4+2x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2}-\frac{11\ln\left(x^2-2\right)}{4}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\int \frac{\sqrt{x}\left(x-\sqrt{2}\right)^{\frac{3}{4}}\left(x+\sqrt{2}\right)^{\frac{3}{4}}}{\left(x^2+1\right)^2}dx\right)
\end{aligned}
$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{5}{2}}+\sqrt{x}}{\left(x-\sqrt{2}\right)^{\frac{3}{8}}\left(x+\sqrt{2}\right)^{\frac{3}{8}}\left(x^2-2\right)^{\frac{11}{8}}}\right) \\
&\quad + c_2 \left(\frac{x^{\frac{5}{2}}+\sqrt{x}}{\left(x-\sqrt{2}\right)^{\frac{3}{8}}\left(x+\sqrt{2}\right)^{\frac{3}{8}}\left(x^2-2\right)^{\frac{11}{8}}}\left(\int \frac{\sqrt{x}\left(x-\sqrt{2}\right)^{\frac{3}{4}}\left(x+\sqrt{2}\right)^{\frac{3}{4}}}{\left(x^2+1\right)^2}dx\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2} + \frac{33}{64\left(x-\sqrt{2}\right)^2} + \frac{33}{64\left(x+\sqrt{2}\right)^2} - \frac{43\sqrt{2}}{128\left(x-\sqrt{2}\right)} + \frac{43\sqrt{2}}{128\left(x+\sqrt{2}\right)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \{-1, 2, 5\} \end{aligned}$$

For the pole at $x = \sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x-\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \left\{2, -\frac{3}{2}, \frac{11}{2}\right\} \end{aligned}$$

For the pole at $x = -\sqrt{2}$ let b be the coefficient of $\frac{1}{\left(x+\sqrt{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{33}{64}$. Hence

$$\begin{aligned} E_c &= \{2, 2+2\sqrt{1+4b}, 2-2\sqrt{1+4b}\} \\ &= \left\{2, -\frac{3}{2}, \frac{11}{2}\right\} \end{aligned}$$

Now since the order of r at ∞ is $4 > 2$ then

$$E_\infty = \{0, 2, 4\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-1, 2, 5\}$
$\sqrt{2}$	2	$\left\{2, -\frac{3}{2}, \frac{11}{2}\right\}$
$-\sqrt{2}$	2	$\left\{2, -\frac{3}{2}, \frac{11}{2}\right\}$

Order of r at ∞	E_∞
4	$\{0,2,4\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -1,\ e_2 = -\frac{3}{2},\ e_3 = -\frac{3}{2},\ e_\infty = 4$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(4 - (-1 + (-3/2) + (-3/2))) \\ &= 4 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{-1}{(x-(0))} + \frac{-\frac{3}{2}}{(x-(\sqrt{2}))} + \frac{-\frac{3}{2}}{(x-(-\sqrt{2}))}\right) \\ &= -\frac{1}{2x} - \frac{3}{4\left(x-\sqrt{2}\right)} - \frac{3}{4\left(x+\sqrt{2}\right)} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 4$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d = 4$, then letting

$$p = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$-\frac{6\left(\left(a_1 - \frac{5a_3}{6}\right)x^4 + 4\left(-\frac{5}{3} + a_0 + \frac{a_2}{3}\right)x^3 + \frac{5x^2a_1}{2} + \frac{2(-2a_0+a_2)x}{3} - \frac{2a_1}{3}\right)(x^2-2)}{\left(x+\sqrt{2}\right)^3\left(x-\sqrt{2}\right)^3x^2} = 0$$

And solving for p gives

$$p = x^4 + 2x^2 + 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{4x^3+4x}{x^4+2x^2+1} - \frac{1}{2x} - \frac{3}{4\left(x-\sqrt{2}\right)} - \frac{3}{4\left(x+\sqrt{2}\right)}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$\begin{aligned}&w^2 - \left(\frac{4x^3+4x}{x^4+2x^2+1} - \frac{1}{2x} - \frac{3}{4\left(x-\sqrt{2}\right)} - \frac{3}{4\left(x+\sqrt{2}\right)}\right)w \\ &+ \frac{\left(x^4-\frac{9}{2}x^2+\frac{1}{2}\right)^2}{\left(x^2+1\right)^2 x^2 \left(x-\sqrt{2}\right)^2 \left(x+\sqrt{2}\right)^2} = 0\end{aligned}$$

Solving for ω gives

$$\omega = \frac{2x^4-9x^2+1}{2\left(x^4-x^2-2\right)x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{2x^4-9x^2+1}{2\left(x^4-x^2-2\right)x} dx} \\ &= \frac{x^2+1}{x^{\frac{1}{4}}\left(x^2-2\right)^{\frac{3}{8}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x^3-3x}{-x^4+2x^2} \, dx} \\ &= z_1 e^{\frac{3\ln(x)}{4} - \frac{11\ln\left(x^2-2\right)}{8}} \\ &= z_1\left(\frac{x^{\frac{3}{4}}}{\left(x^2-2\right)^{\frac{11}{8}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{5}{2}} + \sqrt{x}}{(x^2 - 2)^{\frac{7}{4}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^3-3x}{-x^4+2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3\ln(x)}{2} - \frac{11\ln\left(x^2-2\right)}{4}}}{(y_1)^2}\,dx \\
&= y_1 \left(\int \frac{\sqrt{x}\,(x^2-2)^{\frac{3}{4}}}{(x^2+1)^2}\,dx \right)
\end{aligned}$$

Therefore for kovacic case $n = 2$ the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{5}{2}} + \sqrt{x}}{(x^2-2)^{\frac{7}{4}}} \right) + c_2 \left(\frac{x^{\frac{5}{2}} + \sqrt{x}}{(x^2-2)^{\frac{7}{4}}} \left(\int \frac{\sqrt{x}\,(x^2-2)^{\frac{3}{4}}}{(x^2+1)^2}\,dx \right) \right)
\end{aligned}$$

Attempting to find a solution using $n = 4$.

Unable to find solution using case 3 with $n = 4$.

Attempting to find a solution using $n = 6$.

Unable to find solution using case three with $n = 6$.

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1() + c_2()
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \left(x^{\frac{5}{2}} + \sqrt{x}\right)}{\left(x^2 - 2\right)^{\frac{7}{4}}} + \frac{c_2 \sqrt{x}\,(x^2 + 1) \left(\int \frac{\sqrt{x}\,(x^2-2)^{\frac{3}{4}}}{(x^2+1)^2} dx\right)}{\left(x^2 - 2\right)^{\frac{7}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \left(x^{\frac{5}{2}} + \sqrt{x}\right)}{\left(x^2 - 2\right)^{\frac{7}{4}}} + \frac{c_2 \sqrt{x}\,(x^2 + 1) \left(\int \frac{\sqrt{x}\,(x^2-2)^{\frac{3}{4}}}{(x^2+1)^2} dx\right)}{\left(x^2 - 2\right)^{\frac{7}{4}}}$$

Verified OK.

3.9.1 Maple step by step solution

Let's solve

$(-x^4 + 2x^2)\, y'' + (-4x^3 - 3x)\, y' + (-2x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2-1)y}{x^2(x^2-2)} - \frac{(4x^2+3)y'}{x(x^2-2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(4x^2+3)y'}{x(x^2-2)} + \frac{2(x^2-1)y}{x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+3}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -\frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- ◦ $x = 0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x^2(x^2-2) + x(4x^2+3)\,y' + (2x^2-2)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  ◦ Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

Rewrite ODE with series expansions

$-a_0(-1+2r)\,(-2+r)\,x^r - a_1(1+2r)\,(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (-a_k(2k+2r-1)\,(k+r-2) + a_{\ldots}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-(-1+2r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{2, \frac{1}{2}\right\}$

- Each term must be 0

  $-a_1(1+2r)(-1+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $-2(k+r-2)\left(k+r-\frac{1}{2}\right)a_k + a_{k-2}(k+r)(k+r-1) = 0$
- Shift index using $k- > k+2$

  $-2(k+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k(k+r+2)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k(k+r+2)(k+r+1)}{(k+r)(2k+3+2r)}$
- Recursion relation for $r = 2$

  $a_{k+2} = \frac{a_k(k+4)(k+3)}{(k+2)(2k+7)}$
- Solution for $r = 2$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k(k+4)(k+3)}{(k+2)(2k+7)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{a_k(k+4)(k+3)}{(k+2)(2k+7)}, a_1 = 0, b_{k+2} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}, b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve(x^2*(2-x^2)*diff(y(x), x$2) - x*(3+4*x^2)*diff(y(x), x) + (2-2*x^2)*y(x) = 0,y(x), si
```

$$y(x) = \frac{c_1\sqrt{x}\,(x^2+1)}{(x^2-2)^{\frac{7}{4}}} + \frac{c_2\sqrt{x}\,(x^2+1)\left(\int \frac{(x^2-2)^{\frac{3}{4}}\sqrt{x}}{(x^2+1)^2}dx\right)}{(x^2-2)^{\frac{7}{4}}}$$

✓ Solution by Mathematica

Time used: 20.212 (sec). Leaf size: 86

```
DSolve[x^2*(2-x^2)*y''[x] - x*(3+4*x^2)*y'[x] + (2-2*x^2)*y[x] == 0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{2^{3/4}c_2(x^2+1)\,x^2\,\text{Hypergeometric2F1}\left(\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{x^2}{2}\right)+3c_2(2-x^2)^{3/4}\,x^2+6c_1(x^2+1)\,\sqrt{x}}{6\,(2-x^2)^{7/4}}$$

4 section 3. Problems from Kovacic related papers

4.1 problem Kovacic 1985 paper. page 13. section 3.2, example 1 7869
4.2 problem Kovacic 1985 paper. page 14. section 3.2, example 2 7879
4.3 problem Kovacic 1985 paper. page 15. Weber equation 7889
4.4 problem Kovacic 1985 paper. page 19. section 4.2. Example 1 7899
4.5 problem Kovacic 1985 paper. page 23. section 5.2. Example 1 7908
4.6 problem Kovacic 1985 paper. page 25. section 5.2. Example 2 7921
4.7 problem Kovacic 2005 paper. Example 2 7933
4.8 problem David Saunders 1981 paper. Example 1 7941
4.9 problem David Saunders 1981 paper. Example 3 7949
4.10 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1 7969
4.11 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2 7975
4.12 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3 7982

4.1 problem Kovacic 1985 paper. page 13. section 3.2, example 1

Internal problem ID [8326]
Internal file name [OUTPUT/7259_Sunday_June_05_2022_05_40_31_PM_81695269/index.tex]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 13. section 3.2, example 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - \frac{(4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4)\,y}{4x^4} = 0}$$

Writing the ode as

$$y'' + \left(-x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 + 2x - 3 - \frac{1}{x} - \frac{7}{4x^2} + \frac{5}{x^3} - \frac{1}{x^4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4$$
$$t = 4x^4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1590: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{x^2} - \frac{5}{2x} - \frac{9}{4} - \frac{41x}{8} - \frac{443x^2}{32} - \frac{3017x^3}{64} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = 1$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be -5. Therefore

$$\begin{aligned} b &= (-5) - (0) \\ &= -5 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{-5}{1} + 2\right) &&= -\frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{-5}{1} + 2\right) &&= \frac{7}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x - 1 + \frac{1}{x} + \frac{3}{2x^2} + \frac{15}{8x^3} - \frac{17}{8x^4} - \frac{37}{8x^5} - \frac{85}{16x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x-1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2-2x+1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^6-8x^5+12x^4+4x^3+7x^2-20x+4}{4x^4} \\
&= Q + \frac{R}{4x^4} \\
&= \left(x^2-2x+3\right) + \left(\frac{4x^3+7x^2-20x+4}{4x^4}\right) \\
&= x^2-2x+3+\frac{4x^3+7x^2-20x+4}{4x^4}
\end{aligned}
$$

We see that the coefficient of the term x^3 in the quotient is 3. Now b can be found.

$$
\begin{aligned}
b &= (3)-(1) \\
&= 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x-1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{2}{1}-1\right) &&= \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{2}{1}-1\right) &&= -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{x^2}$	$-\frac{3}{2}$	$\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$x - 1$	$\frac{1}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) \\
&= \frac{1}{2} - \left(-\frac{3}{2}\right) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= \frac{1}{x^2} - \frac{3}{2x} + (x - 1) \\
&= \frac{1}{x^2} - \frac{3}{2x} + x - 1 \\
&= \frac{1}{x^2} - \frac{3}{2x} + x - 1
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(\frac{1}{x^2} - \frac{3}{2x} + x - 1\right)(2x + a_1) + \left(\left(-\frac{2}{x^3} + \frac{3}{2x^2} + 1\right) + \left(\frac{1}{x^2} - \frac{3}{2x} + x - 1\right)^2 - \left(\frac{4x^6 - 8x^5 + 12\cdots}{\cdots}\right.\right.$$

$$\frac{-2x^3 a_1 + (-4a_0 + 2a_1 - 4)\, x^2 \cdots}{x^2}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -1, a_1 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 1$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \left(x^2 - 1\right) e^{\int \left(\frac{1}{x^2} - \frac{3}{2x} + x - 1\right) dx} \\ &= \left(x^2 - 1\right) e^{\frac{x^2}{2} - x - \frac{1}{x} - \frac{3\ln(x)}{2}} \\ &= \frac{\left(x^2 - 1\right) e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{\left(x^2 - 1\right) e^{\frac{x^3 - 2x^2 - 2}{2x}}}{x^{\frac{3}{2}}} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}} \int \frac{1}{\frac{(x^2-1)^2\mathrm{e}^{\frac{x^3-2x^2-2}{x}}}{x^3}}\,dx \\ &= \frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}} \left(\int \frac{x^3\mathrm{e}^{\frac{-x^3+2x^2+2}{x}}}{(x^2-1)^2} dx\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}}\right) + c_2\left(\frac{(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}}\left(\int \frac{x^3\mathrm{e}^{\frac{-x^3+2x^2+2}{x}}}{(x^2-1)^2} dx\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}} + \frac{c_2(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}\left(\int \frac{x^3\mathrm{e}^{\frac{-x^3+2x^2+2}{x}}}{(x^2-1)^2} dx\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}}{x^{\frac{3}{2}}} + \frac{c_2(x^2-1)\,\mathrm{e}^{\frac{x^3-2x^2-2}{2x}}\left(\int \frac{x^3\mathrm{e}^{\frac{-x^3+2x^2+2}{x}}}{(x^2-1)^2} dx\right)}{x^{\frac{3}{2}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
   No special function solution was found.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve(diff(y(x),x$2)= (4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/(4*x^4)*y(x),y(x), singsol=al
```

$$y(x)=\frac{c_1 \mathrm{e}^{\frac{x^3-2x^2-2}{2x}}\left(x^2-1\right)}{x^{\frac{3}{2}}}+\frac{c_2 \mathrm{e}^{\frac{x^3-2x^2-2}{2x}}\left(x^2-1\right)\left(\int \frac{x^3 \mathrm{e}^{-\frac{x^3-2x^2-2}{x}}}{(x-1)^2(x+1)^2}dx\right)}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.629 (sec). Leaf size: 79

```
DSolve[y''[x]== (4*x^6-8*x^5+12*x^4+4*x^3+7*x^2-20*x+4)/(4*x^4)*y[x],y[x],x,IncludeSingularS
```

$$y(x) \to \frac{e^{\frac{x^2}{2}-x-\frac{1}{x}}(x^2-1)\left(c_2 \int_1^x \frac{e^{-K[1]^2+2K[1]+\frac{2}{K[1]}}K[1]^3}{(K[1]^2-1)^2}dK[1]+c_1\right)}{x^{3/2}}$$

4.2 problem Kovacic 1985 paper. page 14. section 3.2, example 2

4.2.1 Maple step by step solution 7886

Internal problem ID [8327]
Internal file name [`OUTPUT/7260_Sunday_June_05_2022_05_40_39_PM_18800528/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 14. section 3.2, example 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(-1 + \frac{6}{x^2}\right) y = 0}$$

Writing the ode as

$$y'' + \left(1 - \frac{6}{x^2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 - \frac{6}{x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -x^2 + 6$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1591: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -1 + \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 3 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -2 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx i - \frac{3i}{x^2} - \frac{9i}{2x^4} - \frac{27i}{2x^6} - \frac{405i}{8x^8} - \frac{1701i}{8x^{10}} - \frac{15309i}{16x^{12}} - \frac{72171i}{16x^{14}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = i$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= i \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = -1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-x^2 + 6}{x^2} \\ &= Q + \frac{R}{x^2} \\ &= (-1) + \left(\frac{6}{x^2}\right) \\ &= -1 + \frac{6}{x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 0. Dividing this by leading coefficient in t which is 1 gives 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= i \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{i} - 0\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{i} - 0\right) &&= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-x^2 + 6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	i	0	0

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 0$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{2}{x} + (-)\,(i) \\
&= -\frac{2}{x} - i \\
&= -\frac{2}{x} - i
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 2$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x^2 + a_1 x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(2) + 2\left(-\frac{2}{x} - i\right)(2x + a_1) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x} - i\right)^2 - \left(\frac{-x^2 + 6}{x^2}\right)\right) = 0$$

$$\frac{2ixa_1 + 4ia_0 - 6x - 4a_1}{x} = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = -3, a_1 = -3i\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^2 - 3ix - 3$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{\int \left(-\frac{2}{x} - i\right) dx} \\
&= \left(x^2 - 3ix - 3\right) \mathrm{e}^{-ix - 2\ln(x)} \\
&= \frac{(x^2 - 3ix - 3)\, \mathrm{e}^{-ix}}{x^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \int \frac{1}{\frac{(x^2-3ix-3)^2\mathrm{e}^{-2ix}}{x^4}}\,dx \\ &= \frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \right) + c_2 \left(\frac{(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} \left(\frac{(ix^2 - 3x - 3i)\,\mathrm{e}^{2ix}}{-2x^2 + 6ix + 6} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} - \frac{c_2 \mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2 - 3ix - 3)\,\mathrm{e}^{-ix}}{x^2} - \frac{c_2 \mathrm{e}^{ix}(ix^2 - 3x - 3i)}{2x^2}$$

Verified OK.

4.2.1 Maple step by step solution

Let's solve

$y'' + \left(1 - \frac{6}{x^2}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-6)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-6)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{x^2-6}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -6$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' + (x^2 - 6)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\left(k+r-1\right) x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(2+r)\left(-3+r\right) x^{r}+a_{1}(3+r)\left(-2+r\right) x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_{k}(k+r+2)\left(k+r-3\right)+a_{k-2}\right) x^{k+r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(2+r)\left(-3+r\right)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,3\}$

- Each term must be 0

  $a_{1}(3+r)\left(-2+r\right)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+2)\left(k+r-3\right)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+4+r)\left(k+r-1\right)+a_{k}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}}{(k+4+r)(k+r-1)}$

- Recursion relation for $r=-2$

  $a_{k+2}=-\frac{a_{k}}{(k+2)(k-3)}$

- Solution for $r=-2$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k-2}, a_{k+2}=-\frac{a_{k}}{(k+2)(k-3)}, a_{1}=0\right]$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_{k}}{(k+7)(k+2)}$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k}{(k+7)(k+2)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{a_k}{(k+2)(k-3)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+7)(k+2)}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)= ( (4*(5/2)^2-1)/(4*x^2)-1)*y(x),y(x), singsol=all)
```

$$y(x)=\frac{c_1(\cos(x)\,x^2-3x\sin(x)-3\cos(x))}{x^2}+\frac{c_2(x^2\sin(x)+3\cos(x)\,x-3\sin(x))}{x^2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[y''[x]== ( (4*(5/2)^2-1)/(4*x^2)-1)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x(c_1 j_2(x)-c_2 y_2(x))$$

4.3 problem Kovacic 1985 paper. page 15. Weber equation

4.3.1 Maple step by step solution 7895

Internal problem ID [8328]
Internal file name [OUTPUT/7261_Sunday_June_05_2022_05_40_44_PM_8687533/index.tex]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 15. Weber equation.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(\frac{x^2}{4} - \frac{11}{2}\right) y = 0}$$

Writing the ode as

$$y'' + \left(-\frac{x^2}{4} + \frac{11}{2}\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -\frac{x^2}{4} + \frac{11}{2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 22}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 22$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{11}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1593: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{11}{2x} - \frac{121}{4x^3} - \frac{1331}{4x^5} - \frac{73205}{16x^7} - \frac{1127357}{16x^9} - \frac{37202781}{32x^{11}} - \frac{643076643}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 22}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{11}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{11}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{11}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{11}{2}\right) - (0) \\
&= -\frac{11}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= -6 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{11}{2}}{\frac{1}{2}} - 1\right) &&= 5
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2}{4} - \frac{11}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}$	-6	5

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 5$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 5 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 5$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(20x^3 + 12x^2a_4 + 6xa_3 + 2a_2) + 2\left(-\frac{x}{2}\right)(5x^4 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^{2}}{4}\right.\right.$$

$$a_4x^4 + 2(10 + a_3)\,x^3 + 3(a_2 + 4a_4)\,x^2 + 2(2a_1 + 3a_3)\,x + 5a_0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0, a_1 = 15, a_2 = 0, a_3 = -10, a_4 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x^5 - 10x^3 + 15x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x^5 - 10x^3 + 15x) \, e^{\int -\frac{x}{2} dx} \\
&= (x^5 - 10x^3 + 15x) \, e^{-\frac{x^2}{4}} \\
&= x(x^4 - 10x^2 + 15) \, e^{-\frac{x^2}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= x(x^4 - 10x^2 + 15) \, e^{-\frac{x^2}{4}}
\end{aligned}$$

Which simplifies to

$$y_1 = x(x^4 - 10x^2 + 15) \, e^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x(x^4 - 10x^2 + 15) \, e^{-\frac{x^2}{4}} \int \frac{1}{x^2 (x^4 - 10x^2 + 15)^2 \, e^{-\frac{x^2}{2}}} \, dx \\
&= x(x^4 - 10x^2 + 15) \, e^{-\frac{x^2}{4}} \left(\int \frac{e^{\frac{x^2}{2}}}{x^2 (x^4 - 10x^2 + 15)^2} dx \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}}\right) \\
&\quad + c_2\left(x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2\left(x^4-10x^2+15\right)^2}dx\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}} \\
&\quad + c_2 x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2\left(x^4-10x^2+15\right)^2}dx\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}} + c_2 x\left(x^4-10x^2+15\right)\mathrm{e}^{-\frac{x^2}{4}}\left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{x^2\left(x^4-10x^2+15\right)^2}dx\right)
$$

Verified OK.

4.3.1 Maple step by step solution

Let's solve

$y'' + \left(-\frac{x^2}{4}+\frac{11}{2}\right)y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Multiply by denominators

  $4y'' + \left(-x^2+22\right)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k-m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$8a_2 + 22a_0 + (24a_3 + 22a_1)\, x + \left(\sum_{k=2}^{\infty} (4a_{k+2}(k+2)\,(k+1) + 22a_k - a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[8a_2 + 22a_0 = 0, 24a_3 + 22a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(k^2 + 3k + 2)\, a_{k+2} + 22a_k - a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$4\left((k+2)^2 + 3k + 8\right) a_{k+4} + 22a_{k+2} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{-22a_{k+2} + a_k}{4(k^2 + 7k + 12)}, a_2 = -\frac{11a_0}{4}, a_3 = -\frac{11a_1}{12}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 67

```
dsolve(diff(y(x),x$2)= (1/4*x^2-1/2-5)*y(x),y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x^2}{4}} x(x^4 - 10x^2 + 15) + c_2 \mathrm{e}^{-\frac{x^2}{4}} x(x^4 - 10x^2 + 15) \left(\int \frac{\mathrm{e}^{\frac{x^2}{2}}}{(x^4 - 10x^2 + 15)^2 x^2} dx\right)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 22

```
DSolve[y''[x]== (1/4*x^2-1/2-5)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \text{ParabolicCylinderD}(-6, ix) + c_1 \text{ParabolicCylinderD}(5, x)$$

4.4 problem Kovacic 1985 paper. page 19. section 4.2. Example 1

4.4.1 Maple step by step solution 7904

Internal problem ID [8329]
Internal file name [`OUTPUT/7262_Sunday_June_05_2022_05_40_48_PM_11536697/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 19. section 4.2. Example 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - \left(\frac{1}{x} - \frac{3}{16x^2}\right) y = 0}$$

Writing the ode as

$$y'' + \frac{(-16x+3)\,y}{16x^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{-16x+3}{16x^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 16x - 3$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{16x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1595: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{x} - \frac{3}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}E_c &= \{2, 2 + 2\sqrt{1+4b}, 2 - 2\sqrt{1+4b}\} \\ &= \{1, 2, 3\}\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1,\ e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1-16x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+4\sqrt{x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+4\sqrt{x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \int \frac{1}{\sqrt{x}\, \mathrm{e}^{4\sqrt{x}}} \, dx \\ &= x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}}\right) + c_2 \left(x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} \left(-\frac{\mathrm{e}^{-4\sqrt{x}}}{2}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} - \frac{c_2 x^{\frac{1}{4}} \mathrm{e}^{-2\sqrt{x}}}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = c_1 x^{\frac{1}{4}} \mathrm{e}^{2\sqrt{x}} - \frac{c_2 x^{\frac{1}{4}} \mathrm{e}^{-2\sqrt{x}}}{2}
$$

Verified OK.

4.4.1 Maple step by step solution

Let's solve

$y'' + \frac{(-16x+3)y}{16x^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(16x-3)y}{16x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(16x-3)y}{16x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = -\frac{16x-3}{16x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $16x^2y'' + (-16x+3)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+4r)\,(-3+4r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(4k+4r-1)\,(4k+4r-3) - 16a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+4r)\,(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $16\left(k+r-\frac{3}{4}\right)\left(k+r-\frac{1}{4}\right) a_k - 16a_{k-1} = 0$

- Shift index using $k- >k+1$

  $16\left(k+\frac{1}{4}+r\right)\left(k+\frac{3}{4}+r\right) a_{k+1} - 16a_k = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{16a_k}{(4k+1+4r)(4k+3+4r)}$

- Recursion relation for $r = \frac{1}{4}$

$a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}$

- Solution for $r = \frac{1}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}\right]$$

- Recursion relation for $r = \frac{3}{4}$

$a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}$

- Solution for $r = \frac{3}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+1} = \frac{16a_k}{(4k+4)(4k+6)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+1} = \frac{16a_k}{(4k+2)(4k+4)}, b_{k+1} = \frac{16b_k}{(4k+4)(4k+6)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)= (1/x-3/(16*x^2))*y(x),y(x), singsol=all)
```

$$y(x) = c_1 x^{\frac{1}{4}} \sinh\left(2\sqrt{x}\right) + c_2 x^{\frac{1}{4}} \cosh\left(2\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 41

```
DSolve[y''[x]== (1/x-3/(16*x^2))*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-2\sqrt{x}} \sqrt[4]{x} \left(2c_1 e^{4\sqrt{x}} - c_2\right)$$

4.5 problem Kovacic 1985 paper. page 23. section 5.2. Example 1

4.5.1 Maple step by step solution 7916

Internal problem ID [8330]
Internal file name [`OUTPUT/7263_Sunday_June_05_2022_05_40_51_PM_69715044/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 23. section 5.2. Example 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - \left(-\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}\right)y = 0}$$

Writing the ode as

$$y'' + \frac{(32x^2 - 27x + 27)\,y}{144x^2(x-1)^2} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{32x^2 - 27x + 27}{144x^2(x-1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-32x^2 + 27x - 27}{144\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -32x^2 + 27x - 27 \\ t &= 144\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-32x^2 + 27x - 27}{144\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1597: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 144(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x} - \frac{2}{9(x-1)^2} - \frac{3}{16x^2} + \frac{3}{16(x-1)}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{3}{16}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{3, 6, 9\}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{2}{9x^2} - \frac{37}{144x^3} - \frac{23}{48x^4} - \frac{101}{144x^5} - \frac{133}{144x^6} - \frac{55}{48x^7} + \cdots$$

The above shows that

$$b = -\frac{2}{9}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{4, 5, 6, 7, 8\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{3, 6, 9\}$
1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{4, 5, 6, 7, 8\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 3,\ e_2 = 4,\ e_\infty = 7$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(7 - (3 + (4))) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned} \theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\ &= \frac{4}{12}\left(\frac{3}{(x-(0))} + \frac{4}{(x-(1))}\right) \\ &= \frac{1}{x} + \frac{4}{3x-3} \end{aligned}$$

And

$$\begin{aligned} S &= \prod_{c\in\Gamma}(x-c) \\ &= x(x-1) \end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned} P_n &= -p(x) \\ &= -1 \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0 \end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r =$

$\frac{-32x^2+27x-27}{144(x^2-x)^2}$).

$$
\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{7x}{3} - 1 \\
P_2 &= -4x^2 + \frac{41}{12}x - \frac{3}{4} \\
P_1 &= \frac{40}{9}x^3 - \frac{409}{72}x^2 + \frac{5}{2}x - \frac{3}{8} \\
P_0 &= -\frac{64}{27}x^4 + \frac{871}{216}x^3 - \frac{257}{96}x^2 + \frac{13}{16}x - \frac{3}{32} \\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0 \\
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0
\end{aligned}
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\begin{aligned}
&-\frac{8x^4}{81} + \frac{871x^3}{5184} - \frac{257x^2}{2304} + \frac{13x}{384} - \frac{1}{256} + \frac{x(x-1)\left(320x^3 - 409x^2 + 180x - 27\right)\omega}{432} \\
&- \frac{x^2(x-1)^2\left(48x^2 - 41x + 9\right)\omega^2}{24} + x^3(x-1)^3\left(\frac{7x}{3} - 1\right)\omega^3 - x^4(x-1)^4\omega^4 = 0
\end{aligned}
$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{12x(x-1)} \left(7x - 3 + \sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x} + \sqrt{-\frac{2\left(\left(-x^2 + x + \frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x} + x^2 (x-1)\right)}{\sqrt{x^2 + \left(x^3 (x-1)^2\right)^{\frac{1}{3}} - x}}} \right) \tag{4A}$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega \, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx \\ &= e^{\int \omega \, dx} \int \frac{e^{\int -\frac{B}{A} \, dx}}{\left(e^{\int \omega \, dx}\right)^2} \, dx \end{aligned}$$

Since $B = 0$ then the above reduces to

$$y_2 = e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx$$

Therefore for kovacic case $n = 4$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\int \omega \, dx}\right) + c_2 \left(e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} \, dx\right)
\end{aligned}
$$

Attempting to find a solution using $n = 6$.

Unable to find solution using case three with $n = 6$.

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1() + c_2()
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}{12x(x-1)}dx} \tag{1}
$$

$$
+c_2 \mathrm{e}^{\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}-x}}}}{x(x-1)}dx\right)}{12}} \left(\int \mathrm{e}^{-\frac{\left(\int \frac{7x-3+\sqrt{x^2+\left(x^3(x- \cdots\right.}{\cdots}\right.}{\cdots}} \cdots\right.
$$

Verification of solutions

$$y = c_1 e^{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x}}}{12x(x-1)}dx}$$

$$+c_2 e^{\frac{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x+\sqrt{-\frac{2\left(\left(-x^2+x+\frac{\left(x^3(x-1)^2\right)^{\frac{1}{3}}}{2}\right)\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x+x^2(x-1)\right)}{\sqrt{x^2+\left(x^3(x-1)^2\right)^{\frac{1}{3}}}-x}}}{x(x-1)}dx}{12}} \left(\int e^{-\frac{\int \frac{7x-3+\sqrt{x^2+\left(x^3(x-\right.}}{}}}\right.$$

Verified OK.

4.5.1 Maple step by step solution

Let's solve

$y'' + \frac{(32x^2-27x+27)y}{144x^2(x-1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{32x^2-27x+27}{144x^2(x-1)^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{3}{16}$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $144y''x^2(x-1)^2 + (32x^2 - 27x + 27)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..4$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $9a_0(-1+4r)\,(-3+4r)\, x^r + (9a_1(3+4r)\,(1+4r) - 9a_0(32r^2 - 32r + 3))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (9a_k(4k\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $9(-1+4r)\,(-3+4r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$

- Each term must be 0

  $9a_1(3+4r)\,(1+4r) - 9a_0(32r^2 - 32r + 3) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = \frac{a_0 (32r^2-32r+3)}{16r^2+16r+3}$

- Each term in the series must be 0, giving the recursion relation

  $144(a_k + a_{k-2} - 2a_{k-1})\, k^2 + 144(2(a_k + a_{k-2} - 2a_{k-1})\, r - a_k - 5a_{k-2} + 6a_{k-1})\, k + 144(a_k + a_{k-}$

- Shift index using $k- >k+2$

  $144(a_{k+2} + a_k - 2a_{k+1})\, (k+2)^2 + 144(2(a_{k+2} + a_k - 2a_{k+1})\, r - a_{k+2} - 5a_k + 6a_{k+1})\, (k+2) + 1$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}+288kra_k-576kra_{k+1}+144r^2a_k-288r^2a_{k+1}-144ka_k-288ka_{k+1}-144ra_k-288ra_{k+1}+32a_k-27}{9(16k^2+32kr+16r^2+48k+48r+35)}$

- Recursion relation for $r = \frac{1}{4}$

  $a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)}$

- Solution for $r = \frac{1}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}, a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\frac{3a_0}{8}\right]$

- Recursion relation for $r = \frac{3}{4}$

  $a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}+72ka_k-720ka_{k+1}+5a_k-405a_{k+1}}{9(16k^2+72k+80)}$

- Solution for $r = \frac{3}{4}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{4}}, a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}+72ka_k-720ka_{k+1}+5a_k-405a_{k+1}}{9(16k^2+72k+80)}, a_1 = -\frac{a_0}{8}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{4}}\right), a_{k+2} = -\frac{144k^2a_k-288k^2a_{k+1}-72ka_k-432ka_{k+1}+5a_k-117a_{k+1}}{9(16k^2+56k+48)}, a_1 = -\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 209

```
dsolve(diff(y(x),x$2)= ( -3/(16*x^2) - 2/(9*(x-1)^2) + 3/(16*x*(x-1))  )*y(x),y(x), singsol=
```

$$y(x) = c_1\sqrt{(x-1)^{\frac{1}{3}}+1}\left((x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}(x-1)^{\frac{1}{3}}\left(\frac{\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}-\sqrt{3}}{-\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}+\sqrt{3}}\right)^{\frac{1}{8}} + \frac{c_2\sqrt{(x-1)^{\frac{1}{3}}+1}\left((x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1\right)^{\frac{1}{4}}(x-1)^{\frac{1}{3}}}{\left(\frac{\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}-\sqrt{3}}{-\sqrt{3}\,(x-1)^{\frac{1}{3}}+2\sqrt{(x-1)^{\frac{2}{3}}-(x-1)^{\frac{1}{3}}+1}+\sqrt{3}}\right)^{\frac{1}{8}}}$$

✓ Solution by Mathematica

Time used: 0.235 (sec). Leaf size: 550

```
DSolve[y''[x]== ( -3/(16*x^2) - 2/(9*(x-1)^2) + 3/(16*x*(x-1))  )*y[x],y[x],x,IncludeSingula
```

$$y(x) \to c_1 \exp\left(\int_1^x \text{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\right] dK[1]\right) + c_2 \exp\left(\int_1^x \text{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 118368K[1]^5 + 120096K[1]^4 - 50976K[1]^3 + 7776K[1]^2\right)\#1^2 + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\right] dK[1]\right) \int_1^x \exp\left(-2\int_1^{K[2]} \text{Root}\left[2048K[1]^4 - 3484K[1]^3 + 2313K[1]^2 - 702K[1] + \left(20736K[1]^8 - 82944K[1]^7 + 124416K[1]^6 - 82944K[1]^5 + 20736K[1]^4\right)\#1^4 + \left(-48384K[1]^7 + 165888K[1]^6 - 207360K[1]^5 + 110592K[1]^4 - 20736K[1]^3\right)\#1^3 + \left(41472K[1]^6 - 11836\right. \right. \right.$$

$$\left.\left. + \left(-15360K[1]^5 + 34992K[1]^4 - 28272K[1]^3 + 9936K[1]^2 - 1296K[1]\right)\#1 + 81\&, 1\right] dK[1]\right) dK[2]$$

4.6 problem Kovacic 1985 paper. page 25. section 5.2. Example 2

4.6.1 Maple step by step solution 7929

Internal problem ID [8331]
Internal file name [`OUTPUT/7264_Sunday_June_05_2022_05_41_00_PM_16839255/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 1985 paper. page 25. section 5.2. Example 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \frac{(5x^2 + 27)\, y}{36\, (x^2 - 1)^2} = 0}$$

Writing the ode as

$$y'' + \frac{(5x^2 + 27)\, y}{36\, (x^2 - 1)^2} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{5x^2 + 27}{36\, (x^2 - 1)^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5x^2 - 27}{36\,(x^2 - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5x^2 - 27$$
$$t = 36(x^2 - 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-5x^2 - 27}{36\,(x^2 - 1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 36(x^2 - 1)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Unable to find solution using case one

Attempting to find a solution using case $n = 2$.

Unable to find solution using case two.

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{2}{9(1+x)^2} - \frac{2}{9(x-1)^2} + \frac{11}{72(x-1)} - \frac{11}{72(1+x)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(1+x)^2}$ in the partial fractions decomposition of r given above. This shows that $b = -\frac{2}{9}$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{4, 5, 6, 7, 8\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} | k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx -\frac{5}{36x^2} - \frac{37}{36x^4} - \frac{23}{12x^6} - \frac{101}{36x^8} - \frac{133}{36x^{10}} - \frac{55}{12x^{12}} + \dots$$

The above shows that

$$b = -\frac{5}{36}$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{2, 4, 6, 8, 10\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
1	2	$\{4, 5, 6, 7, 8\}$
-1	2	$\{4, 5, 6, 7, 8\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{2, 4, 6, 8, 10\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 4, \ e_2 = 4, \ e_\infty = 8$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$
\begin{aligned}
d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\
&= \frac{4}{12}(8 - (4 + (4))) \\
&= 0
\end{aligned}
$$

The following rational function is

$$
\begin{aligned}
\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c} \\
&= \frac{4}{12}\left(\frac{4}{(x-(1))} + \frac{4}{(x-(-1))}\right) \\
&= \frac{8x}{3x^2-3}
\end{aligned}
$$

And

$$
\begin{aligned}
S &= \prod_{c\in\Gamma}(x-c) \\
&= (x-1)(1+x)
\end{aligned}
$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$
p(x) = 1
$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$
\begin{aligned}
P_n &= -p(x) \\
&= -1 \\
P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0
\end{aligned}
\tag{1A}
$$

The coefficients a_i are solved for from

$$
P_{-1} = 0 \tag{2A}
$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r =$

$\frac{-5x^2-27}{36(x^2-1)^2}$).

$$
\begin{aligned}
P_4 &= -p \\
&= -1 \\
P_3 &= \frac{8x}{3} \\
P_2 &= -5x^2 - \frac{1}{3} \\
P_1 &= \frac{50}{9}x^3 + \frac{14}{9}x \\
P_0 &= -\frac{125}{54}x^4 - \frac{67}{27}x^2 + \frac{1}{18} \\
P_{-1} &= 0
\end{aligned}
$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0
$$

$$
\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
\frac{(x-1)^2 (1+x)^2 \left(-5x^2 - \frac{1}{3}\right)\omega^2}{2} + \frac{8(x-1)^3 (1+x)^3 x\,\omega^3}{3} - (1+x)^4 (x-1)^4 \omega^4 - \frac{125x^4}{1296} - \frac{67x^2}{648} + \frac{1}{432} + \frac{25\omega\, x^5}{27} - \frac{2\omega\, x^3}{3} - \frac{7\omega x}{27} = 0
$$

The solution ω of eq. 3A is found as

$$
\omega = \frac{1}{6x^2-6}\left(4x + \sqrt{x^2-1+(x^2-1)^{\frac{2}{3}}} + \sqrt{-\frac{2\left(\left(-x^2 + \frac{(x^2-1)^{\frac{2}{3}}}{2} + 1\right)\sqrt{x^2-1+(x^2-1)^{\frac{2}{3}}} + x^3 - x\right)}{\sqrt{x^2-1+(x^2-1)^{\frac{2}{3}}}}}\right) \tag{4A}
$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Unable to integrate $\int \omega dx$. Leaving the integral unevaluated. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\int \omega \, dx} \end{aligned}$$

Where ω given above. The second solution y_2 to the original ode is found using reduction of order

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx \\ &= e^{\int \omega \, dx} \int \frac{e^{\int -\frac{B}{A}\, dx}}{\left(e^{\int \omega \, dx}\right)^2}\, dx \end{aligned}$$

Since $B = 0$ then the above reduces to

$$y_2 = e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} dx$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\int \omega \, dx}\right) + c_2\left(e^{\int \omega \, dx} \int \left(e^{\int \omega \, dx}\right)^{-2} dx\right) \end{aligned}$$

<u>Attempting to find a solution using $n = 6$.</u>

Unable to find solution using case three with $n = 6$.

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1() + c_2() \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+\frac{\left(x^2-1\right)^{\frac{2}{3}}}{2}+1\right)\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+x^3-x\right)}{\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}}}}{6x^2-6}dx} \tag{1}$$

$$+c_2 \mathrm{e}^{\frac{\left(\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+\frac{\left(x^2-1\right)^{\frac{2}{3}}}{2}+1\right)\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+x^3-x\right)}{\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}}}}{x^2-1}dx\right)}{6}} \left(\int \mathrm{e}^{-\frac{\left(\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+\right.\right.}{}}}{x}\right.}{}}\right.$$

Verification of solutions

$$y = c_1 \mathrm{e}^{\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+\frac{\left(x^2-1\right)^{\frac{2}{3}}}{2}+1\right)\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+x^3-x\right)}{\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}}}}{6x^2-6}dx}$$

$$+c_2 \mathrm{e}^{\frac{\left(\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+\frac{\left(x^2-1\right)^{\frac{2}{3}}}{2}+1\right)\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+x^3-x\right)}{\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}}}}{x^2-1}dx\right)}{6}} \left(\int \mathrm{e}^{-\frac{\left(\int \frac{4x+\sqrt{x^2-1+\left(x^2-1\right)^{\frac{2}{3}}}+\sqrt{-\frac{2\left(\left(-x^2+ \cdots\right.\right.}{\cdots}}}{x \cdots}\right.}{\cdots}} \cdots\right.$$

Verified OK.

4.6.1 Maple step by step solution

Let's solve

$y'' + \frac{(5x^2+27)y}{36(x^2-1)^2} = 0$

- Highest derivative means the order of the ODE is 2

  y''

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{5x^2+27}{36(x^2-1)^2}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\Big|_{x=-1} = 0$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = \frac{2}{9}$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $36y''(x^2-1)^2 + (5x^2+27)\,y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(36u^4 - 144u^3 + 144u^2)\left(\frac{d^2}{du^2}y(u)\right) + (5u^2 - 10u + 32)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $16a_0(-1+3r)(-2+3r)\,u^r + (16a_1(2+3r)(1+3r) - 2a_0(72r^2-72r+5))\,u^{1+r} + \left(\sum_{k=2}^{\infty} (16a_k(\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $16(-1+3r)(-2+3r) = 0$

- Values of r that satisfy the indicial equation

$r \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$

- Each term must be 0

$$16a_1(2+3r)(1+3r) - 2a_0(72r^2 - 72r + 5) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{a_0(72r^2-72r+5)}{8(9r^2+9r+2)}$$

- Each term in the series must be 0, giving the recursion relation

$$36(4a_k + a_{k-2} - 4a_{k-1})k^2 + 36(2(4a_k + a_{k-2} - 4a_{k-1})r - 4a_k - 5a_{k-2} + 12a_{k-1})k + 36(4a_k + a$$

- Shift index using $k- >k+2$

$$36(4a_{k+2} + a_k - 4a_{k+1})(k+2)^2 + 36(2(4a_{k+2} + a_k - 4a_{k+1})r - 4a_{k+2} - 5a_k + 12a_{k+1})(k+2) -$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 72kra_k - 288kra_{k+1} + 36r^2a_k - 144r^2a_{k+1} - 36ka_k - 144ka_{k+1} - 36ra_k - 144ra_{k+1} + 5a_k - 10a_{k+1}}{16(9k^2+18kr+9r^2+27k+27r+20)}$$

- Recursion relation for $r = \frac{1}{3}$

$$a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}$$

- Solution for $r = \frac{1}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}, a_1 = -\frac{11a_0}{48}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{1}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a_{k+1}}{16(9k^2+33k+30)}, a_1 = -\frac{11a_0}{48}\right]$$

- Recursion relation for $r = \frac{2}{3}$

$$a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 12ka_k - 336ka_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}$$

- Solution for $r = \frac{2}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{2}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 12ka_k - 336ka_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}, a_1 = -\frac{11a_0}{96}\right]$$

- Revert the change of variables $u = 1+x$

$$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{2}{3}}, a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} + 12ka_k - 336ka_{k+1} - 3a_k - 170a_{k+1}}{16(9k^2+39k+42)}, a_1 = -\frac{11a_0}{96}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k(1+x)^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k(1+x)^{k+\frac{2}{3}}\right), a_{k+2} = -\frac{36k^2a_k - 144k^2a_{k+1} - 12ka_k - 240ka_{k+1} - 3a_k - 74a}{16(9k^2+33k+30)}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Tetrahedral Galois group A4_SL2.
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 103

```
dsolve(diff(y(x),x$2)= -(5*x^2+27)/(36*(x^2-1)^2)*y(x),y(x), singsol=all)
```

$$y(x) = c_1 (x^2-1)^{\frac{1}{3}} e^{\int \mathrm{RootOf}\left(-1+(432x^4-864x^2+432)\_Z^4+(-72x^2+72)\_Z^2+16x\_Z,\mathrm{index}=1\right)dx} + c_2 (x^2-1)^{\frac{1}{3}} e^{\int \mathrm{RootOf}\left(-1+(432x^4-864x^2+432)\_Z^4+(-72x^2+72)\_Z^2+16x\_Z,\mathrm{index}=2\right)dx}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 38

```
DSolve[y''[x]== -(5*x^2+27)/(36*(x^2-1)^2)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2-1}\left(c_1 P_{-\frac{1}{6}}^{\frac{1}{3}}(x) + c_2 Q_{-\frac{1}{6}}^{\frac{1}{3}}(x)\right)$$

4.7 problem Kovacic 2005 paper. Example 2

4.7.1 Maple step by step solution 7938

Internal problem ID [8332]
Internal file name [`OUTPUT/7265_Sunday_June_05_2022_05_41_10_PM_17690448/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Kovacic 2005 paper. Example 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + \frac{y}{4x^2} = 0}$$

The ode can be written as

$$4x^2y'' + y = 0$$

Which shows it is a Euler ODE. Writing the ode as

$$4x^2y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1601: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \sqrt{x}\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \sqrt{x} \int \frac{1}{x} \, dx \\
&= \sqrt{x}(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\, \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x} + c_2\sqrt{x}\, \ln(x)
$$

Verified OK.

4.7.1 Maple step by step solution

Let's solve

$4x^2y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y'' + y = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{d}{dt}y(t) - \frac{y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + \frac{y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

$$y_2(t) = t\,\mathrm{e}^{\frac{t}{2}}$$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 \mathrm{e}^{\frac{t}{2}} + c_2 t\,\mathrm{e}^{\frac{t}{2}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 \sqrt{x} + c_2 \sqrt{x}\, \ln(x)$$

- Simplify

$$y = (c_1 + c_2 \ln(x))\sqrt{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)= -1/(4*x^2)*y(x),y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\, \ln(x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 24

```
DSolve[y''[x]== -1/(4*x^2)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{x}(c_2 \log(x) + 2c_1)$$

4.8 problem David Saunders 1981 paper. Example 1

4.8.1 Maple step by step solution 7947

Internal problem ID [8333]
Internal file name [`OUTPUT/7266_Sunday_June_05_2022_05_41_13_PM_96761775/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: David Saunders 1981 paper. Example 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - (x^2 + 3)\, y = 0}$$

Writing the ode as

$$y'' + (-x^2 - 3)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 + 3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1603: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2+3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{3}{1} - 1\right) &= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + (x) \\
&= x \\
&= x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(x)\,(1) + \left((1) + (x)^2 - \left(x^2 + 3\right)\right) &= 0 \\
-2a_0 &= 0
\end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (x)\, e^{\int x dx} \\
&= (x)\, e^{\frac{x^2}{2}} \\
&= x\, e^{\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\,dx \\ &= x\,\mathrm{e}^{\frac{x^2}{2}} \left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^2}{2}} \left(\frac{-\sqrt{\pi}\,\operatorname{erf}(x)\,x - \mathrm{e}^{-x^2}}{x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\,\operatorname{erf}(x)\,x\,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right)$$

Verified OK.

4.8.1 Maple step by step solution

Let's solve

$y'' + (-x^2 - 3)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 3a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k - a_{k-2} = 0$

- Shift index using $k- > k + 2$

$$\left((k+2)^2+3k+8\right)a_{k+4}-3a_{k+2}-a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=\frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2=\frac{3a_0}{2}, a_3=\frac{a_1}{2}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)= (x^2+3)*y(x),y(x), singsol=all)
```

$$y(x)=c_1\mathrm{e}^{\frac{x^2}{2}}x+c_2\mathrm{e}^{\frac{x^2}{2}}\left(\sqrt{\pi}\,\mathrm{erf}\,(x)\,x+\mathrm{e}^{-x^2}\right)$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 46

```
DSolve[y''[x]== (x^2+3)*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\mathrm{erf}(x)+c_1e^{x^2}x-c_2\right)$$

4.9 problem David Saunders 1981 paper. Example 3

4.9.1 Maple step by step solution 7966

Internal problem ID [8334]
Internal file name [OUTPUT/7267_Sunday_June_05_2022_05_41_17_PM_70244857/index.tex]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: David Saunders 1981 paper. Example 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' - 2y = 0}$$

Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1605: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 1$ the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{2}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-4, 2, 8\}
\end{aligned}
$$

Since the order of r at ∞ is 2 then let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
E_\infty &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{-4, 2, 8\}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{-4, 2, 8\}$

Order of r at ∞	E_∞
2	$\{-4, 2, 8\}$

Using the family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 2,\ e_\infty = 2$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(2 - (2)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{2}{(x-(0))}\right) \\ &= \frac{1}{x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{x} - \frac{2}{x^2} = 0$$

Solving for ω gives

$$\omega = -\frac{1}{x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3} \right) \end{aligned}$$

Therefore for kovacic case $n = 2$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x} \right) + c_2 \left(\frac{1}{x} \left(\frac{x^3}{3} \right) \right) \end{aligned}$$

Attempting to find a solution using $n = 4$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 2$. Hence

$$E_c = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 4$. Hence the above becomes

$$E_c = \{-12, -3, 6, 15, 24\}$$

Let

$$E_\infty = \left\{ 6 + \frac{12k}{n} \sqrt{1 + 4b} | k = 0, \pm 1, \pm 2, \ldots, \pm \frac{n}{2} \right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{2}{x^2} + \cdots$$

The above shows that

$$b = 2$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 4$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-12, -3, 6, 15, 24\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 4$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-12, -3, 6, 15, 24\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-12, -3, 6, 15, 24\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \dots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = -12,\ e_\infty = -12$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{4}{12}(-12 - (-12)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12} \sum_{c \in \Gamma} \frac{e_c}{x-c} \\ &= \frac{4}{12} \left(\frac{-12}{(x-(0))} \right) \\ &= -\frac{4}{x}\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c \in \Gamma} (x-c) \\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x) \\ &= -1 \\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2 r P_{i+1} \qquad i = n, n-1, \ldots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 4$ and $r = \frac{2}{x^2}$).

$$\begin{aligned}P_4 &= -p \\ &= -1 \\ P_3 &= -4 \\ P_2 &= -12 \\ P_1 &= -24 \\ P_0 &= -24 \\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i = 0$$
$$\sum_{i=0}^{4} S^i \frac{P_i}{(4-i)!}\omega^i = 0$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$-(x\omega + 1)^4 = 0$$

The solution ω of eq. 3A is found as

$$\omega = \frac{1}{x}(-1)$$

This ω is used to find a solution to $z'' = rz$.

$$z_1(x) = e^{\int \omega \, dx} \tag{5A}$$

Doing the integration gives in eq. (4A) gives

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}}\,dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 4$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

Attempting to find a solution using $n = 6$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. This shows that $b = 2$. Hence

$$E_c = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \,|\, k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z}$$

Where n for case 3 is $4, 6$ or 12. For the current case $n = 6$. Hence the above becomes

$$E_c = \{-12, -6, 0, 6, 12, 18, 24\}$$

Let

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} \,|\, k = 0, \pm 1, \pm 2, \dots, \pm \frac{n}{2}\right\} \cap \mathbb{Z} \tag{B1}$$

Where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series for r at ∞ given by

$$r \approx \frac{2}{x^2} + \cdots$$

The above shows that

$$b = 2$$

The value of n in eq. (B1) for case 3 is $4, 6$ or 2.For the current case $n = 6$. eq. (B1) simplifies to the following, after removing any duplicate and non integer entries in the set.

$$E_\infty = \{-12, -6, 0, 6, 12, 18, 24\}$$

The following table summarizes the results found so far for poles and for the order of r at ∞ for case 3 of Kovacic algorithm using $n = 6$.

pole c location	pole order	set $\{E_c\}$
0	2	$\{-12, -6, 0, 6, 12, 18, 24\}$

Order of r at ∞	set $\{E_\infty\}$
2	$\{-12, -6, 0, 6, 12, 18, 24\}$

Now that E_c sets for all poles are found and E_∞ set is found, the next step is to determine a non negative integer d using the following

$$d = \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right)$$

Where in the above e_c is a distinct element from each corresponding E_c. This means all possible tuples $\{e_{c_1}, e_{c_2}, \ldots, e_{c_n}\}$ are tried in the sum above, where e_{c_i} is one element of each E_c found earlier. Using the following family $\{e_1, e_2, \ldots, e_\infty\}$ given by

$$e_1 = 0, \; e_\infty = 0$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{n}{12}\left(e_\infty - \sum_{c \in \Gamma} e_c\right) \\ &= \frac{6}{12}(0 - (0)) \\ &= 0 \end{aligned}$$

The following rational function is

$$\begin{aligned}\theta &= \frac{n}{12}\sum_{c\in\Gamma}\frac{e_c}{x-c}\\ &= \frac{6}{12}\left(\frac{0}{(x-(0))}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}S &= \prod_{c\in\Gamma}(x-c)\\ &= x\end{aligned}$$

The polynomial $p(x)$ is now determined. Since the degree of the polynomial is $d = 0$, then let

$$p(x) = 1$$

The following set of equations are set up in order to determine the coefficients a_i (if any) of the above polynomial

$$\begin{aligned}P_n &= -p(x)\\ &= -1\\ P_{i-1} &= -Sp_i' + ((n-i)S' - S\theta)P_i - (n-1)(i+1)S^2rP_{i+1} \qquad i = n, n-1, \ldots, 0\end{aligned} \tag{1A}$$

The coefficients a_i are solved for from

$$P_{-1} = 0 \tag{2A}$$

By using method of undetermined coefficients. Carrying the above computation in eq. (1A) gives the following sequence of polynomials P_i (noting that $n = 6$ and $r = \frac{2}{x^2}$).

$$\begin{aligned}P_6 &= -p\\ &= -1\\ P_5 &= 0\\ P_4 &= 12\\ P_3 &= 24\\ P_2 &= -216\\ P_1 &= -1440\\ P_0 &= -2880\\ P_{-1} &= 0\end{aligned}$$

Because $P_{-1} = 0$ then $z = e^{\int \omega}$ is a solution. ω is found by finding a solution to the equation generated by the following sum

$$
\begin{aligned}
\sum_{i=0}^{n} S^i \frac{P_i}{(n-i)!}\omega^i &= 0 \\
\sum_{i=0}^{6} S^i \frac{P_i}{(6-i)!}\omega^i &= 0
\end{aligned}
$$

Where the P_i are the polynomials found earlier. Computing the above sum gives

$$
-(x\omega - 2)^2 (x\omega + 1)^4 = 0
$$

The solution ω of eq. 3A is found as

$$
\omega = \frac{1}{x}(-1)
$$

This ω is used to find a solution to $z'' = rz$.

$$
z_1(x) = e^{\int \omega \, dx} \tag{5A}
$$

Doing the integration gives in eq. (4A) gives

$$
\begin{aligned}
z_1(x) &= e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \frac{1}{x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}}\, dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore for kovacic case $n = 6$ the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{3}$$

Verified OK.

4.9.1 Maple step by step solution

Let's solve

$x^2 y'' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- ○ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2)=0$
- Roots of the characteristic polynomial

  $r=(-1,2)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{2t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{2t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x}+c_2x^2$
- Simplify

  $y=\frac{c_1}{x}+c_2x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)= 2*y(x),y(x), singsol=all)
```

$$y(x)=\frac{c_1}{x}+c_2x^2$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]== 2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2x^3 + c_1}{x}$$

4.10 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1

4.10.1 Maple step by step solution 7972

Internal problem ID [8335]
Internal file name [`OUTPUT/7268_Sunday_June_05_2022_05_41_32_PM_68038286/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4xy' + \left(4x^2 + 2\right) y = 0}$$

Writing the ode as

$$y'' + 4xy' + \left(4x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4x \\ C &= 4x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1607: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\ &= z_1 e^{-x^2} \\ &= z_1 \left(e^{-x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2}
$$

Verified OK.

4.10.1 Maple step by step solution

Let's solve

$y'' + 4xy' + (4x^2 + 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(4*x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2} + c_2 x e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(4*x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2 x + c_1)$$

4.11 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2

4.11.1 Maple step by step solution 7978

Internal problem ID [8336]
Internal file name [OUTPUT/7269_Sunday_June_05_2022_05_41_34_PM_49921325/index.tex]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + \left(x^2 + 2\right) y = 0}$$

Writing the ode as

$$x^2y'' - 2xy' + \left(x^2 + 2\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= x^2 + 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1609: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}\\ &= z_1 e^{\ln(x)}\\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = x\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x\cos(x)) + c_2(x\cos(x)(\tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\cos(x) + c_2 x\sin(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\cos(x) + c_2 x\sin(x)
$$

Verified OK.

4.11.1 Maple step by step solution

Let's solve

$x^2 y'' - 2xy' + (x^2+2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+2)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{(x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{x^2+2}{x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 2$

◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2y''-2xy'+(x^2+2)\,y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

$x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

◦ Shift index using $k->k-m$

$x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

◦ Convert $x\cdot y'$ to series expansion

$x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

◦ Convert $x^2\cdot y''$ to series expansion

$x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+r)\,(-2+r)\,x^r+a_1r(-1+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r-1)\,(k+r-2)+a_{k-2})\,x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)\,(-2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{1,2\}$

- Each term must be 0

$a_1r(-1+r)=0$

- Solve for the dependent coefficient(s)

$a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)(k+r-2)+a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+1+r)(k+r)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{a_k}{(k+3)(k+2)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=-\frac{a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+(x^2+2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x \sin(x) + c_2 \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]-2*x*y'[x]+(x^2+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-ix} x - \frac{1}{2} i c_2 e^{ix} x$$

4.12 problem Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3

4.12.1 Maple step by step solution 7987

Internal problem ID [8337]
Internal file name [`OUTPUT/7270_Sunday_June_05_2022_05_41_37_PM_88565773/index.tex`]

Book: Collection of Kovacic problems
Section: section 3. Problems from Kovacic related papers
Problem number: Carolyn J. Smith 1984 paper. Appendix B examples and tests. Example 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x-2)^2 y''-(x-2) y'-3y=0}$$

Writing the ode as

$$(x-2)^2 y''+(-x+2) y'-3y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x-2)^2 \\ B &= -x+2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4(x-2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = 4(x-2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4(x-2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1611: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4 (x-2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2 (x-2)} + (-) (0) \\ &= -\frac{3}{2 (x-2)} \\ &= -\frac{3}{2 (x-2)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\left(x-2\right)}\right)(0) + \left(\left(\frac{3}{2\left(x-2\right)^2}\right) + \left(-\frac{3}{2\left(x-2\right)}\right)^2 - \left(\frac{15}{4\left(x-2\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2(x-2)} dx} \\ &= \frac{1}{\left(x-2\right)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x+2}{(x-2)^2} \, dx} \\ &= z_1 e^{\frac{\ln(x-2)}{2}} \\ &= z_1 \left(\sqrt{x-2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x-2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x+2}{(x-2)^2} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{\ln(x-2)}}{(y_1)^2} dx \\
&= y_1 \left(\frac{(x-2)^4}{4} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x-2} \right) + c_2 \left(\frac{1}{x-2} \left(\frac{(x-2)^4}{4} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x-2} + \frac{c_2 (x-2)^3}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x-2} + \frac{c_2 (x-2)^3}{4}
$$

Verified OK.

4.12.1 Maple step by step solution

Let's solve

$(x-2)^2 y'' + (-x+2) y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x-2} + \frac{3y}{(x-2)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x-2} - \frac{3y}{(x-2)^2} = 0$

- ☐ Check to see if $x_0 = 2$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{1}{x-2}, P_3(x) = -\frac{3}{(x-2)^2}\right]$
  - $(x-2) \cdot P_2(x)$ is analytic at $x = 2$

    $\left.\left((x-2) \cdot P_2(x)\right)\right|_{x=2} = -1$
  - $(x-2)^2 \cdot P_3(x)$ is analytic at $x = 2$

    $\left.\left((x-2)^2 \cdot P_3(x)\right)\right|_{x=2} = -3$
  - $x = 2$is a regular singular point

    Check to see if $x_0 = 2$ is a regular singular point

    $x_0 = 2$
- Multiply by denominators

  $(x-2)^2 y'' + (-x+2) y' - 3y = 0$
- Change variables using $x = u + 2$ so that the regular singular point is at $u = 0$

  $u^2 \left(\frac{d^2}{du^2} y(u)\right) - u\left(\frac{d}{du} y(u)\right) - 3y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite DE with series expansions
  - Convert $u \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r}$
  - Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r}$

    Rewrite DE with series expansions

    $\sum_{k=0}^{\infty} a_k (k+r+1)(k+r-3) u^{k+r} = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+1)(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_k = 0$
- Recursion relation for $r = 0$

  $a_k = 0$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$
- Revert the change of variables $u = x - 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-2)^k, a_k = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve((x-2)^2*diff(y(x),x$2)-(x-2)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x-2} + \frac{c_2(x^4 - 8x^3 + 24x^2 - 32x)}{x-2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 22

```
DSolve[(x-2)^2*y''[x]-(x-2)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-2)^3 + \frac{c_2}{x-2}$$